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# Optimal execution strategies in limit order books with general shape functions

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**Abstract:** We consider optimal execution strategies for block market orders placed in a limit order book (LOB). We build on the resilience model proposed by Obizhaeva and Wang (2005) but allow for a general shape of the LOB defined via a given density function. Thus, we can allow for empirically observed LOB shapes and obtain a nonlinear price impact of market orders. We distinguish two possibilities for modeling the resilience of the LOB after a large market order: the exponential recovery of the number of limit orders, i.e., of the volume of the LOB, or the exponential recovery of the bid-ask spread. We consider both of these resilience modes and, in each case, derive explicit optimal execution strategies in discrete time. Applying our results to a block-shaped LOB, we obtain a new closed-form representation for the optimal strategy, which explicitly solves the recursive scheme given in Obizhaeva and Wang (2005). We also provide some evidence for the robustness of optimal strategies with respect to the choice of the shape function and the resilience-type.

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# 1 Introduction.

A common problem for stock traders consists in unwinding large block orders of shares, which can comprise up to twenty percent of the daily traded volume of shares. Orders of this size create significant impact on the asset price and, to reduce the overall market impact, it is necessary to split them into smaller orders that are subsequently placed throughout a certain time interval. The question at hand is thus to allocate an optimal proportion of the entire order to each individual placement such that the overall price impact is minimized.

Problems of this type were investigated by Bertsimas and Lo [7], Almgren and Chriss [2, 3], Almgren and Lorenz [4], Obizhaeva and Wang [15], and Schied and Schöneborn [17] to mention only a few. For extensions to situations with several competing traders, see [10], [11], [18], and the references therein.

The mathematical formulation of the corresponding optimization problem relies first of all on specifying a stock price model that takes into account the often nonlinear feedback effects resulting from the placement of large orders by a ‘large trader’. In the majority of models in the literature, such orders affect the stock price in the following two ways. A first part of the price impact is permanent and forever pushes the price in a certain direction (upward for buy orders, downward for sell orders). The second part, which is usually called the temporary impact, has no duration and only instantaneously affects the trade that has triggered it. It is therefore equivalent to a (possibly nonlinear) penalization by transaction costs. Models of this type underlie the above-mentioned papers [7], [2], [3], [4], [10], [11], and [18]. Also the market impact models described in Bank and Baum [6], Cetin *et al.* [12], Frey [13], and Frey and Patie [14] fall into that category. While most of these models start with the dynamics of the asset price process as a given fundamental, Obizhaeva and Wang [15] recently proposed a market impact model that derives its dynamics from an underlying model of a limit order book (LOB). In this model, the ask part of the LOB consists of a uniform distribution of shares offered at prices higher than the current best ask price. When the large trader is not active, the mid price of the LOB fluctuates according to the actions of noise traders, and the bid-ask spread remains constant. A buy market order of the large trader, however, consumes a block of shares located immediately to the right of the best ask and thus increase the ask price by a linear proportion of the size of the order. In addition, the LOB will recover from the impact of the buy order, i.e., it will show a certain resilience. The resulting price impact will neither be instantaneous nor entirely permanent but will decay on an exponential scale.

The model from [15] is quite close to descriptions of price impact on LOBs found in empirical studies such as Biais *et al.* [8], Potters and Bouchaud [16], Bouchaud *et al.* [9], and Weber and Rosenow [19]. In particular, the existence of a strong resilience effect, which stems from the placement of new limit orders close to the bid-ask spread, seems to be a well established fact, although its quantitative features seem to be the subject of an ongoing discussion.

In this paper, we will pick up the LOB-based market impact model from [15] and generalize it by allowing for a *nonuniform* price distribution of shares within the LOB. The resulting nonconstant LOB shape conforms to empirical observations made in [8, 16, 9, 19]. It also leads

completely naturally to a *nonlinear* price impact of market orders as found in an empirical study by Almgren *et al.* [5]; see also Almgren [1] and the references therein. In this generalized model, we will also consider the following two distinct possibilities for modeling the resilience of the LOB after a large market order: the exponential recovery of the number of limit orders, i.e., of the volume of the LOB (Model 1), or the exponential recovery of the bid-ask spread (Model 2). While one can imagine also other possibilities, we will focus on these two obvious resilience modes.

After introducing the generalized LOB with its two resilience modes, we consider the problem of optimally executing a buy order for  $X_0$  shares within a certain time frame  $[0, T]$ . The focus on buy orders is for the simplicity of the presentation only, completely analogous results hold for sell orders as well. While most other papers, including [15], focus on optimization within the class of *deterministic* strategies, we will here allow for dynamic updating of trading strategies, that is, we optimize over the larger class of adapted strategies. We will also allow for intermediate sell orders in our strategies. Our main results, Theorem 4.1 and Theorem 5.1, will provide explicit solutions of this problem in Model 1 and Model 2, respectively. Applying our results to a block-shaped LOB, we obtain a new closed-form representation for the corresponding optimal strategy, which explicitly solves the recursive scheme given in [15]. Looking at several examples, we will also find some evidence for the robustness of the optimal strategy.

The method we use in our proofs is different from the approach used in [15]. Instead of using dynamic programming techniques, we will first reduce the model of a full LOB with nontrivial bid-ask spreads to a simplified model, for which the bid-ask spreads have collapsed but the optimization problem is equivalent. The minimization of the simplified cost functional is then reduced to the minimization of certain functions that are defined on an affine space. This latter minimization is then carried out by means of the Lagrange multiplier method and explicit calculations.

The paper is organized as follows. In Section 2, we explain the two market impact models that we derive from the generalized LOB model with different resilience modes. In Section 3, we set up the resulting optimization problem. The main results for Models 1 and 2 are presented in the respective Sections 4 and 5. In Section 6, we consider the special case of a uniform distribution of shares in the LOB as considered in [15]. In particular, we provide our new explicit formula for the optimal strategy in a block-shaped LOB as obtained in [15]. Section 7 contains numerical and theoretical studies of the optimization problem for various nonconstant shape functions. The proofs of our main results are given in the remaining Sections 8 through 11. More precisely, in Section 8 we reduce the optimization problem for our two-sided LOB models to the optimization over deterministic strategies within a simplified model with a collapsed bid-ask spread. The derivations of the explicit forms of the optimal strategies in Models 1 and 2 are carried out in the respective Sections 9 and 10. In Section 11 we prove the results for block-shaped LOBs from Section 6.

## 2 Two market impact models with resilience.

In this section, we aim at modeling the dynamics of a LOB that is exposed to repeated market orders by a large trader. The overall goal of the large trader will be to purchase a large amount  $X_0 > 0$  of shares within a certain time period  $[0, T]$ . Hence, emphasis is on buy orders, and we concentrate first on the upper part of the LOB, which consists of shares offered at various ask prices. The lowest ask price at which shares are offered is called the *best ask price*.

Suppose first that the large trader is not active, so that the dynamics of the limit order book are determined by the actions of noise traders only. We assume that the corresponding unaffected best ask price  $A^0$  is a martingale on a given filtered probability space  $(\Omega, (\mathcal{F}_t), \mathcal{F}, \mathbb{P})$  and satisfies  $A_0^0 = A_0$ . This assumption includes in particular the case in which  $A^0$  is a Bachelier model, i.e.,  $A_t^0 = A_0 + \sigma W_t$  for an  $(\mathcal{F}_t)$ -Brownian motion  $W$ , as considered in [15]. We emphasize, however, that we can take any martingale and hence use, e.g., a geometric Brownian motion, which avoids the counterintuitive negative prices of the Bachelier model. Moreover, we can allow for jumps in the dynamics of  $A^0$  so as to model the trading activities of other large traders in the market.

Above the unaffected best ask price  $A_t^0$ , we assume a continuous ask price distribution for available shares in the LOB: the number of shares offered at price  $A_t^0 + x$  is given by  $f(x) dx$  for a continuous density function  $f : \mathbb{R} \rightarrow ]0, \infty[$ . We will say that  $f$  is the *shape function* of the LOB. The choice of a constant shape function corresponds to the block-shaped LOB model of Obizhaeva and Wang [15].

The shape function determines the impact of a market order placed by our large trader. Suppose for instance that the large trader places a buy market order for  $x_0 > 0$  shares at time  $t = 0$ . This market order will consume all shares located at prices between  $A_0$  and  $A_0 + D_{0+}^A$ , where  $D_{0+}^A$  is determined by

$$\int_0^{D_{0+}^A} f(x) dx = x_0.$$

Consequently, the ask price will be shifted up from  $A_0$  to

$$A_{0+} := A_0 + D_{0+}^A;$$

see Figure 1 for an illustration.

Let us denote by  $A_t$  the actual ask price at time  $t$ , i.e., the ask price after taking the price impact of previous buy orders of the large trader into account, and let us denote by

$$D_t^A := A_t - A_t^0$$

the *extra spread* caused by the actions of the large trader. Another buy market order of  $x_t > 0$  shares will now consume all the shares offered at prices between  $A_t$  and

$$A_{t+} := A_t + D_{t+}^A - D_t^A = A_t^0 + D_{t+}^A,$$

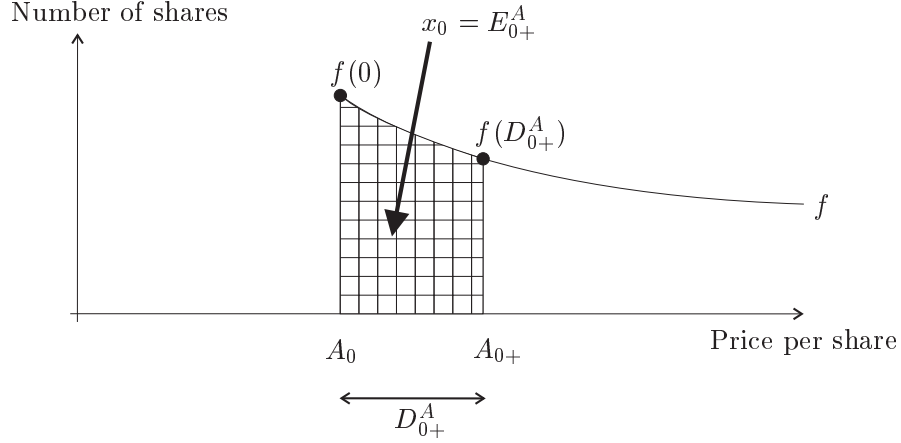


Figure 1: The impact of a buy market order of  $x_0$  shares .

where  $D_{t+}^A$  is determined by the condition

$$\int_{D_t^A}^{D_{t+}^A} f(x) dx = x_t. \quad (1)$$

Thus, the process  $D^A$  captures the impact of market orders on the current best ask price. Clearly, the price impact  $D_{t+}^A - D_t^A$  will be a nonlinear function of the order size  $x_t$  unless  $f$  is constant between  $D_t^A$  and  $D_{t+}^A$ . Hence, our model includes the case of *nonlinear impact functions*; see, e.g., Almgren [1] and Almgren *et al.* [5] for a discussion.

