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Determinants related to binomial coefficients modulo 2 and 4

Roland Bacher

Abstract: We study a few determinants related to reductions modulo 2 and 4 of binomial and q -binomial coefficients evaluated at $q = 1, -1$ and i .¹

1 Introduction

The main results of this paper are computations of a few determinants related to binomial coefficients. The most interesting example, discovered after browsing through [5], is obtained by considering binomial coefficients modulo 4. We include related results contained in [2] and [3].

The next section recalls mostly well-known facts concerning binomial and q -binomial coefficients and states the main results. It contains also a (new?) formula for evaluating q -binomials coefficients at roots of unity.

Section 3 describes the algebra of recurrence matrices which is a convenient tool for proofs.

Section 4 proves formulae for the determinant of the reduction modulo 2 of the symmetric Pascal matrix (already contained in [2]) and of a determinant related to the 2-valuation of the binomial coefficients (essentially contained in [3]).

Section 5 is devoted to the proof of the main result, a determinant associated to the “Beeblebrox reduction” (defined as $\beta(n) = 0$ if $n \in 2\mathbb{Z}$ and $\beta(n) = \pm 1 \equiv n \pmod{4}$ for odd n) of binomial coefficients. It contains also a digression on the “lower triangular Beeblebrox matrix” and an associated group.

Section 6 contains a proof of a (new?) formula for evaluating q -binomial coefficients at roots of unity. This formula yields easily some determinants associated to the reduction modulo 2 and the Beeblebrox reduction of (real and imaginary parts) of q -binomial coefficients evaluated at $q = -1$ and $q = i$.

¹Math. Class: 05A10,05A30,11B65,11B85,15-99 Keywords: binomial coefficient, q -binomial coefficient, determinant, recurrence matrix

2 Main results

2.1 Reductions modulo 2

Let $P(n)$ be the integral symmetric $n \times n$ matrix with coefficients $P_{s,t} \in \{0, 1\}$, $0 \leq s, t < n$ defined by

$$P_{s,t} \equiv \binom{s+t}{s} \pmod{2}$$

where $\binom{s+t}{s} = \frac{(s+t)!}{s!t!}$ denotes the usual binomial coefficient involved in the expansion $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$.

The evaluation $\binom{s+t}{s} \pmod{2}$ can be computed using a Theorem of Lucas (see [9], page 52) which yields the equality

$$\binom{n}{k} \equiv \prod_{j=0}^{\infty} \binom{\nu_j}{\kappa_j} \pmod{p}$$

where p is an arbitrary prime number and where $n = \sum_{j=0}^{\infty} \nu_j p^j$, $k = \sum_{j=0}^{\infty} \kappa_j p^j$ with $\nu_j, \kappa_j \in \{0, 1, \dots, p-1\}$. Another formula (due to Kummer) for $\binom{n}{k} \pmod{2}$ will be presented in Section 2.2 below.

Let $ds(n) = \sum_{j=0}^{\infty} \nu_j \in \mathbb{N}$ denote the digit-sum of a natural integer with binary expansion $n = \sum_{j=0}^{\infty} \nu_j 2^j$, $\nu_0, \nu_1, \dots \in \{0, 1\}$.

Theorem 2.1. *We have*

$$\det(P(2n)) = (-1)^n$$

and

$$\det(P(2n+1)) = (-1)^{n+ds(n)} .$$

Remark 2.2. *The infinite symmetric integral matrix \tilde{P} with coefficients $\tilde{P}_{s,t} = \binom{s+t}{s}$ given by the binomial coefficients is sometimes called the Fermat matrix. The trivial identities $\sum_{k=0}^s \binom{s}{k} \binom{t}{k} = \sum_{k=0}^s \binom{s}{k} \binom{t}{t-k} = \binom{s+t}{t}$ show $\det(\tilde{P}(n)) = 1$ where $\tilde{P}(n)$ is the symmetric $n \times n$ submatrix with coefficients $\binom{s+t}{s}$, $0 \leq s, t < n$ of \tilde{P} .*

2.2 2-valuations

We denote by $v_2 : \mathbb{Q}^* \rightarrow \mathbb{N}$ the 2-valuation of a rational number. We have thus $\alpha = 2^{-v_2(\alpha)} \frac{n}{m}$ with odd $n, m \in \mathbb{Z}$ for $\alpha \in \mathbb{Q}$. Let $V(n)$ be the symmetric $n \times n$ matrix with coefficients $V_{s,t} \in \{\pm 1, \pm i\}$ given by

$$V_{s,t} = i^{v_2(\binom{s+t}{s})}, \quad 0 \leq s, t < n .$$

The p -valuation $v_p(\binom{s+t}{s})$ of a binomial coefficient can be computed using a Theorem of Kummer stating that $v_p(\binom{s+t}{s})$ equals the number of

carries occurring during the addition of the integers $s = \sum_{j=0} \sigma_j p^j$ and $t = \sum_{j=0} \tau_j p^j$ (with $\sigma_j, \tau_j \in \{0, 1, \dots, p-1\}$), written in base p . More precisely, Kummer deduces the formula

$$v_p\left(\binom{s+t}{s}\right) = \frac{1}{p-1} \sum_{j=0} (\sigma_j + \tau_j - u_j)$$

(see [8], page 116) where $s = \sum_{j=0} \sigma_j p^j, t = \sum_{j=0} \tau_j p^j, s+t = \sum_{j=0} u_j p^j$ (with $\sigma_j, \tau_j, u_j \in \{0, 1, \dots, p-1\}$) from the identity

$$v_p(x!) = \frac{1}{p-1} \left(x - \sum_{j=0} \xi_j p^j \right)$$

(see Lehrsatz, page 115 of [8]) where $x = \sum_{j=0} \xi_j p^j \in \mathbb{N}$ with $\xi_j \in \{0, 1, \dots, p-1\}$ and where p is a prime number.

In order to state the next result, we need the regular folding sequence $f : \{1, 2, \dots\} \rightarrow \{\pm 1\}$ defined recursively by $f(2^n) = 1$ and $f(2^n + a) = -f(2^n - a)$ for $1 \leq a < 2^n$, see [1].

Theorem 2.3. *We have*

$$\det(V(2n)) = (-1)^n \prod_{k=1}^{2n-1} (1 - f(k)i) \in \mathbb{Z}[i]$$

and

$$\det(V(2n+1)) = (-1)^{n+ds(n)} \prod_{k=1}^{2n} (1 - f(k)i) \in \mathbb{Z}[i]$$

(with $ds(\sum_{j=0} \nu_j 2^j) = \sum_{j=0} \nu_j$ denoting the binary digit-sum).