Another important quantity is the process

$$E_t^A = \int_0^{D_t^A} f(x) dx, \quad (2)$$

of the number of shares ‘already eaten up’ at time  $t$ . It quantifies the impact of the large trader on the *volume* of the LOB. By introducing the antiderivative

$$F(y) = \int_0^y f(x) dx \quad (3)$$

of  $f$ , the relation (2) can also be expressed as

$$E_t^A = F(D_t^A) \quad \text{and} \quad D_t^A = F^{-1}(E_t^A), \quad (4)$$

where we have used our assumption that  $f$  is strictly positive to obtain the second identity. The relation (1) is equivalent to

$$E_{t+}^A = E_t^A + x_t. \quad (5)$$

We still need to specify how  $D^A$  and, equivalently,  $E^A$  evolve when the large trader is inactive in between market orders. It is a well established empirical fact that order books exhibit a certain resilience as to the price impact of a large buy market orders, i.e., after the

initial impact the best ask price reverts back to its previous position; cf. Biais *et al.* [8], Potters and Bouchaud [16], Bouchaud *et al.* [9], and Weber and Rosenow [19] for empirical studies. That is, at least a part of the price impact will only be temporary. For modeling this resilience, we follow Obizhaeva and Wang [15] in proposing an exponential recovery of the LOB. While in the case of a block-shaped LOB as considered in [15] the respective assumptions of exponential recovery for  $D^A$  and for  $E^A$  coincide, they provide two distinct possibilities for the case of a general shape function. Since either of them appears to be plausible, we will discuss them both in the sequel. More precisely, we will consider the following two models for the resilience of the market impact:

**Model 1:** The volume of the order book recovers exponentially, i.e.,  $E$  evolves according to

$$E_{t+s}^A = e^{-\rho s} E_t^A \quad (6)$$

if the large investor is inactive during the time interval  $[t, t + s[$ .

**Model 2:** The extra spread  $D_t^A$  decays exponentially, i.e.,

$$D_{t+s}^A = e^{-\rho s} D_t^A \quad (7)$$

if the large investor is inactive during the time interval  $[t, t + s[$ .

Here the *resilience speed*  $\rho$  is a positive constant, which for commonly traded blue chip shares will often be calibrated such that the half-life time of the exponential decay is in the order of a few minutes; see, e.g., [16, 9, 19]. Note that the dynamics of both  $D^A$  and  $E^A$  are now completely specified in either model.

Up to now, we have only described the effect of buy orders on the upper half of the LOB. Since the overall goal of the larger trader is to buy  $X_0 > 0$  shares up to time  $T$ , a restriction to buy orders would seem to be reasonable. However, we do not wish to exclude the *a priori* possibility that, under certain market conditions, it could be beneficial to also *sell* some shares and to buy them back at a later point in time. To this end, we also need to model the impact of sell market orders on the lower part of the LOB, which consists of a certain number of bids for shares at each price below the *best bid price*. As for ask prices, we will distinguish between an unaffected best bid price,  $B_t^0$ , and the actual best bid price,  $B_t$ , for which the price impact of previous sell orders of the large trader is taken into account. All we assume on the dynamics of  $B^0$  is

$$B_t^0 \leq A_t^0 \quad \text{at all times } t. \quad (8)$$

The distribution of bids below  $B_t^0$  is modeled by the restriction of the shape function  $f$  to the domain  $] -\infty, 0]$ . More precisely, for  $x < 0$ , the number of bids at price  $B_t^0 + x$  is equal to  $f(x) dx$ . The quantity

$$D_t^B := B_t - B_t^0,$$

which usually will be negative, is called the *extra spread* in the bid price distribution. A sell market order of  $x_t < 0$  shares placed at time  $t$  will consume all the shares offered at prices between  $B_t$  and

$$B_{t+} := B_t + D_{t+}^B - D_t^B = B_t^0 + D_{t+}^B,$$

where  $D_{t+}^B$  is determined by the condition

$$x_t = \int_{D_t^B}^{D_{t+}^B} f(x) dx = F(D_{t+}^B) - F(D_t^B) = E_{t+}^B - E_t^B, \quad (9)$$

for  $E_s^B := F(D_s^B)$ . Note that  $F$  is defined via (3) also for negative arguments. If the large trader is inactive during the time interval  $[t, t+s[$ , then the processes  $D^B$  and  $E^B$  behave just as their counterparts  $D^A$  and  $E^A$ , i.e.,

$$\begin{aligned} E_{t+s}^B &= e^{-\rho s} E_t^B & \text{in Model 1,} \\ D_{t+s}^B &= e^{-\rho s} D_t^B & \text{in Model 2.} \end{aligned} \quad (10)$$

### 3 The cost minimization problem.

When placing a single buy market order of size  $x_t \geq 0$  at time  $t$ , the large trader will purchase  $f(x) dx$  shares at price  $A_t^0 + x$ , with  $x$  ranging from  $D_t^A$  to  $D_{t+}^A$ . Hence, the total cost of the buy market order amounts to

$$\pi_t(x_t) := \int_{D_t^A}^{D_{t+}^A} (A_t^0 + x) f(x) dx = A_t^0 x_t + \int_{D_t^A}^{D_{t+}^A} x f(x) dx. \quad (11)$$

For a sell market order  $x_t \leq 0$ , we have

$$\pi_t(x_t) := B_t^0 x_t + \int_{D_t^B}^{D_{t+}^B} x f(x) dx. \quad (12)$$

In practice, very large orders are often split into a number of consecutive market orders to reduce the overall price impact. Hence, the question at hand is to determine the size of the individual orders so as to minimize a cost criterion. So let us assume that the large trader needs to buy a total of  $X_0 > 0$  shares until time  $T$  and that trading can occur at  $N + 1$  equidistant times  $t_n = n\tau$  for  $n = 0, \dots, N$  and  $\tau := T/N$ . An *admissible strategy* will be a sequence  $\xi = (\xi_0, \xi_1, \dots, \xi_N)$  of random variables such that

- $\sum_{n=0}^N \xi_n = X_0$ ,
- each  $\xi_n$  is measurable with respect to  $\mathcal{F}_{t_n}$ ,
- each  $\xi_n$  is bounded from below.

The quantity  $\xi_n$  corresponds to the size of the market order placed at time  $t_n$ . Note that we do not *a priori* require  $\xi_n$  to be positive, i.e., we also allow for intermediate sell orders, but we assume that there is some lower bound on sell orders.



The *average cost*  $\mathcal{C}(\xi)$  of an admissible strategy  $\xi$  is defined as the expected value of the total costs incurred by the consecutive market orders:

$$\mathcal{C}(\xi) = \mathbb{E} \left[ \sum_{n=0}^N \pi_{t_n}(\xi_n) \right]. \quad (13)$$

Our goal in this paper consists in finding admissible strategies that minimize the average cost within the class of all admissible strategies. Note that the value of  $\mathcal{C}(\xi)$  depends on whether we choose Model 1 or Model 2, and it will turn out that also the quantitative—though not the qualitative—features of the optimal strategies will be slightly model-dependent.

Before turning to the statements of our results, let us introduce the following standing assumption for our further analysis: the function  $F$  is supposed to be unbounded in the sense that

$$\lim_{x \uparrow \infty} F(x) = \infty \quad \text{and} \quad \lim_{x \downarrow -\infty} F(x) = -\infty. \quad (14)$$

This assumption of unlimited order book depth is of course an idealization of reality and is for convenience only. It should not make a difference, however, as soon as the depth of the real LOB is big enough to accommodate every market order of our optimal strategy.

## 4 Main theorem for Model 1.

We will now consider the minimization of the cost functional  $\mathcal{C}(\xi)$  in Model 1, in which we assume an exponential recovery of the LOB volume; cf. (6).

**Theorem 4.1** (Optimal strategy in Model 1).

*Suppose that the function*

$$h_1(u) := F^{-1}(u) - e^{-\rho\tau} F^{-1}(e^{-\rho\tau} u)$$

*is one-to-one. Then there exists a unique optimal strategy  $\xi^{(1)} = (\xi_0^{(1)}, \dots, \xi_N^{(1)})$ . The initial market order  $\xi_0^{(1)}$  is the unique solution of the equation*

$$F^{-1} \left( X_0 - N \xi_0^{(1)} (1 - e^{-\rho\tau}) \right) = \frac{h_1(\xi_0^{(1)})}{1 - e^{-\rho\tau}}, \quad (15)$$

*the intermediate orders are given by*

$$\xi_1^{(1)} = \dots = \xi_{N-1}^{(1)} = \xi_0^{(1)} (1 - e^{-\rho\tau}), \quad (16)$$

*and the final order is determined by*

$$\xi_N^{(1)} = X_0 - \xi_0^{(1)} - (N-1) \xi_0^{(1)} (1 - e^{-\rho\tau}).$$

*In particular, the optimal strategy is deterministic. Moreover, it consists only of nontrivial buy orders, i.e.,  $\xi_n^{(1)} > 0$  for all  $n$ .*

Some remarks on this result are in order. First, the optimal strategy  $\xi^{(1)}$  consists only of buy orders and so the bid price remains unaffected, i.e., we have  $E_t^B \equiv 0 \equiv D_t^B$ . It follows moreover that the process  $E := E^A$  is recursively given by the following Model 1 dynamics:

$$\begin{aligned} E_0 &= 0, \\ E_{t_n+} &= E_{t_n} + \xi_n^{(1)}, \quad n = 0, \dots, N, \\ E_{t_{k+1}} &= e^{-\rho\tau} E_{t_k+} = e^{-\rho\tau} (E_{t_k} + \xi_k^{(1)}), \quad k = 0, \dots, N-1. \end{aligned} \tag{17}$$

Hence, by (15) and (16),

$$E_{t_n+} = \xi_0^{(1)} \quad \text{and} \quad E_{t_{n+1}} = e^{-\rho\tau} \xi_0^{(1)} \quad \text{for } n = 0, \dots, N-1. \tag{18}$$

That is, once  $\xi_0^{(1)}$  has been determined via (15), the optimal strategy consists in a sequence of market orders that consume exactly that amount of shares by which the LOB has recovered since the preceding market order, due to the resilience effect. At the terminal time  $t_N = T$ , all remaining shares are bought. In the case of a block-shaped LOB, this qualitative pattern was already observed by Obizhaeva and Wang [15]. Our Theorem 4.1 now shows that this optimality pattern is actually independent of the LOB shape, thus indicating a certain robustness of optimal strategies.

**Remark 4.2** According to (4) and (18), the extra spread  $D := D^A$  of the optimal strategy  $\xi^{(1)}$  satisfies

$$D_{t_n+} = F^{-1}(E_{t_n+}) = F^{-1}(\xi_0^{(1)}).$$

For  $n = N$  we moreover have that

$$\begin{aligned} D_{t_N+} &= F^{-1}(E_{t_N+}) = F^{-1}(E_{t_N} + \xi_N^{(1)}) \\ &= F^{-1}(\xi_0^{(1)} e^{-\rho\tau} + X_0 - \xi_0^{(1)} - (N-1)\xi_0^{(1)}(1 - e^{-\rho\tau})) \\ &= F^{-1}(X_0 - N\xi_0^{(1)}(1 - e^{-\rho\tau})). \end{aligned}$$

Hence, the left-hand side of (15) is equal to  $D_{t_N+}$ .