Remark 2.4. *The paper [3] deals with the Hankel matrix H defined by $H_{s,t} = i^{v_2(s+t)}$, $0 \leq s, t$ related by $H = D\bar{V}D$ to the complex conjugate \bar{V} of the matrix V involved in Theorem 2.3 where D is the diagonal matrix with diagonal entries $i^{ds(0)}, i^{ds(1)}, i^{ds(2)}, \dots$.*

Let us also mention that slight extensions of the computations occurring in our proof of Theorem 2.3 establish the existence of nice continued J -fraction expansions for the formal power series (cf. [3])

$$\prod_{k=0}^{\infty} (1 + ix^{2^k}),$$

$$\frac{1}{x} \left(\frac{1+i}{2} + \frac{1-x}{i-1} \prod_{k=0}^{\infty} (1 + ix^{2^k}) \right),$$

$$\frac{1}{x^2} \left(\frac{1+i}{2} + \frac{i-1}{2}x + \frac{1-x^2}{i-1} \prod_{k=0}^{\infty} (1 + ix^{2^k}) \right).$$

2.3 Beeblebrox reduction

The idea (and the ‘‘Beeblebrox’’ terminology) of considering the ‘‘Beeblebrox reduction’’ of binomial coefficients are due to Granville, see [5] and [6].

We define the ‘‘Beeblebrox reduction’’ as the multiplicative map $\beta : \mathbb{Z} \longrightarrow \{0, \pm 1\}$ given by

$$\beta(x) = \begin{cases} 0 & \text{if } x \equiv 0 \pmod{2} \\ 1 & \text{if } x \equiv 1 \pmod{4} \\ -1 & \text{if } x \equiv 3 \pmod{4} \end{cases}$$

or equivalently by $\beta(2\mathbb{Z}) = 0, \beta(4\mathbb{Z} \pm 1) = \pm 1$.

The following result allows fast computations of $\beta\left(\binom{n}{k}\right)$.

Theorem 2.5. *We have*

$$\begin{aligned} \beta\left(\binom{2n}{2k}\right) &= \beta\left(\binom{n}{k}\right) \\ \beta\left(\binom{2n}{2k+1}\right) &= 0 \\ \beta\left(\binom{2n+1}{2k}\right) &= (-1)^k \beta\left(\binom{n}{k}\right) \\ \beta\left(\binom{2n+1}{2k+1}\right) &= (-1)^{n(k+1)} \beta\left(\binom{n}{k}\right) \end{aligned}$$

We denote by $Z(n)$ (where the letter Z stands for Zaphod Beeblebrox, following the amusing terminology of [5] and [6]) the symmetric *Beeblebrox matrix* of size $n \times n$ with coefficients $Z_{s,t} \in \{-1, 0, 1\}$ for $0 \leq s, t < n$ given by the Beeblebrox reduction $Z_{s,t} = \beta\left(\binom{s+t}{s}\right)$ of binomial coefficients.

Define $f : \mathbb{N} \longrightarrow \pm 3^{\mathbb{Z}}$ by $f(0) = 1, f(1) = -1$ and recursively by

$$f(2^a + b) = \begin{cases} 3f(b) & \text{if } 2b < 2^a \\ \frac{1}{3}f(b) & \text{otherwise} \end{cases}$$

for $n = 2^a + b \geq 2$ where $0 \leq b < 2^a$.

Theorem 2.6. *We have*

$$\det(Z(n)) = \prod_{k=0}^{n-1} f(k) \in \pm 3^{\mathbb{N}}.$$

Remark 2.7. *One has the equality*

$$f(n) = (-1)^n 3^{e_1(n)}, n \in \mathbb{N}$$

for $e_1, e_2, e_3 : \mathbb{N} \longrightarrow \mathbb{Z}$ recursively given by $e_1(0) = e_2(0) = 0, e_3(0) = 1$ and

$$\begin{aligned} e_1(2n) &= e_2(n) & e_1(2n+1) &= (2e_1 - e_2)(n) \\ e_2(2n) &= e_2(n) & e_2(2n+1) &= e_3(n) \\ e_3(2n) &= (-2e_1 + 2e_2 + e_3)(n) & e_3(2n+1) &= (2e_1 - e_2)(n) \end{aligned}$$

for $n > 0$.

2.4 q -binomials

The expansion $(x + y)^n = \sum_{k=0}^n \binom{n}{k}_q x^k y^{n-k}$ involving two non-commuting variables x, y related by $yx = qxy$ where q is a central variable defines the q -binomials coefficients

$$\binom{n}{k}_q = \frac{\prod_{j=1}^n (1 - q^j)}{\left(\prod_{j=1}^k (1 - q^j)\right) \left(\prod_{j=1}^{n-k} (1 - q^j)\right)} \in \mathbb{N}[q].$$

An ordinary binary coefficient $\binom{s+t}{s}$ can be identified with the number of lattice paths with steps $(1, 0)$ and $(0, -1)$ starting at $(0, s)$ and ending at $(t, 0)$. Similarly, the coefficient of q^c in the q -binomial $\binom{s+t}{s}_q$ counts the number of such paths delimiting a polygon of area c in the first quadrant $\{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0\}$.

Reflecting all paths contributing to $\binom{s+t}{s}_q$ with respect to the diagonal line $x = y$ yields the equality

$$\binom{s+t}{s}_q = \binom{s+t}{t}_q.$$

Rotating all paths contributing to $\binom{s+t}{s}_q$ by a half-turn centered on $\frac{1}{2}(t, s)$ shows the identity

$$\binom{s+t}{s}_q = q^{st} \binom{s+t}{s}_{q^{-1}}.$$

Partitioning all paths contributing to $\binom{s+t}{s}_q$ accordingly to the nature of their first step (horizontal or vertical) shows the recursive formula

$$\binom{s+t}{s}_q = q^s \binom{s+t-1}{s}_q + \binom{s+t-1}{s-1}_q$$

or equivalently $\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q$ which is the q -version of the celebrated recurrence relation $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for ordinary binomial coefficients.

Cutting all lattice paths γ contributing to $\binom{s+t}{s}_q$ along the line $s = t$ in two lattice paths shows the formula

$$\sum_k q^{k^2} \binom{s}{k}_q \binom{t}{k}_q = \binom{s+t}{s}_q$$

where $k \in \{0, 1, \dots, \min(s, t)\}$. This identity amounts to the matrix identity $P_q = L_q D_q L_q^t$ where P_q is the infinite symmetric matrix with coefficients $\binom{s+t}{s}_q$, $0 \leq s, t$, where L_q is the lower triangular unipotent matrix with coefficients $\binom{s}{t}_q$, $0 \leq s, t$ and where D_q is diagonal with diagonal coefficients $1, q, q^4, q^9, q^{16}, q^{25}, \dots$. Denoting by $P_q(n)$ the submatrix

$\binom{s+t}{s}_q$, $0 \leq s, t < n$ formed by the first n rows and columns of P_q we have the identity $\det(P_q(n)) = q^{\sum_{j=0}^{n-1} j^2}$ which specialises to the identity $\det(P_1(n)) = 1$ of Remark 2.2.

The following result which I have been unable to locate in the literature is somewhat analogous to the formula

$$\binom{a}{b} \equiv \binom{\lfloor a/p \rfloor}{\lfloor b/p \rfloor} \binom{a \pmod{p}}{b \pmod{p}} \pmod{p}$$

(where p is a prime number and where $a \pmod{p}, b \pmod{p} \in \{0, 1, \dots, p-1\}$) due to Lucas (see [9] and Section 2.1. It shows that the evaluation of q -binomial coefficients at roots of 1 reduces essentially to evaluations of ordinary binomial coefficients.

Theorem 2.8. *If $\omega = e^{2i\pi k/n}$ is a primitive n -th root of 1 (ie. $(k, n) = 1$ with $k \in \mathbb{Z}$ and $n \in \mathbb{N}$) then*

$$\binom{a}{b}_\omega = \binom{\lfloor a/n \rfloor}{\lfloor b/n \rfloor}_1 \binom{a \pmod{n}}{b \pmod{n}}_\omega$$

for all $a, b \in \mathbb{N}$ where $a \pmod{n}, b \pmod{n} \in \{0, 1, \dots, n-1\}$.

This Theorem can be used to obtain formulae for determinants of the symmetric matrices obtained by considering the reduction modulo 2 or the Beeblebrox reduction of (the real and imaginary part of) $\binom{s+t}{s}_q$, $0 \leq s, t < n$ evaluated at $q = -1$ and $q = i$.

3 The algebra of recurrence matrices

Recurrence matrices, introduced in [3], are a convenient tool for proving our main results. Recurrence matrices are closely related to rational formal power series in free non-commutative variables and can be considered as generalisations of finite state automata or of iterated tensor products. They arise also naturally in the context of “automata groups”, a notion generalising a famous group of Grigorchuk, see [7]. The following exposition is limited to the strictly necessary technicalities for our purpose. Generalisations (e.g. by replacing the field of complex numbers by an arbitrary commutative field or by considering sequences of square matrices of size $k^n \times k^n$, $n \in \mathbb{N}$ for $k \in \{1, 2, 3, \dots\}$) are fairly straightforward and contained in [3] or with more details in the draft [4].

3.1 Recurrence matrices

Consider the vector space

$$\mathcal{A} = \prod_{n=0}^{\infty} M_{2^n \times 2^n}(\mathbb{C})$$

whose elements are sequences $A = A[0], A[1], A[2], \dots$ with $A[n] \in M_{2^n \times 2^n}(\mathbb{C})$ denoting a complex square matrix of size $2^n \times 2^n$. The obvious product

$$AB = A[0]B[0], A[1]B[1], A[2]B[2], \dots$$

turns \mathcal{A} into an associative algebra. Denoting by

$$\rho(0,0)A, \rho(0,1)A, \rho(1,0)A, \rho(1,1)A \in \mathcal{A}$$

the four ‘‘corners’’ of

$$A = A[0], \left(\begin{array}{cc} (\rho(0,0)A)[0] & (\rho(0,1)A)[0] \\ (\rho(1,0)A)[0] & (\rho(1,1)A)[0] \end{array} \right), \left(\begin{array}{cc} (\rho(0,0)A)[1] & (\rho(0,1)A)[1] \\ (\rho(1,0)A)[1] & (\rho(1,1)A)[1] \end{array} \right), \dots$$

obtained (after deletion of the 1×1 matrix $A[0]$) by considering for all $n \geq 1$ the $2^{n-1} \times 2^{n-1}$ submatrix defined by the first or last 2^{n-1} rows and by the first or last 2^{n-1} columns of $A[n]$ we get four linear endomorphisms $\rho(s, t)$, $0 \leq s, t \leq 1$ of the vector space \mathcal{A} . The endomorphisms $\rho(s, t) \in \text{End}(\mathcal{A})$ are called *shift maps*. Using a hopefully suggestive synthetic notation, an element $A \in \mathcal{A}$ can thus be written as

$$A = A[0], \left(\begin{array}{cc} \rho(0,0)A & \rho(0,1)A \\ \rho(1,0)A & \rho(1,1)A \end{array} \right)$$

with $A[0] \in \mathbb{C}$ and $\rho(0,0)A, \rho(0,1)A, \rho(1,0)A, \rho(1,1)A \in \mathcal{A}$.

Definition A subspace $\mathcal{V} \subset \mathcal{A}$ is *recursively closed* if $\rho(s, t)\mathcal{V} \subset \mathcal{V}$ for all four shift maps $\rho(0,0), \rho(0,1), \rho(1,0), \rho(1,1)$.

The *recursive closure* \overline{A} of an element $A \in \mathcal{A}$ is the intersection of all recursively closed subspaces containing A . The *complexity* of $A \in \mathcal{A}$ is the dimension $\dim(\overline{A}) \in \mathbb{N} \cup \{\infty\}$ of the recursive closure $\overline{A} \subset \mathcal{A}$.

An element $A \in \mathcal{A}$ is a *recurrence matrix* if its recursive closure \overline{A} is of finite dimension. We denote by $\mathcal{R} \subset \mathcal{A}$ the subset of all recurrence matrices.

Proposition 3.1. (i) We have $\dim(\overline{\lambda A}) = \dim(\overline{A})$ for all $\lambda \in \mathbb{C}^*$ and for all $A \in \mathcal{A}$.

(ii) We have $\dim(\overline{A+B}) \leq \dim(\overline{A}) + \dim(\overline{B})$ for all $A, B \in \mathcal{A}$.

(iii) We have $\dim(\overline{AB}) \leq \dim(\overline{A}) \dim(\overline{B})$ for all $A, B \in \mathcal{A}$.

Proof (i) and (ii) are obvious.

Denoting by $\Sigma \overline{A} \overline{B} = \{\sum X_i Y_i \mid X_i \in \overline{A}, Y_i \in \overline{B}\}$ the vector space spanned by all products XY , $X \in \overline{A}, Y \in \overline{B}$, we have $AB \in \Sigma \overline{A} \overline{B}$.

For

$$XY = (X[0]Y[0]), \left(\begin{array}{cc} \rho(0,0)(XY) & \rho(0,1)(XY) \\ \rho(1,0)(XY) & \rho(1,1)(XY) \end{array} \right) \in \Sigma \overline{A} \overline{B}$$

with $X \in \overline{A}, Y \in \overline{B}$, the computation

$$\begin{aligned}\rho(0,0)(XY) &= (\rho(0,0)X)(\rho(0,0)Y) + (\rho(0,1)X)(\rho(1,0)Y) \\ \rho(0,1)(XY) &= (\rho(0,0)X)(\rho(0,1)Y) + (\rho(0,1)X)(\rho(1,1)Y) \\ \rho(1,0)(XY) &= (\rho(1,0)X)(\rho(0,0)Y) + (\rho(1,1)X)(\rho(1,0)Y) \\ \rho(1,1)(XY) &= (\rho(1,0)X)(\rho(0,1)Y) + (\rho(1,1)X)(\rho(1,1)Y)\end{aligned}$$

shows that $XY \in \Sigma \overline{A} \overline{B}$ is recursively closed of dimension $\leq \dim(\overline{A}) \dim(\overline{B})$. The obvious inclusion $\overline{AB} \subset \Sigma \overline{A} \overline{B}$ implies now assertion (iii). \square

Corollary 3.2. *The set \mathcal{R} of recurrence matrices is a subalgebra of \mathcal{A} .*

3.2 Presentations

Setting $X_{s,t}A = \rho(s,t)A$ for $0 \leq s, t \leq 1$, the shift maps $\rho(s,t) \in \text{End}(\mathcal{A})$, $0 \leq s, t \leq 1$ define a representation of the free non-commutative monoid $\{X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}\}^*$ in four generators $X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}$ on the vector space \mathcal{A} . Restrictions to recursively closed subspaces $\mathcal{V} \subset \mathcal{A}$ yield subrepresentations. In particular, a recurrence matrix A is completely determined by the description of the action of the four generating shift maps $\rho(s,t)$, $0 \leq s, t \leq 1$ on \overline{A} together with the linear *augmentation map* $X \mapsto X[0] \in \mathbb{C}$ for $X \in \overline{A}$.