We now comment on the conditions in Theorem 4.1.

**Remark 4.3** (When is  $h_1$  one-to-one?) The function  $h_1$  is continuous with  $h_1(0) = 0$  and  $h_1(u) > 0$  for  $u > 0$ . Hence,  $h_1$  is one-to-one if and only if  $h_1$  is strictly increasing. We want to consider when this is the case. To this end, note that the condition

$$h_1'(u) = \frac{1}{f(F^{-1}(u))} - \frac{e^{-2\rho\tau}}{f(F^{-1}(e^{-\rho\tau}u))} > 0$$

is equivalent to

$$\ell(u) := f(F^{-1}(e^{-\rho\tau}u)) - e^{-2\rho\tau} f(F^{-1}(u)) > 0. \tag{19}$$

That is, the function  $h_1$  will be one-to-one if, for instance, the shape function  $f$  is decreasing for  $u > 0$  and increasing for  $u < 0$ . In fact, it has been observed in the empirical studies [8, 16, 9, 19] that average shapes of typical order books have a maximum at or close to the best quotes and then decay as a function of the distance to the best quotes, which would conform to our assumption.

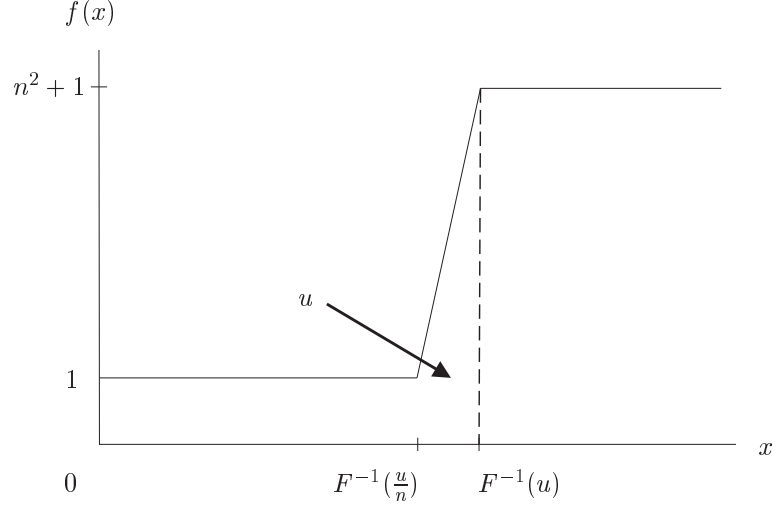


Figure 2: A shape function  $f$  for which the corresponding function  $h_1$  is not one-to-one.

We now want to give an example of a shape function  $f$  such that the corresponding function  $h_1$  is not one-to-one. To this end, let us assume for simplicity that there exists an  $n \in \{2, 3, \dots\}$  such that  $e^{-\rho\tau} = \frac{1}{n}$ . We set

$$f(x) := \begin{cases} 1 & x \in [0, \frac{\frac{1}{2}n^2+1}{n-1}) \\ 1 + n^2(x - \frac{\frac{1}{2}n^2+1}{n-1}) & x \in [\frac{\frac{1}{2}n^2+1}{n-1}, \frac{\frac{1}{2}n^2+1}{n-1} + 1] \\ n^2 + 1 & x \in (\frac{\frac{1}{2}n^2+1}{n-1} + 1, \infty) \end{cases};$$

see Figure 2. For  $u := (\frac{\frac{1}{2}n^2+1}{n-1} + 1) + \frac{1}{2}n^2$  we then have  $F^{-1}(u) = \frac{\frac{1}{2}n^2+1}{n-1} + 1$  as well as

$$F^{-1}(e^{-\rho\tau}u) = F^{-1}\left(\frac{u}{n}\right) = \frac{\frac{1}{2}n^2+1}{n-1}.$$

Hence, we get

$$f(F^{-1}(e^{-\rho\tau}u)) = 1 < \left(1 + \frac{1}{n^2}\right) = e^{-2\rho\tau} f(F^{-1}(u)),$$

which tells us according to (19) that the corresponding function  $h_1$  is not strictly increasing.

**Remark 4.4** (Continuous-time limit of the optimal strategy). One can also investigate the asymptotic behavior of the optimal strategy when the number  $N$  of trades in  $]0, T]$  tends to infinity. It is not difficult to see that  $h_1/(1 - e^{-\rho\tau})$  converges pointwise to

$$h_1^\infty(x) := F^{-1}(x) + \frac{x}{f(F^{-1}(x))}.$$

Observe also that  $N(1 - e^{-\rho\tau}) \rightarrow \rho T$ . Since for any  $N$  we have  $\xi_0^{(1)} \in ]0, X_0[$ , we can extract a subsequence that converges and its limit is then necessarily solution of the equation

$$F^{-1}(X_0 - \rho T x) = h_1^\infty(x).$$

If this equation has a unique solution  $\xi_0^{(1),\infty}$  we deduce that the optimal initial trade converges to  $\xi_0^{(1),\infty}$  when  $N \rightarrow \infty$ . This is the case, for example, if  $h_1^\infty$  is strictly increasing and especially when  $f$  is decreasing. In that case,  $N\xi_1^{(1)}$  converges to  $\rho T \xi_0^{(1),\infty}$  and  $\xi_N^{(1)}$  to  $\xi_T^{(1),\infty} := X_0 - \xi_0^{(1),\infty}(1 + \rho T)$ . Thus, in the continuous-time limit, the optimal strategy consists in an initial block order of  $\xi_0^{(1),\infty}$  shares at time 0, continuous buying at the constant rate  $\rho \xi_0^{(1),\infty}$  during  $]0, T[$ , and a final block order of  $\xi_T^{(1),\infty}$  shares at time  $T$ .

## 5 Main theorem for Model 2.

We will now consider the minimization of the cost functional

$$\mathcal{C}(\xi) = \mathbb{E} \left[ \sum_{n=0}^N \pi_{t_n}(\xi_n) \right]$$

in Model 2, where we assume an exponential recovery of the extra spread; cf. (7).

**Theorem 5.1** (Optimal strategy in Model 2).

Suppose that the function

$$h_2(x) := x \frac{f(x) - e^{-2\rho\tau} f(e^{-\rho\tau} x)}{f(x) - e^{-\rho\tau} f(e^{-\rho\tau} x)}$$

is one-to-one and that the shape function satisfies

$$\lim_{|x| \rightarrow \infty} x^2 \inf_{y \in [e^{-\rho\tau} x, x]} f(y) = \infty. \quad (20)$$

Then there exists a unique optimal strategy  $\xi^{(2)} = (\xi_0^{(2)}, \dots, \xi_N^{(2)})$ . The initial market order  $\xi_0^{(2)}$  is the unique solution of the equation

$$F^{-1} \left( X_0 - N[\xi_0^{(2)} - F(e^{-\rho\tau} F^{-1}(\xi_0^{(2)}))] \right) = h_2(F^{-1}(\xi_0^{(2)})), \quad (21)$$

the intermediate orders are given by

$$\xi_1^{(2)} = \dots = \xi_{N-1}^{(2)} = \xi_0^{(2)} - F(e^{-\rho\tau} F^{-1}(\xi_0^{(2)})), \quad (22)$$

and the final order is determined by

$$\xi_N^{(2)} = X_0 - N\xi_0^{(2)} + (N-1)F(e^{-\rho\tau} F^{-1}(\xi_0^{(2)})).$$

In particular, the optimal strategy is deterministic. Moreover, it consists only of nontrivial buy orders, i.e.,  $\xi_n^{(2)} > 0$  for all  $n$ .

Since the optimal strategy  $\xi^{(2)}$  consists only of buy orders, the processes  $D^B$  and  $E^B$  vanish, and  $D := D^A$  is given by

$$\begin{aligned} D_0 &= 0, \\ D_{t_n+} &= F^{-1}(\xi_n^{(2)} + F(D_{t_n})), \quad n = 0, \dots, N \\ D_{t_{k+1}} &= e^{-\rho\tau} D_{t_k+}, \quad k = 0, \dots, N-1. \end{aligned} \tag{23}$$

Hence, induction shows that

$$D_{t_n+} = F^{-1}(\xi_0^{(2)}) \quad \text{and} \quad D_{t_{n+1}} = e^{-\rho\tau} F^{-1}(\xi_0^{(2)}) \quad \text{for } n = 0, \dots, N-1.$$

By (4), the process  $E := E^A$  satisfies

$$E_{t_n+} = \xi_0^{(2)} \quad \text{and} \quad E_{t_{n+1}} = F(e^{-\rho\tau} F^{-1}(\xi_0^{(2)})) \quad \text{for } n = 0, \dots, N-1.$$

This is very similar to our result (18) in Model 1: once  $\xi_0^{(1)}$  has been determined via (15), the optimal strategy consists in a sequence of market orders that consume exactly that amount of shares by which the LOB has recovered since the preceding market order. At the terminal time  $t_N = T$ , all remaining shares are bought. The only differences are in the size of the initial market order and in the mode of recovery. This qualitative similarity between the optimal strategies in Models 1 and 2 again confirms our observation of the robustness of the optimal strategy.

**Remark 5.2** *At the terminal time  $t_N = T$ , the extra spread is given by*

$$\begin{aligned} D_{t_N+} &= F^{-1}(E_{t_N+}) = F^{-1}(E_{t_N} + \xi_N^{(2)}) \\ &= F^{-1}\left(X_0 - N[\xi_0^{(2)} - F(e^{-\rho\tau} F^{-1}(\xi_0^{(2)}))]\right), \end{aligned}$$

*and this expression coincides with the left-hand side in (21).*

Let us now comment on the conditions assumed in Theorem 5.1. To this end, we first introduce the function

$$\tilde{F}(x) := \int_0^x y f(y) dy. \tag{24}$$

**Remark 5.3** *If  $\tilde{F}$  is convex then condition (20) in Theorem 5.1 is satisfied. This fact admits the following short proof. Take  $y^* \in [e^{-\rho\tau}x, x]$  realizing the infimum of  $f$  in  $[e^{-\rho\tau}x, x]$ . Then*

$$x^2 \inf_{y \in [e^{-\rho\tau}x, x]} f(y) = x^2 f(y^*) \geq y^* (y^* f(y^*)). \tag{25}$$

*Due to the convexity of  $\tilde{F}$ , its derivative  $\tilde{F}'(x) = xf(x)$  is increasing. It is also nonzero iff  $x \neq 0$ . Therefore the right-hand side of (25) tends to infinity for  $|x| \rightarrow \infty$ .*

However, the convexity of  $\tilde{F}$  is not necessary for condition (20) as is illustrated by the following simple example.

**Example 5.4** *Let us construct a shape function for which (20) is satisfied even though  $\tilde{F}$  need not be convex. To this end, take any continuous function  $b : \mathbb{R} \rightarrow ]0, \infty[$  that is bounded away from zero. Then let*

$$f(x) := \begin{cases} b(1) & |x| \leq 1 \\ \frac{b(x)}{\sqrt{|x|}} & |x| > 1. \end{cases}$$

*This shape function clearly satisfies condition (20). Taking for example  $b(x) = 1 + \varepsilon \cos(x)$  with  $0 < \varepsilon < 1$ , however, gives a nonconvex function  $\tilde{F}$ . Moreover, by choosing  $\varepsilon$  small enough, we can obtain  $h'_2(x) > 0$  so that the shape function  $f$  satisfies the assumptions of Theorem 5.1.*

We now comment on the condition that  $h_2$  is one-to-one. The following example shows that this is indeed a nontrivial assumption.

**Example 5.5** *We now provide an example of a shape function  $f$  for which the corresponding function  $h_2$  is not one-to-one. First note that  $h_2(0) = 0$  and*

$$\lim_{\epsilon \downarrow 0} \frac{h_2(\epsilon) - h_2(0)}{\epsilon} = \frac{1 - e^{-2\rho\tau}}{1 - e^{-\rho\tau}} > 0. \quad (26)$$

*Since  $h_2$  is continuous, it cannot be one-to-one if we can find  $x_-, x_+ > 0$  such that  $h_2(x_-) < 0$  and  $h_2(x_+) > 0$ . To this end, we assume that there exist  $n \in \{2, 3, \dots\}$  such that  $e^{-\rho\tau} = \frac{1}{n}$  and take*

$$f(x) := \begin{cases} (n+1) & x \in [0, \frac{1}{n}) \\ (n+1) - \frac{n^2}{n-1} (x - \frac{1}{n}) & x \in [\frac{1}{n}, 1] \\ 1 & x \in (1, \infty); \end{cases}$$

*see Figure 3. Furthermore, we define  $x_- := 1$  and  $x_+ := \frac{1}{n}$  to obtain*

$$h_2(x_-) = \frac{n^2 - (n+1)}{-n} < 0 \quad \text{and} \quad h_2(x_+) = \frac{n+1}{n^2} > 0.$$

**Remark 5.6** (Continuous-time limit of the optimal strategy). *As in Remark 4.4, we can study the asymptotic behavior of the optimal strategy as the number  $N$  of trades in  $]0, T[$  tends to infinity. First, we can check that  $h_2$  converges pointwise to*

$$h_2^\infty(x) := x \left( 1 + \frac{f(x)}{f(x) + x f'(x)} \right),$$

*and that  $N(x - F(e^{-\rho\tau} F^{-1}(x)))$  tends to  $\rho T F^{-1}(x) f(F^{-1}(x))$ , provided that  $f$  is continuously differentiable. Now, suppose that the equation*

$$F^{-1}(X_0 - \rho T F^{-1}(x) f(F^{-1}(x))) = h_2^\infty(F^{-1}(x))$$

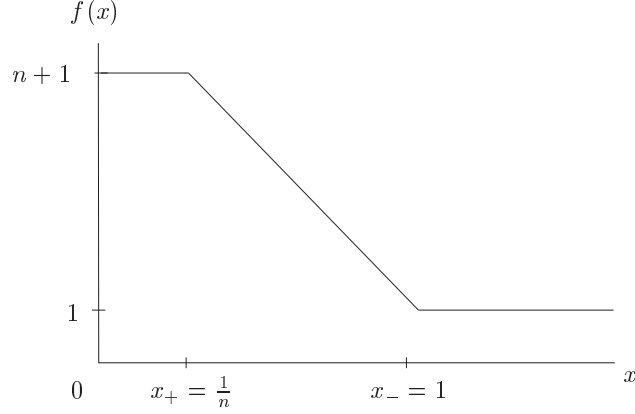


Figure 3: A shape function  $f$  for which the function  $h_2$  is not one-to-one.

has a unique solution on  $]0, X_0[$ , which we will call  $\xi_0^{(2),\infty}$ . We can check that  $\xi_0^{(2),\infty}$  is the only one possible limit for a subsequence of  $\xi_0^{(2)}$ , and it is therefore its limit. We can then show that  $N\xi_1^{(2)}$  converges to  $\rho TF^{-1}(\xi_0^{(2),\infty})f(F^{-1}(\xi_0^{(2),\infty}))$  and  $\xi_N^{(2)}$  to

$$\xi_T^{(2),\infty} := X_0 - \xi_0^{(2),\infty} - \rho TF^{-1}(\xi_0^{(2),\infty})f(F^{-1}(\xi_0^{(2),\infty})).$$

Thus, in the continuous-time limit, the optimal strategy consists in an initial block order of  $\xi_0^{(2),\infty}$  shares at time 0, continuous buying at the constant rate  $\rho F^{-1}(\xi_0^{(2),\infty})f(F^{-1}(\xi_0^{(2),\infty}))$  during  $]0, T[$ , and a final block order of  $\xi_T^{(2),\infty}$  shares at time  $T$ .

## 6 Closed form solution for block-shaped LOBs and additional permanent impact.

In this first example section, we consider a block-shaped LOB corresponding to a constant shape function  $f(x) \equiv q$  for some  $q > 0$ . In this case, there is no difference between Models 1 and 2. Apart from our more general dynamics for  $A^0$ , the main difference to the market impact model introduced by Obizhaeva and Wang [15] is that, for the moment, we do not consider a permanent impact of market orders. In Corollary 6.4, we will see, however, that our results yield a closed-form solution even in the case of nonvanishing permanent impact.

By applying either Theorem 4.1 or Theorem 5.1 we obtain the following Corollary; a detailed proof can be found in Section 11.

**Corollary 6.1** (Closed-form solution for block-shaped LOB).

*In a block-shaped LOB, the unique optimal strategy  $\xi^*$  is*

$$\xi_0^* = \xi_N^* = \frac{X_0}{(N-1)(1-e^{-\rho\tau})+2} \quad \text{and} \quad \xi_1^* = \dots = \xi_{N-1}^* = \frac{X_0 - 2\xi_0^*}{N-1}. \quad (27)$$

The preceding result extends [15, Proposition 1] in several aspects. First, we do not focus on the Bachelier model but admit arbitrary martingale dynamics for our unaffected best ask price  $A^0$ . Second, only static, deterministic buy order strategies are considered in [15], while we here allow our admissible strategies to be adapted and to include sell orders. Since, *a posteriori*, our optimal strategy turns out to be deterministic and positive, it is clear that it must coincide with the optimal strategy from [15, Proposition 1]. Our strategy (27) therefore also provides an *explicit closed-form solution* of the recursive scheme obtained in [15]. We recall this recursive scheme in (31) below.

On the other hand, Obizhaewa and Wang [15] allow for an additional *permanent impact* of market orders. Intuitively, in a block-shaped LOB with  $f \equiv q > 0$ , the permanent impact of a market order  $x_t$  means that only a certain part of the impact of  $x_t$  decays to zero, while the remaining part remains forever present in the LOB. More precisely, the impact of an admissible buy order strategy  $\xi$  on the extra spread  $D^A$  is given by the dynamics

$$D_t^A = \lambda \sum_{t_k < t} \xi_k + \sum_{t_k < t} \kappa e^{-\rho(t-t_k)} \xi_k, \quad (28)$$

where  $\lambda < 1/q$  is a constant quantifying the permanent impact and

$$\kappa := \frac{1}{q} - \lambda \quad (29)$$

is the proportion of the temporary impact. Note that, for  $\lambda = 0$ , we get back our dynamics (6) and (7), due to the fact that we consider a block-shaped LOB. It will be convenient to introduce the process  $X_t$  of the still outstanding number of shares at time  $t$  when using an admissible strategy:

$$X_t := X_0 - \sum_{t_k < t} \xi_k. \quad (30)$$

We can now state the result by Obizhaewa and Wang.

**Proposition 6.2** [15, Proposition 1] *In a block-shaped LOB with permanent impact  $\lambda$ , the optimal strategy  $\xi^{OW}$  in the class of deterministic strategies is determined by the forward scheme*

$$\begin{aligned} \xi_n^{OW} &= \frac{1}{2} \delta_{n+1} [\epsilon_{n+1} X_{t_n} - \phi_{n+1} D_{t_n}], \quad n = 0, \dots, N-1, \\ \xi_N^{OW} &= X_T, \end{aligned} \quad (31)$$

where  $\delta_n$ ,  $\epsilon_n$  and  $\phi_n$  are defined by the backward scheme

$$\begin{aligned} \delta_n &:= \left( \frac{1}{2q} + \alpha_n - \beta_n \kappa e^{-\rho\tau} + \gamma_n \kappa^2 e^{-2\rho\tau} \right)^{-1} \\ \epsilon_n &:= \lambda + 2\alpha_n - \beta_n \kappa e^{-\rho\tau} \\ \phi_n &:= 1 - \beta_n e^{-\rho\tau} + 2\gamma_n \kappa e^{-2\rho\tau}. \end{aligned} \quad (32)$$



with  $\alpha_n$ ,  $\beta_n$  and  $\gamma_n$  given by

$$\begin{aligned}\alpha_N &= \frac{1}{2q} - \lambda \quad \text{and} \quad \alpha_n = \alpha_{n+1} - \frac{1}{4}\delta_{n+1}\epsilon_{n+1}^2, \\ \beta_N &= 1 \quad \text{and} \quad \beta_n = \beta_{n+1}e^{-\rho\tau} + \frac{1}{2}\delta_{n+1}\epsilon_{n+1}\phi_{n+1}, \\ \gamma_N &= 0 \quad \text{and} \quad \gamma_n = \gamma_{n+1}e^{-2\rho\tau} - \frac{1}{4}\delta_{n+1}\phi_{n+1}^2.\end{aligned}\tag{33}$$

It is *a priori* clear that for  $\lambda = 0$  the explicit optimal strategy obtained in Corollary 6.1 must coincide with the strategy  $\xi^{OW}$  obtained via the recursive scheme (31) in Proposition 6.2. To cross-check our results with the ones in [15], we will nevertheless provide an explicit and independent proof of the following proposition. It can be found in Section 11.