Remark 3.3. *The above action (recursively defined by*

$$(X_{s_1, j_1} X_{s_2, t_2} \cdots X_{s_l, t_l})A = (X_{s_1, j_1} X_{s_2, t_2} \cdots X_{s_{l-1}, t_{l-1}})(\rho(s_l, t_l)A),$$

associates an element $A \in \mathcal{A}$ bijectively with a formal power series

$$\sum_{\mathbf{X} \in \{X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}\}^*} ((\mathbf{X}A)[0])\mathbf{X} \in \mathbb{C}\langle\langle X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1} \rangle\rangle$$

in four free non-commuting variables $X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}$. Recursive matrices correspond under this bijection to rational formal power series in $\mathbb{C}\langle\langle X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1} \rangle\rangle$.

The recursive closure \overline{A} of $A \in \mathcal{A}$ corresponds to the linear span of $\mathbf{X}A$, $\mathbf{X} \in \{X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}\}^$.*

3.3 Recursive presentations

In the sequel of this paper we use recursive presentations in order to describe a recurrence matrix $A \in \mathcal{R}$. A *recursive presentation* of $A \in \mathcal{A}$ is given by choosing a basis $A_1 = A, \dots, A_a$ of \overline{A} (or more generally of any finite-dimensional recursively closed vector space containing \overline{A}) and by the recursive identities for A_j , $1 \leq j \leq a$ given by

$$A_j[0] \in \mathbb{C}, \left(\begin{array}{ll} \rho(0,0)A_j = \sum_{k=1}^a \rho(0,0)_{k,j} A_k & \rho(0,1)A_j = \sum_{k=1}^a \rho(0,1)_{k,j} A_k \\ \rho(1,0)A_j = \sum_{k=1}^a \rho(1,0)_{k,j} A_k & \rho(1,1)A_j = \sum_{k=1}^a \rho(1,1)_{k,j} A_k \end{array} \right)$$

which define $A_1, \dots, A_a \in \mathcal{R}$ uniquely.

3.4 Saturation level

Given an element $A = A[0], A[1], A[2], \dots \in \mathcal{A}$, we denote by $\pi_l(A) = A[l]$ the projection of A onto its square matrix of size $2^l \times 2^l$. Similarly,

$$\pi_{\leq l}(A) = \pi_0(A), \pi_1(A), \dots, \pi_l(A) = A[0], A[1], \dots, A[l] \in \bigoplus_{j=0}^l M_{2^j \times 2^j}(\mathbb{C})$$

denotes the projection of the sequence A onto its first $l + 1$ matrices.

The *saturation level* of a finite dimensional subspace $\mathcal{V} \subset \mathcal{A}$ is the smallest integer $N \in \mathbb{N}$ such that $\mathcal{K}_{\leq N}(\mathcal{V}) = \mathcal{K}_{\leq N+1}(\mathcal{V})$ where $\mathcal{K}_{\leq l}(\mathcal{V}) \subset \mathcal{V}$ is the kernel of the projection $\pi_{\leq l} : \mathcal{V} \rightarrow \bigoplus_{j=0}^l M_{2^j \times 2^j}$.

Proposition 3.4. *We have $\mathcal{K}_{\leq N}(\mathcal{V}) = \{0\}$ for the saturation level N of a finite-dimensional subspace $\mathcal{V} \subset \mathcal{A}$ which is recursively closed.*

In particular, $\pi_{\leq N} : \mathcal{V} \rightarrow \bigoplus_{j=0}^N M_{2^j \times 2^j}$ is injective.

Proof The obvious inclusion $\rho(s, t)\mathcal{K}_{\leq l+1}(\mathcal{V}) \subset \mathcal{K}_{\leq l}(\mathcal{V})$ implies that $\mathcal{K}_{\leq N}(\mathcal{V}) = \mathcal{K}_{\leq N+1}(\mathcal{V}) \subset \mathcal{V}$ is recursively closed. Since the restriction to $\mathcal{K}_{\leq N+1}(\mathcal{V})$ of the augmentation map $\pi_0 : \mathcal{A} \rightarrow \mathbb{C}$ is obviously trivial, we have $(\mathbf{X}K)[0] = 0$ for all $\mathbf{X} \in \{X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}\}^*$ and for all $K \in \mathcal{K}_{\leq N}(\mathcal{V})$. This shows $\mathcal{K}_{\leq N}(\mathcal{V}) = \{0\}$. \square

Proposition 3.4 enables us to do effective computations in the algebra \mathcal{R} of recurrence matrices: It allows to extract a basis from a finite (linearly dependent) generating set \mathcal{S} spanning a recursively closed vector space $\mathcal{V} \subset \mathcal{R}$. Similarly, given an element $A \in \mathcal{V}$ with \mathcal{V} recursively closed given by a finite generating set, it allows the construction of a basis for \overline{A} . For elements $A, B \in \mathcal{R}$ described by (recursive) presentations with respect to bases A_1, \dots, A_a and B_1, \dots, B_b of \overline{A} and \overline{B} , it allows the algorithmic construction of bases of the recursively closed vector spaces $\overline{A \pm B}$, respectively \overline{AB} , since they are included in the recursively closed vector spaces spanned by $A \pm B, A_1, \dots, A_a, B_1, \dots, B_b$, respectively $AB, A_i B_j, 1 \leq i \leq a, 1 \leq j \leq b$ which contain $\overline{A \pm B}$, respectively \overline{AB} .

Moreover, identities in \mathcal{R} can be proven by finite computations: proving for example the identity $AB = C$ (with suitable $A, B, C \in \mathcal{R}$ given by finite presentations) boils down to the computation of the saturation level N of the vector space $\overline{AB} + \overline{C}$ followed by the verification of the identity $\pi_{\leq N}(A)\pi_{\leq N}(B) = \pi_{\leq N}(C)$ in the finite-dimensional quotient $\pi_{\leq N}(\mathcal{A})$ of the algebra \mathcal{A} .

Remark 3.5. *Determining the inversibility (in \mathcal{R}) of a recursive matrix A seems to be a difficult problem which has perhaps no algorithmic solution: The element*

$$1, \begin{pmatrix} 1 & & & \\ \omega & 1 & & \\ & & & \end{pmatrix}, \begin{pmatrix} 1 & & & \\ \omega & 1 & & \\ 0 & \omega & 1 & \\ 0 & 0 & \omega & 1 \end{pmatrix}, \dots \in \mathcal{R}$$

consisting of lower triangular unipotent matrices with constant subdiagonal $\omega \in \mathbb{C}^*$ has complexity 2. It is invertible in the algebra \mathcal{R} for $\omega = e^{2i\pi/N}$ (with $N \in \mathbb{N}$) a root of unity. The complexity of the inverse $A^{-1} \in \mathcal{R}$ depends on N (for $\omega = e^{2i\pi/N}$) and tends to infinity if $N \rightarrow \infty$.

Concretely, determining the invertibility of an element $A \in \mathcal{R}$ amounts to guessing a (recursive) presentation of $B = A^{-1}$ using a finite number of matrices $A[0]^{-1}, A[1]^{-1}, A[2]^{-1}, \dots$ followed by a proof of the identity $AB = 1$ (which is easy for the correct guess or indicates a wrong guess in case of failure).

3.5 The LU decomposition of a convergent non-singular element in \mathcal{R}

An element $P \in \mathcal{A}$ such that $P = \rho(0,0)P$ is called *convergent*. It is given by considering the sequence

$$P_{0,0}, \left(\begin{array}{cc} P_{0,0} & P_{0,1} \\ P_{1,0} & P_{1,1} \end{array} \right), \left(\begin{array}{cccc} P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} \\ P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\ P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3} \end{array} \right), \dots$$

of all square submatrices formed by the first 2^j rows and columns (for $j = 0, 1, \dots$) of an infinite “limit” matrix

$$\left(\begin{array}{cccc} P_{0,0} & P_{0,1} & P_{0,2} & \dots \\ P_{1,0} & P_{1,1} & P_{1,2} & \dots \\ \vdots & & & \end{array} \right).$$

Henceforth we denote generally a convergent element in \mathcal{A} and the associated infinite matrix by the same letter. This should not lead to confusions except in cases where both interpretations are correct.

We call an infinite matrix P *non-singular* if the $k \times k$ square matrix $P(k)$ formed by its first k rows and columns has non-zero determinant for all $k \geq 1$. Such a non-singular matrix P has an LU -decomposition: It can be written as $P = LU$ with L lower triangular unipotent (1's on the diagonal) and U upper triangular non-singular. The identity $P = LU$ implies the equality $\det(P(k)) = \det(U(k))$ for all $k \geq 1$ and gives rise to an LU -decomposition in \mathcal{A} by considering as above for $j = 0, 1, 2, \dots$ the submatrices formed by the first 2^j rows and columns of P, L and U . If P is symmetric we have moreover $U = DL^t$ where D is diagonal non-singular and L^t is obtained by transposing the matrix L .

All proofs of the results presented in Section 2 boil down to LU -decompositions with $P = P^t = LU, L, U = DL^t \in \mathcal{R}$.

4 Modulo 2 and 2-valuations

Proof of Theorem 2.1 The infinite symmetric Pascal matrix P with coefficients $\binom{i+j}{i} \pmod{2} \in \{0, 1\}$ for $0 \leq i, j$ defines a convergent element (still denoted) $P \in \mathcal{A}$. It follows from Lucas's formula (see Section 2.1) that P is a recurrence matrix of complexity 1 recursively presented by

$$P = 1, \begin{pmatrix} P & P \\ P & P \end{pmatrix}$$

(zero-entries are omitted). The recurrence matrix $P \in \mathcal{R}$ has an LU decomposition in \mathcal{R} given by the equality $P = LDL^t$ with $L, D \in \mathcal{R}$ of complexity 1 defined by the recursive presentations

$$L = 1, \begin{pmatrix} L & \\ L & L \end{pmatrix} \text{ and } D = 1, \begin{pmatrix} D & \\ & -D \end{pmatrix}$$

where L is lower triangular unipotent and D is diagonal. An easy analysis of the coefficients of the diagonal recurrence matrix D ends the proof. \square

Remark 4.1. *The convergent lower triangular recurrence matrix L and the convergent diagonal recurrence matrix D correspond to the infinite limit-matrices (still denoted) L, D with coefficients given by $L_{i,j} = \binom{i+j}{i} \pmod{2} \in \{0, 1\}$ (for $0 \leq i, j$) and $D_{n,n} = (-1)^{\nu_0 + \nu_1 + \nu_2 + \dots} = (-1)^{ds(n)}$ where $n = \sum_{j=0}^{\infty} \nu_j 2^j \geq 0$ is a binary integer.*

Remark 4.2. *Recurrence matrices of complexity 1 are of the form*

$$\lambda, \lambda M, \lambda M \otimes M, \lambda M \otimes M \otimes M, \dots$$

where $\lambda \in \mathbb{C}$ and where M is a complex 2×2 matrix.

It follows that the matrices L and D (and thus also $P = LDL^t$) involved in the proof of Theorem 2.1 are invertible in \mathcal{R} . The recurrence matrix D is its own inverse. The inverse L^{-1} of L is recursively presented by $L^{-1} = 1, \begin{pmatrix} L^{-1} & \\ -L^{-1} & L^{-1} \end{pmatrix}$.

Proof of Theorem 2.3 The infinite matrix V with coefficients

$$V_{s,t} = i^{\nu_2} \binom{s+t}{s} = i^{ds(s)+ds(t)-ds(s+t)}$$

gives rise to a convergent element $V \in \mathcal{A}$. A bit of work using Kummer's formulae (see Section 2.2) shows that $V = V_1$ is recurrence matrix given by the recursive presentation

$$V_1 = 1, \begin{pmatrix} V_1 & V_2 \\ V_2 & iV_1 \end{pmatrix}$$

$$V_2 = 1, \begin{pmatrix} V_1 & -iV_1 + (1+i)V_2 \\ -iV_1 + (1+i)V_2 & -V_1 \end{pmatrix}$$

We have $V = LU \in \mathcal{R}$ with $L = L_1 \in \mathcal{R}$ recursively presented by

$$\begin{aligned} L_1 &= 1, \begin{pmatrix} L_1 & & & \\ & L_3 & & L_4 \\ & & & \\ & & & \end{pmatrix} \\ L_2 &= 0, \begin{pmatrix} & 0 & & -iL_2 \\ & -L_1 + L_3 & & -iL_2 - iL_4 \\ & & & \\ & & & \end{pmatrix} \\ L_3 &= 1, \begin{pmatrix} & & & L_2 \\ & L_1 & & \\ & -iL_1 + (1+i)L_3 & & L_2 + (1+i)L_4 \\ & & & \end{pmatrix} \\ L_4 &= 1, \begin{pmatrix} & & & \\ & L_1 & & \\ & (1-i)L_1 + iL_3 & & L_4 \\ & & & \end{pmatrix} \end{aligned}$$

and $U = DL^t$ with $D = D_1 \in \mathcal{R}$ diagonal recursively presented by

$$\begin{aligned} D_1 &= 1, \begin{pmatrix} D_1 & & \\ & D_2 & \\ & & \end{pmatrix} \\ D_2 &= -1 + i, \begin{pmatrix} & D_3 & \\ & & 2D_1 - D_2 + 2D_3 \\ & & \end{pmatrix} \\ D_3 &= -1 + i, \begin{pmatrix} & & D_3 \\ & & \\ & & -D_2 \end{pmatrix} \end{aligned}$$

An analysis (left to the reader) of the diagonal entries of D_1 ends the proof. \square

Remark 4.3. *The recurrence matrices L, D and $V = LDL^t$ are invertible in \mathcal{R} , see [3].*

5 Beeblebrox reduction

This section is devoted to proofs and complements involving the Beeblerox reduction $\beta\left(\binom{n}{k}\right)$ of binomial coefficients.