**Proposition 6.3** *For  $\lambda = 0$ , the optimal strategy (27) of Corollary 6.1 solves the recursive scheme (31) in Proposition 6.2.*

Let us now extend our results so as to obtain the explicit solution of (31) even with nonvanishing permanent impact. To this end, we note that the optimal strategy  $\xi^{OW} = (\xi_0^{OW}, \dots, \xi_N^{OW})$  is obtained in [15] as the unique minimizer of the cost functional

$$C_{\lambda,q}^{OW} : \mathbb{R}^{N+1} \rightarrow \mathbb{R}$$

defined by

$$\begin{aligned}C_{\lambda,q}^{OW}(x_0, \dots, x_N) \\ = A_0 \sum_{i=0}^N x_i + \frac{\lambda}{2} \left( \sum_{i=0}^N x_i \right)^2 + \kappa \sum_{k=0}^N \left( \sum_{i=0}^{k-1} x_i e^{-\rho(k-i)\tau} \right) x_k + \frac{\kappa}{2} \sum_{i=0}^N x_i^2,\end{aligned}$$

where  $\kappa$  is as in (29). Now we just have to observe that

$$C_{\lambda,q}^{OW}(x_0, \dots, x_N) = \frac{\lambda}{2} \left( \sum_{i=0}^N x_i \right)^2 + C_{0,\kappa^{-1}}^{OW}(x_0, \dots, x_N).$$

Therefore, under the constraint  $\sum_{i=0}^N x_i = X_0$ , it is equivalent to minimize either  $C_{\lambda,q}^{OW}$  or  $C_{0,\kappa^{-1}}^{OW}$ . We already know that the optimal strategy  $\xi^*$  of Corollary 6.1 minimizes  $C_{0,q}^{OW}$ . But  $\xi^*$  is in fact independent of  $q$ . Hence,  $\xi^*$  also minimizes  $C_{0,\kappa^{-1}}^{OW}$  and in turn  $C_{\lambda,q}^{OW}$ . We have therefore proved:

**Corollary 6.4** *The optimal strategy  $\xi^*$  of Corollary 6.1 is the unique optimal strategy in any block-shaped LOB with permanent impact  $\lambda < 1/q$ . In particular, it solves the recursive scheme (31).*

The last part of the assertion of Corollary 6.4 is remarkable insofar as the recursive scheme (31) depends on both  $q$  and  $\lambda$  whereas the optimal strategy  $\xi^*$  does not.

Example number	$f(x)$	Model 1			Model 2		
		$\xi_0^{(1)}$	$\xi_1^{(1)}$	$\xi_N^{(1)}$	$\xi_0^{(2)}$	$\xi_1^{(2)}$	$\xi_N^{(2)}$
0	$q$	10,223	8,839	10,223	10,223	8,839	10,223
1	$\frac{q}{\sqrt{ x +1}}$	10,257	8,869	9,925	10,756	8,724	10,726
2	$\frac{q}{ x +1}$	10,303	8,909	9,520	13,305	8,154	13,305
3	$qe^{ x }$	10,139	8,767	10,962	9,735	8,947	9,741
4	$\frac{q}{10} x  + q$	10,211	8,829	10,326	10,130	8,860	10,131
5	$\frac{q}{10}x^2 + q$	10,192	8,812	10,498	10,101	8,868	10,091

Table 1: The table shows optimal strategies for various choices of the shape function  $f$ . We set  $X_0 = 100,000$  and  $q = 5,000$  shares,  $\rho = 20$ ,  $T = 1$  and  $N = 10$ .

## 7 Examples.

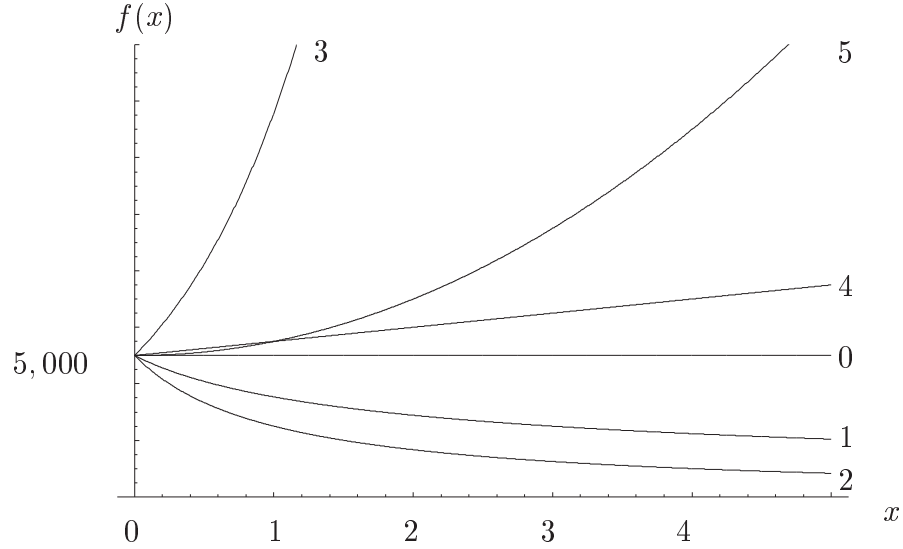


Figure 4: Plots of the shape functions in Table 1 for  $q = 5,000$  shares. The numbers on the right-hand side refer to the example numbers in Table 1.

In this section, we consider various example shape functions, which are listed in Table 1 together with numerical values for their optimal strategies. It is remarkable that the optimal strategies vary only very slightly when changing the shape function or the resilience mode. This observation provides further evidence for the robustness and stability of the optimal strategy, and this time not only on a qualitative but also on a quantitative level.

We check in Subsection 7.1 that the shape functions in Table 1 satisfy the assumptions of both Theorem 4.1 and Theorem 5.1. In particular, the existence of an optimal strategy is guaranteed in each case.

## 7.1 Assumption Checking.

First note that our standing assumption (14) is satisfied, i.e., all shape functions in Table 1 have unbounded antiderivatives  $F$ . Let us continue by checking the assumptions of Theorem 4.1, where  $h_1$  has to be one-to-one. According to Remark 4.3, there is nothing to show in case of Example 1 and 2. For Example 3, 4 and 5, we only need to show that  $h_1$  is increasing on  $[0, \infty)$  due to the fact that the examples satisfy  $h_1(u) = -h_1(|u|)$  for  $u < 0$ . For the third shape function in Table 1, we have

$$h_1'(x) = \frac{e^{-\rho\tau}x + q - e^{-2\rho\tau}(x + q)}{(x + q)(e^{-\rho\tau}x + q)} \quad \text{for } x > 0,$$

which is obviously positive since all variables are positive. For Examples 4 and 5 we consider the positivity of the function  $\ell$  from (19) on  $[0, \infty)$ . In the case of Example 4 we get

$$\ell(x) = \sqrt{\frac{q}{5}} \left[ \sqrt{5q + e^{-\rho\tau}x} - e^{-2\rho\tau} \sqrt{5q + x} \right] > 0,$$

and although  $\ell$  for Example 5 gets quite complicated, its positivity can be shown by plotting it for different choices of  $q$  and  $\rho$ .

We now turn to Theorem 5.1. First, condition (20) is clearly satisfied by each example. We still have to verify that  $h_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is one-to-one. This is enough because in our examples we have  $h_2(u) = -h_2(|u|)$  for  $u \leq 0$ . In case of Example 2,  $h_2$  is a parabola. For the remaining examples we write

$$h_2(x) = x \frac{1 - e^{-2\rho\tau}g(x)}{1 - e^{-\rho\tau}g(x)} \quad \text{with} \quad g(x) := \frac{f(e^{-\rho\tau}x)}{f(x)}.$$

Hence,  $h_2$  is strictly increasing, and therefore one-to-one, if

$$(1 - e^{-2\rho\tau}g(x))(1 - e^{-\rho\tau}g(x)) > -g'(x)xe^{-\rho\tau}(1 - e^{-\rho\tau}) \quad \text{for } x > 0. \quad (34)$$

In case of Example 1 we have  $g(x) = \sqrt{\frac{x+1}{e^{-\rho\tau}x+1}}$  which is increasing and satisfies (34). For Examples 3 to 5 we use that  $g$  as stated in Table 2 is decreasing and therefore we get (34) provided that

$$e^{\rho\tau} - e^{-\rho\tau} > -g'(x)x \quad \text{for } x > 0. \quad (35)$$

According to Table 2, (35) is satisfied for Example 4 and 5. In case of Example 3, (35) is true if  $\rho\tau > 0.2$ .

Example	$g(x)$	$-g'(x)x$
3	$\exp(-x(1 - e^{-\rho\tau}))$	$\leq e^{-1}$
4	$\frac{e^{-\rho\tau}x+10}{x+10}$	$\leq \frac{1}{4}(1 - e^{-\rho\tau})$
5	$\frac{e^{-2\rho\tau}x^2+10}{x^2+10}$	$\leq \frac{1}{2}(1 - e^{-2\rho\tau})$

Table 2:  $h_2$  is one-to-one for Example 3 to 5.

## 8 Reduction to the case of deterministic strategies.

In this section, we prepare for the proofs of Theorems 4.1 and 5.1 by reducing the minimization of the cost functional

$$\mathcal{C}(\xi) = \mathbb{E} \left[ \sum_{n=0}^N \pi_{t_n}(\xi_n) \right]$$

with respect to all admissible strategies  $\xi$  to the minimization of certain cost functions  $C^{(i)} : \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ , where  $i = 1, 2$  refers to the model under consideration.

To this end, we introduce simplified versions of the model dynamics by collapsing the bid-ask spread into a single value. More precisely, for any admissible strategy  $\xi$ , we introduce a new pair of processes  $D$  and  $E$  that react on both sell and buy orders according to the following dynamics.

- We have  $E_0 = D_0 = 0$  and

$$E_t = F(D_t) \quad \text{and} \quad D_t = F^{-1}(E_t). \quad (36)$$

- For  $n = 0, \dots, N$ , regardless of the sign of  $\xi_n$ ,

$$E_{t_n+} = E_{t_n} + \xi_n \quad \text{and} \quad D_{t_n+} = F^{-1}(\xi_n + F(D_{t_n})). \quad (37)$$

- For  $k = 0, \dots, N-1$ ,

$$\begin{aligned} E_{t_{k+1}} &= e^{-\rho\tau} E_{t_k+} && \text{in Model 1,} \\ D_{t_{k+1}} &= e^{-\rho\tau} D_{t_k+} && \text{in Model 2.} \end{aligned} \quad (38)$$

The values of  $E_t$  and  $D_t$  for  $t \notin \{t_0, \dots, t_N\}$  will not be needed in the sequel. Note that  $E = E^A$  and  $D = D^A$  if  $\xi$  consists only of buy orders, while  $E = E^B$  and  $D = D^B$  if  $\xi$  consists only of sell orders. In general, we will only have

$$E_t^B \leq E_t \leq E_t^A \quad \text{and} \quad D_t^B \leq D_t \leq D_t^A. \quad (39)$$

We now introduce the *simplified price* of  $\xi_n$  at time  $t_n$  by

$$\bar{\pi}_{t_n}(\xi_n) := A_{t_n}^0 \xi_n + \int_{D_{t_n}}^{D_{t_n+}} x f(x) dx, \quad (40)$$

regardless of the sign of  $\xi_n$ . Using (39) and (8), we easily get

$$\bar{\pi}_{t_n}(\xi_n) \leq \pi_{t_n}(\xi_n) \quad \text{with equality if } \xi_k \geq 0 \text{ for all } k \leq n. \quad (41)$$

The *simplified price functional* is defined as

$$\bar{\mathcal{C}}(\xi) := \mathbb{E} \left[ \sum_{n=0}^N \bar{\pi}_{t_n}(\xi_n) \right].$$

We will show that, in Model  $i \in \{1, 2\}$ , the simplified price functional  $\bar{\mathcal{C}}$  has a unique minimizer, which coincides with the corresponding optimal strategy  $\xi^{(i)}$  as described in the respective theorem. We will also show that  $\xi^{(i)}$  consists only of buy orders, so that (41) will yield  $\mathcal{C}(\xi^{(i)}) = \bar{\mathcal{C}}(\xi^{(i)})$ . Consequently,  $\xi^{(i)}$  must be the unique minimizer of  $\mathcal{C}$ .

Let us now reduce the minimization of  $\bar{\mathcal{C}}$  to the minimization of functionals  $C^{(i)}$  defined on deterministic strategies. To this end, let us use the notation

$$X_t := X_0 - \sum_{t_k < t} \xi_k \text{ for } t \leq T \text{ and } X_{t_{N+1}} := 0. \quad (42)$$

The accumulated simplified price of an admissible strategy  $\xi$  is

$$\sum_{n=0}^N \bar{\pi}_{t_n}(\xi_n) = \sum_{n=0}^N A_{t_n}^0 \xi_n + \sum_{n=0}^N \int_{D_{t_n}}^{D_{t_{n+1}}} x f(x) dx.$$

Integrating by parts yields

$$\sum_{n=0}^N A_{t_n}^0 \xi_n = - \sum_{n=0}^N A_{t_n}^0 (X_{t_{n+1}} - X_{t_n}) = X_0 A_0 + \sum_{n=1}^N X_{t_n} (A_{t_n}^0 - A_{t_{n-1}}^0). \quad (43)$$

Since  $\xi$  is admissible,  $X_t$  is a bounded predictable process. Hence, due to the martingale property of the unaffected best ask process  $A^0$ , the expectation of (43) is equal to  $X_0 A_0$ .

Next, observe that, in each Model  $i = 1, 2$ , the simplified extra spread process  $D$  evolves deterministically once the values  $\xi_0, \xi_1(\omega), \dots, \xi_N(\omega)$  are given. Hence, there exists a deterministic function  $C^{(i)} : \mathbb{R}^{N+1} \rightarrow \mathbb{R}$  such that

$$\sum_{n=0}^N \int_{D_{t_n}}^{D_{t_{n+1}}} x f(x) dx = C^{(i)}(\xi_0, \dots, \xi_N). \quad (44)$$

It follows that

$$\bar{\mathcal{C}}(\xi) = A_0 X_0 + \mathbb{E} [C^{(i)}(\xi_0, \dots, \xi_N)].$$

We will show in the respective Sections 9 and 10 that the functions  $C^{(i)}$ ,  $i = 1, 2$ , have unique minima within the set

$$\Xi := \left\{ (x_0, \dots, x_N) \in \mathbb{R}^{N+1} \mid \sum_{n=0}^N x_n = X_0 \right\},$$

and that these minima coincide with the values of the optimal strategies  $\xi^{(i)}$  as provided in Theorems 4.1 and 5.1. This concludes the reduction to the case of deterministic strategies. We will now turn to the minimization of the functions  $C^{(i)}$  over  $\Xi$ . To simplify the exposition, let us introduce the following shorthand notation in the sequel:

$$a := e^{-\rho\tau}. \quad (45)$$

## 9 The optimal strategy in Model 1.

In this section, we will minimize the function  $C^{(1)}$  of (44) over the set  $\Xi$  of all deterministic strategies and thereby complete the proof of Theorem 4.1. To this end, recall first the definition of the two processes  $E$  and  $D$  as given in (36)–(38). Based on their Model 1 dynamics, we will now obtain a formula of the cost function  $C^{(1)}$  of (44) in terms of the functions  $F$  and  $\tilde{F}$ . It will be convenient to introduce also the function

$$G(x) := \tilde{F}(F^{-1}(x)). \quad (46)$$

Then we have for any deterministic strategy  $\xi = (x_0, \dots, x_N) \in \Xi$  that

$$\begin{aligned} C^{(1)}(x_0, \dots, x_N) &= \sum_{n=0}^N \int_{D_{t_n}}^{D_{t_n+}} x f(x) dx \\ &= \sum_{n=0}^N \left( \tilde{F}(F^{-1}(E_{t_n+})) - \tilde{F}(F^{-1}(E_{t_n})) \right) \\ &= \sum_{n=0}^N \left( G(E_{t_n} + x_n) - G(E_{t_n}) \right) \end{aligned} \quad (47)$$

$$\begin{aligned} &= G(x_0) - G(0) \\ &\quad + G(ax_0 + x_1) - G(ax_0) \\ &\quad + G(a^2x_0 + ax_1 + x_2) - G(a^2x_0 + ax_1) \\ &\quad + \dots \\ &\quad + G(a^Nx_0 + \dots + x_N) - G(a^Nx_0 + \dots + ax_{N-1}). \end{aligned} \quad (48)$$

The derivative of  $G$  is

$$G'(x) = \tilde{F}'(F^{-1}(x)) (F^{-1})'(x) = F^{-1}(x) f(F^{-1}(x)) \frac{1}{f(F^{-1}(x))} = F^{-1}(x). \quad (49)$$

Hence,  $G$  is twice continuously differentiable, positive and convex. The cost function  $C^{(1)}$  is also twice continuously differentiable.

**Lemma 9.1** *We have  $C^{(1)}(x_0, \dots, x_N) \rightarrow +\infty$  for  $|\xi| \rightarrow \infty$ , and therefore there exists a local minimum of  $C^{(1)}$  in  $\Xi$ .*

*Proof:* Using (49) and the fact that  $F^{-1}(x)$  is increasing, we get that for all  $x \in \mathbb{R}$  and  $c \in (0, 1]$

$$G(x) - G(cx) \geq (1 - c) \cdot |F^{-1}(cx)| \cdot |x|. \quad (50)$$

Let us rearrange the sum in (48) in order to use inequality (50). We obtain

$$\begin{aligned} C^{(1)}(x_0, \dots, x_N) &= G(a^N x_0 + a^{N-1} x_1 + \dots + x_N) - G(0) \\ &\quad + \sum_{n=0}^{N-1} \left[ G(a^n x_0 + \dots + x_n) - G(a(a^n x_0 + \dots + x_n)) \right] \\ &\geq G(a^N x_0 + a^{N-1} x_1 + \dots + x_N) - G(0) \\ &\quad + (1 - a) \sum_{n=0}^{N-1} |F^{-1}(a(a^n x_0 + \dots + x_n))| |a^n x_0 + \dots + x_n|. \end{aligned}$$

Let us denote by  $T_1 : \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{N+1}$  the linear mapping

$$T_1(x_0, \dots, x_N) = (x_0, ax_0 + x_1, \dots, a^N x_0 + x_1 a^{N-1} + \dots + x_N).$$

It is non trivial and therefore the norm of  $T_1(x_0, \dots, x_N)$  tends to infinity as the norm of its argument goes to infinity. Because  $F$  is unbounded, we know that both  $G(x)$  and  $|F^{-1}(ax)||x|$  tend to infinity for  $|x| \rightarrow \infty$ . Let us introduce

$$H(x) = \min(G(x), |F^{-1}(ax)||x|).$$

Then also  $H(x) \rightarrow +\infty$  for  $|x| \rightarrow \infty$ , and we conclude that

$$C^{(1)}(x_0, \dots, x_N) \geq (1 - a)H(|T_1(x_0, \dots, x_N)|_\infty) - G(0),$$

where  $|\cdot|_\infty$  denotes the  $\ell^\infty$ -norm on  $\mathbb{R}^{N+1}$ . Hence, the assertion follows.  $\square$

We now consider Equation (15) in Theorem 4.1, which we recall here for the convenience of the reader:

$$F^{-1}(X_0 - Nx_0(1 - a)) = \frac{h_1(x_0)}{1 - a}.$$

This equation is solved by  $x_0$  if and only if  $x_0$  is a zero of the function

$$\hat{h}_1(x) := h_1(x) - (1 - a)F^{-1}(X_0 - Nx(1 - a)). \quad (51)$$

**Lemma 9.2** *Under the assumptions of the Theorem 4.1,  $\hat{h}_1$  has at most one zero  $x_0$ , which, if it exists, is necessarily positive.*

*Proof:* It is sufficient to show that  $\hat{h}_1$  is strictly increasing. We know that  $h_1(0) = 0$ ,  $h_1(x) > 0$  for  $x > 0$ , and  $h_1$  is continuous and one-to-one. Consequently,  $h_1$  must be strictly increasing and therefore

$$\hat{h}'_1(x) = h'_1(x) + \frac{N(a-1)^2}{f(F^{-1}(X_0 + Nx(a-1)))} > 0.$$

Furthermore, if there exists a solution  $x_0$ , then it must be positive since

$$\hat{h}_1(0) = (a-1)F^{-1}(X_0) < 0.$$

□

Theorem 4.1 will now follow by combining the following proposition with the arguments explained in Section 8.

**Proposition 9.3** *The function  $C^{(1)} : \Xi \rightarrow \mathbb{R}$  has the strategy  $\xi^{(1)}$  from Theorem 4.1 as its unique minimizer. Moreover, the components of  $\xi^{(1)}$  are all strictly positive.*

*Proof:* Thanks to Lemma 9.1, there is at least one optimal strategy  $\xi^* = (x_0^*, \dots, x_N^*) \in \Xi$ , and standard results give the existence of a Lagrange multiplier  $\nu \in \mathbb{R}$  such that

$$\frac{\partial}{\partial x_j} C^{(1)}(x_0^*, \dots, x_N^*) = \nu \quad \text{for } j = 0, \dots, N.$$

Now we use the form of  $C^{(1)}$  as given in (48) to obtain the following relation between the partial derivatives of  $C^{(1)}$  for  $j = 0, \dots, N-1$ :

$$\begin{aligned} \frac{\partial}{\partial x_j} C^{(1)}(x_0, \dots, x_N) &= a \left[ \frac{\partial}{\partial x_{j+1}} C^{(1)}(x_0, \dots, x_N) - G'(a(a^j x_0 + \dots + x_j)) \right] \\ &\quad + G'(a^j x_0 + \dots + x_j) \end{aligned}$$

Recalling (49), we obtain

$$h_1(a^j x_0^* + \dots + x_j^*) = \nu(1-a) \quad \text{for } j = 0, \dots, N-1.$$

Since  $h_1$  is one-to-one we must have

$$\begin{aligned} x_0^* &= h_1^{-1}(\nu(1-a)) \\ x_j^* &= x_0^*(1-a) \quad \text{for } j = 1, \dots, N-1 \\ x_N^* &= X_0 - x_0^* - (N-1)x_0^*(1-a). \end{aligned} \tag{52}$$

Note that these equations link all the trades to the initial trade  $x_0$ . Due to the dynamics (37) and (38), it follows that the process  $E$  of  $\xi^*$  is given by

$$E_{t_n} = a(ax_0 + x_0(1-a)) = ax_0. \tag{53}$$



Consequently, by (47),

$$\begin{aligned}
C^{(1)}(x_0^*, \dots, x_N^*) &= G(x_0^*) - G(0) + (N-1)[G(ax_0^* + x_0^*(1-a)) - G(ax_0^*)] \\
&\quad + G(ax_0^* + X_0 - x_0^* - (N-1)x_0^*(1-a)) - G(x_0^*a) \\
&= N[G(x_0^*) - G(x_0^*a)] + G(X_0 + Nx_0^*(a-1)) - G(0) \\
&=: C_0^{(1)}(x_0^*).
\end{aligned}$$

It thus remains to minimize the function  $C_0^{(1)}(x)$  with respect to  $x$ . Thanks to the existence of an optimal strategy in  $\Xi$  for  $C^{(1)}$ , we know that  $C_0^{(1)}(x)$  has at least one local minimum. Differentiating with respect to  $x_0$  gives

$$\begin{aligned}
\frac{\partial C_0^{(1)}(x)}{\partial x} &= N[F^{-1}(x) - aF^{-1}(ax) + (a-1)F^{-1}(X_0 + Nx(a-1))] \\
&= N\hat{h}_1(x).
\end{aligned} \tag{54}$$

Lemma 9.2 now implies that  $C_0^{(1)}$  can only have one local minimum, which is also positive if it exists. This local minimum must hence be equal to  $x_0^*$ , which establishes both the uniqueness of the optimal strategy as well as our representation.