Proof of Theorem 2.5 We have

$$\binom{2n}{2k} = \frac{(2n) \cdots (2n - 2k + 1)}{(2k) \cdots 1} = \binom{n}{k} \frac{(2n-1)(2n-3) \cdots (2n-2k+1)}{(2k-1)(2k-3) \cdots 1}$$

where both the numerator and the denominator of the fraction

$$F = \frac{(2n-1)(2n-3) \cdots (2n-2k+1)}{(2k-1)(2k-3) \cdots 1}$$

contain k terms. If k is even, we have $F \equiv 1 \pmod{4}$ since the numerator and denominator of the fraction F contain both $k/2$ factors $\equiv 1 \pmod{4}$ and $k/2$ factors $\equiv -1 \pmod{4}$. If k and n are both odd, the numerator and denominator of F contain both $(k+1)/2$ factors $\equiv 1 \pmod{4}$ and $(k-1)/2$ factors $\equiv -1 \pmod{4}$ and we have again $F \equiv 1 \pmod{4}$. If k is odd and n is even, then both binomial coefficients $\binom{2n}{2k}$ and $\binom{n}{k}$ are even and we have thus $\beta\left(\binom{2n}{2k}\right) = \beta\left(\binom{n}{k}\right) = 0$. This proves the first equality.

The binomial coefficient $\binom{2n}{2k+1}$ is obviously even and this implies the second equality.

In the next case we have

$$\binom{2n+1}{2k} = \binom{n}{k} \frac{(2n+1) \cdots (2n-2k+3)}{(2k-1) \cdots (1)}$$

and the last fraction equals 1 (mod 4) if k is even.

For k odd and n even we have $\beta(\binom{2n+1}{2k}) = \beta(\binom{n}{k}) = 0$ since $\binom{2n+1}{2k} \equiv \binom{n}{k} \equiv 0 \pmod{2}$.

For n, k both odd the correction $(-1)^k = -1$ equals the fraction modulo 4. This ends the proof of the third equality.

In the case of the last equality, we have

$$\binom{2n+1}{2k+1} = \frac{2n+1}{2k+1} \binom{2n}{2k}$$

which is even if $n \equiv 0 \pmod{2}$ and $k \equiv 1 \pmod{2}$. If $n \equiv k \pmod{2}$ then $\frac{2n+1}{2k+1} \equiv 1 \pmod{4}$. For n odd and k even we have $\frac{2n+1}{2k+1} \equiv -1 = (-1)^{n(k+1)} \pmod{4}$. The first equality and these observations complete the proof. \square

5.1 Proof of Theorem 2.6

Proof Some work shows again that the convergent element of \mathcal{A} associated to the infinite symmetric matrix Z with coefficients $Z_{s,t} = \beta(\binom{s+t}{s})$, $0 \leq s, t$ is a recurrence matrix $Z = Z_1$ given by the recursive presentation

$$\begin{aligned} Z_1 &= 1, \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & 0 \end{pmatrix} \\ Z_2 &= 1, \begin{pmatrix} Z_1 & Z_2 \\ -Z_3 & 0 \end{pmatrix} \\ Z_3 &= 1, \begin{pmatrix} Z_1 & -Z_2 \\ Z_3 & 0 \end{pmatrix} \end{aligned}$$

We have the identity $Z = LDL^t$ with $L = L_1 \in \mathcal{R}$ lower triangular unipotent given by the recursive presentation

$$\begin{aligned} L_1 &= 1, \begin{pmatrix} L_1 & \\ L_3 & L_4 \end{pmatrix} & L_2 &= 2, \begin{pmatrix} -\frac{2}{3}L_1 & 2L_2 \\ \frac{3}{3}L_3 & 2L_4 \end{pmatrix} \\ L_3 &= 1, \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} & L_4 &= 1, \begin{pmatrix} L_1 & \\ \frac{1}{3}L_3 & L_4 \end{pmatrix} \end{aligned}$$

The diagonal matrix $D = D_1 \in \mathcal{R}$ has recursive presentation

$$D_1 = 1, \begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix}, D_2 = -1, \begin{pmatrix} 3D_1 & \\ & \frac{1}{3}D_2 \end{pmatrix}$$

An easy inspection of the diagonal entries of D_1 completes the proof. \square

5.1.1 Complement: The recurrence matrices L_i^{-1}, D_i^{-1} and Z_i^{-1}

The recurrence matrices L_1, L_2, L_3, L_4 are invertible in \mathcal{R} . Their inverses are given by

$$L_1^{-1} = M_1, \quad L_2^{-1} = -\frac{1}{2}M_3, \quad L_3^{-1} = -\frac{1}{2}M_2, \quad L_4^{-1} = M_4$$

with M_1, M_2, M_3, M_4 recursively presented by

$$\begin{aligned} M_1 &= 1, \begin{pmatrix} M_1 & \\ M_3 & M_4 \end{pmatrix} & M_2 &= -2, \begin{pmatrix} 2M_1 & 2M_2 \\ 2M_3 & 2M_4 \end{pmatrix} \\ M_3 &= -1, \begin{pmatrix} M_1 & M_2 \\ \frac{1}{3}M_3 & -\frac{1}{3}M_4 \end{pmatrix} & M_4 &= 1, \begin{pmatrix} M_1 & \\ \frac{1}{3}M_3 & M_4 \end{pmatrix} \end{aligned}$$

It would perhaps be interesting to understand the group generated by

$$\begin{aligned} a &= L_1 = 1, \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 1 & \frac{1}{3} & 1 \end{pmatrix}, \dots \\ b &= \frac{1}{2}L_2 = 1, \begin{pmatrix} -\frac{1}{3} & 2 \\ -\frac{1}{3} & 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{3} & 0 & -\frac{2}{3} & 4 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 2 \\ -\frac{1}{3} & -\frac{1}{3} & 1 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}, \dots \\ c &= L_3 = 1, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -\frac{2}{3} & 4 \\ 1 & 1 & \frac{2}{3} & 2 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & \frac{1}{3} & 1 \end{pmatrix}, \dots \\ d &= L_4 = 1, \begin{pmatrix} 1 & \\ \frac{1}{3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ \frac{1}{3} & \frac{2}{3} & 1 & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}, \dots \end{aligned}$$

A few relations are $(ac^{-1})^2, ac^{-1} = bd^{-1}, a^{-1}b = c^{-1}d, a^{-1}d = c^{-1}b$.

The inverses of the diagonal recurrence matrices D_1, D_2 are $\tilde{D}_1 = D_1^{-1}, \tilde{D}_2 = D_2^{-1}$ recursively presented by

$$\tilde{D}_1 = 1, \begin{pmatrix} \tilde{D}_1 & \\ & \tilde{D}_2 \end{pmatrix}, \tilde{D}_2 = -1, \begin{pmatrix} \frac{1}{3}\tilde{D}_1 & \\ & 3\tilde{D}_2 \end{pmatrix}$$

The inverses of the matrices Z_1, Z_2, Z_3 are $U_1 = Z_1^{-1}, U_3 = Z_2^{-1}, U_2 =$

Z_3^{-1} recursively presented by

$$\begin{aligned} U_1 &= 1, \begin{pmatrix} 0 & U_2 \\ U_3 & U_1 - U_2 - U_3 \end{pmatrix}, \\ U_2 &= 1, \begin{pmatrix} 0 & U_2 \\ -U_3 & -U_1 + U_2 + U_3 \end{pmatrix}, \\ U_3 &= 1, \begin{pmatrix} 0 & -U_2 \\ U_3 & -U_1 + U_2 + U_3 \end{pmatrix}. \end{aligned}$$

Since $Z_1 = Z_1^t$ and $Z_2 = Z_3^t$, the subgroup of \mathcal{R} generated by Z_1, Z_2, Z_3 has an involutive automorphism given by

$$Z_1 \mapsto Z_1^{-1} = U_1, \quad Z_2 \mapsto Z_3^{-1} = U_2, \quad Z_3 \mapsto Z_2^{-1} = U_3.$$

A few relations among the elements $a = Z_1, b = Z_2, c = Z_3$ are $(ab^{-1})^2, a^2 = cb, ab = ca, ab^{-1} = c^{-1}a$.

Remark 5.1. *The square matrix $A(n)$ of size $n \times n$ with coefficient $A_{s,t} = 1$ if $\binom{s+t}{s} \equiv 1 \pmod{4}$ and $A_{s,t} = 0$ otherwise (for $0 \leq s, t < n$) has $\det(A(1)) = 1, \det(A(2)) = \det(A(6)) = \det(A(10)) = \det(A(22)) = -1$ and $\det(A(n)) = 0$ for all other natural numbers $n \leq 60$.*

Moreover, for the exceptional values $n = 2, 6, 10, 22$, the Hadamard product of $A(n)$ with its inverse $A(n)^{-1}$ is an involutive permutation matrix without fixpoints.

5.2 The triangular Beeblebrox matrix

We define the lower triangular Beeblebrox matrix as the infinite lower triangular matrix with coefficients $L_{s,t} = \beta\left(\binom{s}{t}\right), 0 \leq s, t$ given by the Beeblebrox reduction of binomial coefficients.

One of the main results of [5] states that any fixed row of L contains either no coefficients -1 or the same number (given by a power of 2) of coefficients 1 and -1 . This can of course also be deduced from Theorem 2.5 or by computing LJ where J is the ‘‘recurrence vector’’ obtained by considering the sequence of column vectors

$$(1), (1, 1)^t, (1, 1, 1, 1)^t, (1, 1, 1, 1, 1, 1, 1, 1)^t, \dots$$

The triangular Beeblebrox matrix L defines a recurrence matrix (still denoted) $L = L_1 \in \mathcal{R}$ recursively presented by

$$L_1 = 1, \begin{pmatrix} L_1 \\ L_2 \quad L_3 \end{pmatrix}, L_2 = 1, \begin{pmatrix} L_1 \\ L_2 \quad -L_3 \end{pmatrix}, L_3 = 1, \begin{pmatrix} L_1 \\ -L_2 \quad L_3 \end{pmatrix}.$$

5.2.1 The recurrence matrices L_i^{-1}

The inverse elements L_1^{-1}, L_3^{-1} are given by $L_1^{-1} = M_1, L_3^{-1} = M_3$ with M_1, M_2, M_3 recursively presented by

$$M_1 = 1, \begin{pmatrix} M_1 & \\ M_2 & M_3 \end{pmatrix}, M_2 = -1, \begin{pmatrix} -M_1 & \\ 3M_2 & M_3 \end{pmatrix}, M_3 = 1, \begin{pmatrix} M_1 & 0 \\ -M_2 & M_3 \end{pmatrix}$$

The computation

$$\begin{pmatrix} L_1 & \\ L_2 & -L_3 \end{pmatrix} \begin{pmatrix} L_1^{-1} & \\ L_3^{-1}L_2L_1^{-1} & -L_3^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

(with 1 denoting the multiplicative identity in \mathcal{R}) shows that

$$L_2^{-1} = 1, \begin{pmatrix} L_1^{-1} & \\ L_3^{-1}L_2L_1^{-1} & -L_3^{-1} \end{pmatrix} \in \mathcal{R}.$$

A small amount of work shows that the map which sends L_1, L_2, L_3 to the three matrices

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

(corresponding to the three affine maps $(x, y) \mapsto (x + 2, y), (x, y) \mapsto (y + 1, x + 1), (x, y) \mapsto (x, y + 2)$ of \mathbb{R}^2) defines a faithful representation of the group $\langle L_1, L_2, L_3 \rangle \subset \mathcal{R}$ generated by L_1, L_2, L_3 .

6 q -binomials

Proof of Theorem 2.8 The result holds for $b = 0$ and for $a \leq b$. An induction on $a + b$ (splitting into four subcases) ends the proof:

If $a, b \not\equiv 0 \pmod{n}$:

$$\begin{aligned} \binom{a}{b}_q &= \omega^b \binom{a-1}{b}_\omega + \binom{a-1}{b-1}_\omega \\ &= \binom{[a/n]}{[b/n]} \left(\omega^b \binom{a-1 \pmod{n}}{b \pmod{n}}_\omega + \binom{a-1 \pmod{n}}{b-1 \pmod{n}}_\omega \right) \\ &= \binom{[a/n]}{[b/n]} \binom{a \pmod{n}}{b \pmod{n}}_\omega \end{aligned}$$

If $a \equiv 0 \pmod{n}, b \not\equiv 0 \pmod{n}$:

$$\begin{aligned} \binom{a}{b}_q &= \omega^b \binom{a-1}{b}_\omega + \binom{a-1}{b-1}_\omega \\ &= \binom{a/n-1}{[b/n]} \left(\omega^b \binom{n-1}{b \pmod{n}}_\omega + \binom{n-1}{b-1 \pmod{n}}_\omega \right) \\ &= \binom{a/n-1}{[b/n]} \binom{n}{b \pmod{n}}_\omega \end{aligned}$$

and $\binom{n}{b \pmod{n}}_\omega = 0$ since it is divisible by the n -th cyclotomic polynomial.