Finally, it remains to prove that all market orders in the optimal strategy are strictly positive. Lemma 9.2 gives  $\xi_0^{(1)} = x_0^* > 0$  and then (52) gives  $\xi_n^{(1)} = x_n^* > 0$  for  $n = 1, \dots, N-1$ . As for the final market order, using the facts that (54) vanishes at  $x = x_0^*$  and  $F^{-1}$  is strictly increasing gives

$$\begin{aligned}
0 &= F^{-1}(x_0^*) - aF^{-1}(ax_0^*) - (1-a)F^{-1}(ax_0^* + x_N^*) \\
&> (1-a)[F^{-1}(ax_0^*) - F^{-1}(ax_0^* + x_N^*)],
\end{aligned}$$

which in turn implies  $x_N^* > 0$ . □

## 10 The optimal strategy in Model 2.

In this section, we will minimize the function  $C^{(2)}$  of (44) over the set  $\Xi$  of all deterministic strategies and thereby complete the proof of Theorem 4.1. To this end, recall first that the definitions of  $D$  and  $E$  are given by (36)–(38). Based on their Model 2 dynamics dynamics, we will now obtain a formula of the cost function  $C^{(2)}$  of (44) in terms of the functions  $F$ ,  $\tilde{F}$ , and  $G$ , where  $G$  is as in (46). For any deterministic strategy  $\xi = (x_0, \dots, x_N) \in \Xi$ ,

$$\begin{aligned}
C^{(2)}(x_0, \dots, x_N) &= \sum_{n=0}^N \int_{D_{t_n}}^{D_{t_n+}} x f(x) dx \\
&= \sum_{n=0}^N \left( G(x_n + F(D_{t_n})) - \tilde{F}(D_{t_n}) \right).
\end{aligned} \tag{55}$$

**Lemma 10.1** *We have  $C^{(2)}(x_0, \dots, x_N) \rightarrow +\infty$  for  $|\xi| \rightarrow \infty$ , and therefore there exists a local minimum of  $C^{(2)}$  in  $\Xi$ .*

*Proof:* We rearrange the sum in (55):

$$\begin{aligned} C^{(2)}(x_0, \dots, x_N) &= \tilde{F}(aF^{-1}(x_N + F(D_{t_N}))) \\ &\quad + \sum_{n=0}^N \left[ \tilde{F}(F^{-1}(x_n + F(D_{t_n}))) - \tilde{F}(aF^{-1}(x_n + F(D_{t_n}))) \right] \\ &\geq \sum_{n=0}^N \left[ \tilde{F}(F^{-1}(x_n + F(D_{t_n}))) - \tilde{F}(aF^{-1}(x_n + F(D_{t_n}))) \right]. \end{aligned} \quad (56)$$

For the terms in (56), we have the lower bound

$$\tilde{F}(x) - \tilde{F}(ax) = \left| \int_{ax}^x y f(y) dy \right| \geq \frac{1}{2}(1 - a^2)x^2 \inf_{y \in [ax, x]} f(y) \geq 0.$$

Let

$$H(x) = \frac{1}{2}(1 - a^2)F^{-1}(x)^2 \inf_{y \in [aF^{-1}(x), F^{-1}(x)]} f(y).$$

Then we have  $H(x) \rightarrow +\infty$  for  $|x| \rightarrow \infty$ , due to (20) and (14). Besides, we have

$$C^{(2)}(x_0, \dots, x_N) \geq H(|T_2(\xi)|_\infty)$$

where  $|\cdot|_\infty$  denotes again the  $\ell^\infty$ -norm on  $\mathbb{R}^{N+1}$ , and  $T_2$  is the (nonlinear) transformation

$$T_2(\xi) = (x_0, x_1 + F^{-1}(D_{t_1}), \dots, x_N + F^{-1}(D_{t_N})).$$

It is sufficient to show that  $|T_2(\xi)|_\infty \rightarrow \infty$  when  $|\xi| \rightarrow \infty$ . To prove this, we suppose by way of contradiction that there is a sequence  $\xi^k$  such that  $|\xi^k|_\infty \rightarrow \infty$  and  $T_2(\xi^k)$  stays bounded. Then, all coordinates in the sequence  $(T_2(\xi^k))_k$  are bounded, and in particular  $(x_0^k)_k$  is a bounded sequence. Therefore,  $D_{t_1}^k = aF^{-1}(x_0^k)$  is also a bounded sequence. The second coordinate  $x_1^k + F^{-1}(D_{t_1}^k)$  being also bounded, we get that  $(x_1^k)_k$  is a bounded sequence. In that manner, we get that  $(x_n^k)_k$  is a bounded sequence for any  $n = 0, \dots, N$ , which is the desired contradiction.  $\square$

**Lemma 10.2** (Partial derivatives of  $C^{(2)}$ ).

*We have the following recursive scheme for the derivatives of  $C^{(2)}(x_0, \dots, x_N)$  for  $i = 0, \dots, N-1$ :*

$$\frac{\partial}{\partial x_i} C^{(2)} = F^{-1}(x_i + F(D_{t_i})) + \frac{af(D_{t_{i+1}})}{f(F^{-1}(x_i + F(D_{t_i})))} \left[ \frac{\partial}{\partial x_{i+1}} C^{(2)} - D_{t_{i+1}} \right]. \quad (57)$$

*Proof:* From (23) we get the following scheme for  $D_{t_n}$  for a fixed  $n \in 1, \dots, N$ :

$$\begin{array}{c}
D_{t_n} \\
\parallel \\
aF^{-1}(x_{n-1} + F(D_{t_{n-1}})) \\
\parallel \\
\vdots \\
aF^{-1}(x_{i+1} + F(D_{t_{i+1}})) \\
\parallel \\
aF^{-1}(x_i + F(D_{t_i})) \\
\parallel \\
\vdots \\
aF^{-1}(x_0).
\end{array}$$

Therefore the following relation holds for the partial derivatives of  $D_{t_n}$ :

$$\frac{\partial}{\partial x_i} D_{t_n} = \frac{af(D_{t_{i+1}})}{f(F^{-1}(x_i + F(D_{t_i})))} \frac{\partial}{\partial x_{i+1}} D_{t_n}, \quad i = 0, \dots, n-2. \quad (58)$$

Furthermore, according to (55) and (49),

$$\begin{aligned}
\frac{\partial}{\partial x_i} C^{(2)} &= F^{-1}(x_i + F(D_{t_i})) + \\
&+ \sum_{n=i+1}^N f(D_{t_n}) \frac{\partial}{\partial x_i} D_{t_n} [F^{-1}(x_n + F(D_{t_n})) - D_{t_n}]
\end{aligned} \quad (59)$$

for  $i = 0, \dots, N$ . Combining (59) and (58) yields (57).  $\square$

**Lemma 10.3** *Under the assumptions of the Theorem 5.1, equation (21) has at most one solution  $x_0 > 0$ . Besides, the function  $g(x) := f(x) - af(ax)$  is positive.*

*Proof:* Uniqueness will follow if we can show that both  $h_2 \circ F^{-1}$  and

$$\hat{h}_2(x) := -F^{-1}(X_0 - N[x - F(aF^{-1}(x))])$$

are strictly increasing. Moreover,  $h_2 \circ F^{-1}(0) = 0$  and  $\hat{h}_2(0) < 0$  so that any zero of  $h_2 \circ F^{-1} + \hat{h}_2$  must be strictly positive.

The function  $h_2$  is one-to-one, has zero as fixed point, and satisfies (26). It is therefore strictly increasing, and since  $F^{-1}$  is also strictly increasing, we get that  $h_2 \circ F^{-1}$  is strictly increasing. It remains to show that  $\hat{h}_2$  is strictly increasing. We have that

$$\hat{h}_2'(x) = N \frac{f(F^{-1}(x)) - af(aF^{-1}(x))}{f(F^{-1}(x)) f(F^{-1}(X_0 - N[x - F(aF^{-1}(x))]))},$$

is strictly positive, because, as we will show now, the numerator of this term is positive. The numerator can be expressed as  $g(F^{-1}(x))$  for  $g$  as in the assertion. Hence, establishing strict positivity of  $g$  will conclude the proof. To prove this we also define  $g_2(x) := f(x) - a^2 f(ax)$  so that

$$h_2(x) = x \frac{g_2(x)}{g(x)}.$$

Both functions  $g$  and  $g_2$  are continuous and have the same sign for all  $x \in \mathbb{R}$  due to the properties of  $h_2$  explained at the beginning of this proof. Because of  $g(x) < g_2(x)$  for all  $x \in \mathbb{R}$ , we infer that there can be no change of signs, i.e., either  $g(x) > 0$  and  $g_2(x) > 0$  for all  $x \in \mathbb{R}$  or  $g(x) < 0$  and  $g_2(x) < 0$  everywhere. With  $g(0) = f(0)(1 - a) > 0$  we obtain the positivity of  $g$ .  $\square$

Theorem 5.1 will now follow by combining the following proposition with the arguments explained in Section 8.

**Proposition 10.4** *The function  $C^{(2)} : \Xi \rightarrow \mathbb{R}$  has the strategy  $\xi^{(2)}$  from Theorem 5.1 as its unique minimizer. Moreover, the components of  $\xi^{(2)}$  are all strictly positive.*

*Proof:* The structure of the proof is similar to the one of Theorem 4.1 although the computations are different. Thanks to Lemma 10.1, we know that there exists an optimal strategy  $\xi^* = (x_0^*, \dots, x_N^*) \in \Xi$ . There also exists a corresponding Lagrange multiplier  $\nu$  such that

$$\frac{\partial}{\partial x_i} C^{(2)}(x_0^*, \dots, x_N^*) = \nu, \quad i = 0, \dots, N.$$

From (57), we get

$$\nu = h_2 \left( F^{-1} (x_i^* + F(D_{t_i})) \right), \quad i = 0, \dots, N-1.$$

Since  $h_2$  is one-to one, this implies in particular that  $x_i^* + F(D_{t_i})$  does not depend on  $i = 0, \dots, N-1$ . It follows from (23) also  $D_{t_i+} = F^{-1}(x_i^* + F(D_{t_i}))$  is constant in  $i$ , and so

$$D_{t_i+} = D_{t_0+} = F^{-1}(x_0^*) \quad \text{and} \quad D_{t_{i+1}} = aF^{-1}(x_0^*). \quad (60)$$

Hence,

$$\begin{aligned} x_0^* &= F(h_2^{-1}(\nu)), \\ x_i^* &= x_0^* - F(D_{t_i}) = x_0^* - F(aF^{-1}(x_0^*)) \quad \text{for } i = 1, \dots, N-1, \\ x_N^* &= X_0^* - x_0^* - (N-1) [x_0^* - F(aF^{-1}(x_0^*))]. \end{aligned} \quad (61)$$

These equations link all market orders to the initial trade  $x_0^*$ . Using (61) and once again (60), we find that  $C^{(2)}(x_0^*, \dots, x_N^*)$  is equal to

$$\begin{aligned} C_0^{(2)}(x_0^*) &:= C^{(2)} \left( x_0^*, x_0^* - F(aF^{-1}(x_0^*)), \dots, X_0 - Nx_0^* + (N-1)F(aF^{-1}(x_0^*)) \right) \\ &= N \left[ G(x_0^*) - \tilde{F}(aF^{-1}(x_0^*)) \right] + G(X_0 + N[F(aF^{-1}(x_0^*)) - x_0^*]). \end{aligned}$$

The initial trade  $x_0^*$  must clearly be a local minimum of  $C_0^{(2)}$  and thus  $\frac{\partial}{\partial x} C_0^{(2)}(x_0^*) = 0$ . Therefore,

$$0 = N \left[ D_{0+} - a^2 D_{0+} \frac{f(D_{t_1})}{f(D_{0+})} + D_{t_N+} \left( a \frac{f(D_{t_1})}{f(D_{0+})} - 1 \right) \right],$$

which is equivalent to

$$D_{t_N+} = D_{0+} \frac{f(D_{0+}) - a^2 f(D_{t_1})}{f(D_{0+}) - a f(D_{t_1})}. \quad (62)$$

This is just equation (21), which has at most one solution, due to Lemma 10.3. This concludes the proof of the existence and the representation of the optimal strategy  $\xi^{(2)}$  in Theorem 5.1.

Finally, we need to show the strict positivity of the optimal strategy. Thanks to the positivity of the optimal  $x_0^*$ , we get

$$x_i^* = x_0^* - F(aF^{-1}(x_0^*)) > 0$$

for  $i = 1, \dots, N-1$ . So it only remains to show that  $x_N^* > 0$ . We infer from (62) and (60) that

$$D_{t_N+} = D_{0+} \frac{f(D_{0+}) - a^2 f(aD_{0+})}{f(D_{0+}) - a f(aD_{0+})} = D_{0+} \left[ 1 + \frac{a f(aD_{0+}) - a^2 f(aD_{0+})}{f(D_{0+}) - a f(aD_{0+})} \right].$$

The fraction on the right is strictly positive due to Lemma 10.3. Hence,

$$D_{t_N+} > D_{0+} = \frac{1}{a} D_{t_N} > D_{t_N},$$

which implies  $x_N^* > 0$ . □

## 11 Optimal strategy for block-shaped LOB.

Here we prove the results of Section 6.

*Proof of Corollary 6.1:* For a constant LOB shape  $f \equiv q$  we have  $F(x) = qx$  and  $F^{-1}(x) = \frac{x}{q}$ . Since the corresponding functions  $h_1$  and  $h_2$  are one-to-one and  $\tilde{F}(x) = \frac{q}{2}x^2$  is convex, we can apply either Theorem 4.1 or Theorem 5.1. Hence, the optimal initial trade  $\xi_0^*$  can be computed by solving the equations (15) or (21). As explained in the remarks following Theorems 4.1 and 5.1, the left-hand sides of these equations are equal to  $D_{t_N+}$ . We have

$$\begin{aligned} D_{0+} &= \frac{\xi_0^*}{q}, \\ D_{t_1} &= a \frac{\xi_0^*}{q}, \\ D_{t_N+} &= a \frac{\xi_0^*}{q} + \frac{X_0 - \xi_0^* - (N-1)\xi_0^*(1-a)}{q}. \end{aligned} \quad (63)$$

Equating the left-hand side of (15) or (21) with  $D_{t_N+}$  of (63), we get

$$\xi_0^* = \frac{X_0}{(N-1)(1-a) + 2}.$$

This is equivalent to

$$X_0 = \xi_0^* [(N-1)(1-a) + 2]. \quad (64)$$

Applying Theorem 4.1 and 5.1, we know that  $\xi_1^* = \dots = \xi_{N-1}^* = \xi_0^*(1-a)$ . With this fact and (64) we can compute

$$\xi_N^* = X_0 - (N-1)(1-a)\xi_0^* - \xi_0^* = \xi_0^*.$$

□

Our next aim is to prove Proposition 6.2, i.e., to show that the strategy (27) satisfies the recursion (31). The key point is that we have indeed explicit formulas for the coefficients in the backward schemes of Proposition 6.2:

**Lemma 11.1** *The coefficients  $\alpha_n$ ,  $\beta_n$ , and  $\gamma_n$  from (33) are explicitly given by*

$$\begin{aligned} \alpha_n &= \frac{(1+a^{-1}) - q\lambda[(N-n)(a^{-1}-1) + 2(1+a^{-1})]}{2q[(N-n)(a^{-1}-1) + (1+a^{-1})]} \\ \beta_n &= \frac{1+a^{-1}}{[(N-n)(a^{-1}-1) + (1+a^{-1})]} \\ \gamma_n &= \frac{(N-n)(1-a^{-1})}{2\kappa[(N-n)(a^{-1}-1) + (1+a^{-1})]}. \end{aligned} \quad (65)$$

*The explicit form of the sequences  $\delta_n$ ,  $\epsilon_n$  and  $\phi_n$  from (32) is*

$$\begin{aligned} \delta_n &= \frac{2a^{-2}[(N-n)(a^{-1}-1) + (1+a^{-1})]}{\kappa[(N-n)(1-a^{-2}) + (N-n+2)(a^{-3}-a^{-1})]} \\ \epsilon_n &= \frac{\kappa(a^{-1}-a)}{[(N-n)(a^{-1}-1) + (1+a^{-1})]} \\ \phi_n &= \frac{(N-n+1)(a^{-1}-a) - (N-n)(1-a^2)}{[(N-n)(a^{-1}-1) + (1+a^{-1})]}. \end{aligned} \quad (66)$$

*Proof:* We show by backward induction that the explicit formulas given in (65) for  $\alpha$ ,  $\beta$  and  $\gamma$  follow from (33) and (32). For  $n = N$  it emanates from (65) that

$$\alpha_N = \frac{(1+a^{-1}) - 2q\lambda(1+a^{-1})}{2q(1+a^{-1})} = \frac{1}{2q} - \lambda, \quad \beta_N = 1, \quad \text{and} \quad \gamma_N = 0,$$

as desired.

As induction hypothesis we take our expressions for  $\alpha_{n+1}$ ,  $\beta_{n+1}$  and  $\gamma_{n+1}$  from (65). We show first that they imply (66). Indeed, by (32) we have

$$\begin{aligned}
\delta_{n+1}^{-1} &= \frac{1}{2q} + \alpha_{n+1} - \beta_{n+1}\kappa a + \gamma_{n+1}\kappa^2 a^2 \\
&= \frac{\frac{1}{q}a^{-2}[(N-n+1)a^{-1} - (N-n-3)] - \lambda a^{-2}[(N-n+1)a^{-1} - (N-n-3)]}{2a^{-2}[(N-n-1)(a^{-1}-1) + (1+a^{-1})]} \\
&\quad + \frac{-2\kappa(a^{-1} + a^{-2}) + \kappa(N-n-1)(1-a^{-1})}{2a^{-2}[(N-n-1)(a^{-1}-1) + (1+a^{-1})]} \\
&= \frac{\kappa[(N-n-1)(1-a^{-2}) + (N-n+1)(a^{-3}-a^{-1})]}{2a^{-2}[(N-n-1)(a^{-1}-1) + (1+a^{-1})]} \\
\epsilon_{n+1} &= \lambda + 2\alpha_{n+1} - \beta_{n+1}\kappa a \\
&= \frac{\frac{1}{q}(1+a^{-1}) - \lambda(1+a^{-1}) - \kappa(a+1)}{[(N-n-1)(a^{-1}-1) + (1+a^{-1})]} \\
&= \frac{\kappa(a^{-1}-a)}{[(N-n-1)(a^{-1}-1) + (1+a^{-1})]} \\
\phi_{n+1} &= 1 - \beta_{n+1}a + 2\gamma_{n+1}\kappa a^2 \\
&= \frac{[(N-n-1)(a^{-1}-1) + (1+a^{-1})] - (a+1) + (N-n-1)(a^2-a)}{[(N-n-1)(a^{-1}-1) + (1+a^{-1})]} \\
&= \frac{(N-n)(a^{-1}-a) - (N-n-1)(1-a^2)}{[(N-n-1)(a^{-1}-1) + (1+a^{-1})]}
\end{aligned}$$

Putting these terms into the three equations in (33), we get the desired result (65) by another longish calculation.  $\square$

*Proof of Proposition 6.2.* We can deduce the following formulas from the preceding lemma:

$$\delta_n \epsilon_n = \frac{2}{(N-n)(1-a) + 2}, \quad \delta_n \phi_n = \frac{2(N-n)(1-a) + 1}{\kappa(N-n)(1-a) + 2}. \quad (67)$$

They will turn out to be convenient in (31).

Let us now consider the optimal strategy  $(\xi_0^*, \dots, \xi_N^*)$  from (27). We consider the associated processes  $D_t := D_t^A$  and  $X_t$  as defined in (28) and (30). For  $n = 0$ , we have

$$\xi_0^* = \frac{X_0}{(N-1)(1-a) + 2} = \frac{1}{2}\delta_1\epsilon_1$$

and it satisfies (31) because  $D_0 = 0$ . For  $n \geq 1$ , we can show easily by induction on  $n$  that  $D_{t_n} = a\kappa\xi_0^*$ . From (27), we get that  $\xi_n^* = (1-a)\xi_0^*$  for  $n \notin \{0, N\}$ , and therefore we get

$$X_{t_n} = X_0 - \xi_0^* - (n-1)(1-a)\xi_0^* = [(N-n)(1-a) + 1]\xi_0^*.$$

Using these formulas, and combining with (67), it is now easy to check that for  $n \in \{1, \dots, N - 1\}$ ,

$$\xi_n^* = \frac{1}{2} [\delta_{n+1} \epsilon_{n+1} X_{t_n} - \delta_{n+1} \phi_{n+1} D_{t_n}],$$

which shows that the optimal strategy given in (27) solves (31).  $\square$

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