If $a \not\equiv 0, b \equiv 0 \pmod{n}$:

$$\begin{aligned}
\binom{a}{b}_q &= \omega^b \binom{a-1}{b}_\omega + \binom{a-1}{b-1}_\omega \\
&= \binom{a-1}{b}_\omega + \binom{\lfloor a/n \rfloor}{\lfloor b/n-1 \rfloor} \binom{a-1 \pmod{n}}{n-1}_\omega \\
&= \binom{\lfloor a/n \rfloor}{\lfloor b/n \rfloor} \binom{a-1 \pmod{n}}{0}_\omega + 0 \\
&= \binom{\lfloor a/n \rfloor}{\lfloor b/n \rfloor} \binom{a \pmod{n}}{0}_\omega
\end{aligned}$$

If $a \equiv b \equiv 0 \pmod{n}$:

$$\begin{aligned}
\binom{a}{b}_q &= \omega^b \binom{a-1}{b}_\omega + \binom{a-1}{b-1}_\omega \\
&= \binom{a/n-1}{b/n}_\omega \binom{n-1}{0}_\omega + \binom{a/n-1}{b/n-1}_\omega \binom{n-1}{n-1}_\omega \\
&= \binom{a/n}{b/n} \binom{0}{0}_\omega
\end{aligned}$$

□

Theorem 2.8 implies in particular that the complex numbers $\binom{a}{b}_{e^{2i\pi k/n}}$, $a, b \in \mathbb{N}$ belong to a finite number of real (half-)lines of the complex plane (determined by all non-zero values of $\binom{a \pmod{n}}{b \pmod{n}}_\omega$).

For $n = 2$, the matrix $\binom{a}{b}_{-1}$, $0 \leq a, b < 2$ is given by

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

This implies that $\binom{a}{b}_{-1} \in \mathbb{N}$ for all $a, b \in \mathbb{N}$.

For $n = 4$ we get $\binom{a}{b}_i \in \mathbb{N} \cup i\mathbb{N} \cup (1+i)\mathbb{N}$ since we have

$$\begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1+i & 1 & \\ 1 & i & i & 1 \end{pmatrix}$$

for the matrix with coefficients $\binom{a}{b}_i$, $0 \leq a, b < 4$.

6.1 Recurrence matrices with finite support and the quotient algebra $\tilde{\mathcal{R}} = \mathcal{R}/\mathcal{FS}$

We call an element $A = A[0], A[1], \dots \in \mathcal{R}$ of *finite support* if there exists a natural integer K such that all matrices $A[K], A[K+1], \dots$ of size $\geq 2^K \times 2^K$ are identically zero. The set \mathcal{FS} of all elements of finite support forms a two-sided ideal in \mathcal{R} (and in \mathcal{A}). We denote by $\tilde{\mathcal{R}}$ the quotient algebra \mathcal{R}/\mathcal{FS} .

Lifting an element $\tilde{A} \in \tilde{\mathcal{R}}$ into an element A such that $\tilde{A} \equiv A \pmod{\mathcal{FS}}$, the vector space

$$\overline{A}/(\overline{A} \cap \mathcal{FS}) \subset \tilde{\mathcal{R}}$$

is well-defined. We call it the stable recursive closure $\overline{\tilde{A}}$ of \tilde{A} . Its dimension is the *stable complexity* of $\tilde{A} \in \tilde{\mathcal{R}}$.

An element \tilde{A} of stable complexity \tilde{a} has a *recursive presentation* of the form

$$\tilde{A}_j = \tilde{A}_j[K], \begin{pmatrix} \tilde{\rho}(0,0)\tilde{A}_j & \tilde{\rho}(0,1)\tilde{A}_j \\ \tilde{\rho}(1,0)\tilde{A}_j & \tilde{\rho}(1,1)\tilde{A}_j \end{pmatrix}$$

(for $\tilde{A}_1, \dots, \tilde{A}_{\tilde{a}}$ a basis of $\overline{\tilde{A}}$ and for $\tilde{\rho}(s,t)$ the obvious shift maps of $\tilde{\mathcal{R}}$) where $\tilde{A}_j[K] \in M_{2^K \times 2^K}$ for some natural integer $K = K(\tilde{A})$ which is large enough.

Computations in the quotient algebra $\tilde{\mathcal{R}}$ can be either done by considering lifts into $\tilde{\mathcal{R}}$ (there is however no natural section $\tilde{\mathcal{R}} \rightarrow \mathcal{R}$ of the quotient algebra $\tilde{\mathcal{R}}$ into \mathcal{R} yielding a homomorphism of algebras) or by working with a modified version of the saturation level: Given $\tilde{A} \in \tilde{\mathcal{R}}$ consider the smallest integer N such that there exists an integer $K \leq N$ associated to a recursive basis $\tilde{A}_1, \dots, \tilde{A}_{\tilde{a}} \subset \tilde{\mathcal{R}}$ of $\overline{\tilde{A}}$ as above and such that the projection $\pi_{K,N} = \pi_K \oplus \dots \oplus \pi_N : \bigoplus_{j=1}^{\tilde{a}} \mathbb{C}\tilde{A}_j \rightarrow \bigoplus_{j=K}^N M_{2^j \times 2^j}$ is injective. (The integer N thus defined can also be characterised by the equality $\mathcal{K}_{K,N} = \mathcal{K}_{K,N+1}$ where $\mathcal{K}_{K,l} \subset \bigoplus_{j=1}^{\tilde{a}} \mathbb{C}\tilde{A}_j$ for $l \geq K$ is the kernel of the projection $\pi_K \oplus \pi_{K+1} \oplus \dots \oplus \pi_l$.)

6.1.1 Tensor products and elements of stable complexity 1 in $\tilde{\mathcal{R}}$

Elements of $\tilde{\mathcal{A}}$ admitting a recursive presentation of the form

$$\tilde{A}_j = \lambda_j X, \begin{pmatrix} \tilde{\rho}(0,0)\tilde{A}_j & \tilde{\rho}(0,1)\tilde{A}_j \\ \tilde{\rho}(1,0)\tilde{A}_j & \tilde{\rho}(1,1)\tilde{A}_j \end{pmatrix}$$

(for $\tilde{A}_1, \dots, \tilde{A}_{\tilde{a}}$ a basis of $\overline{\tilde{A}}$) with $\lambda_j \in \mathbb{C}$ and X a fixed matrix of size $2^K \times 2^K$ not depending on j , can be considered as elements in $\mathcal{R} \otimes X$. In particular, such a matrix has an *LU* decomposition in $\tilde{\mathcal{R}}$ if and only if the correspond element of \mathcal{R} and the matrix $X \in M_{2^K \times 2^K}$ have an *LU* decomposition.

6.2 The Beeblebrox reduction of $\binom{s+t}{s}_{-1}$

We denote by Z' the infinite symmetric matrix with coefficients $Z'_{s,t} = \beta\left(\binom{s+t}{s}_{-1}\right)$ given by the Beeblebrox reduction of q -binomials evaluated at $q = -1$.

Theorem 2.8 and Section 6.1.1 imply that $Z' = L'D'(L')^t$ where

$$Z' = Z \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad L' = L \otimes \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad D' = D \otimes \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

(the tensor product $X \otimes M$ denotes the matrix(-sequence) obtained by replacing a scalar entry λ of Z by the 2×2 matrix M) with Z, L, D as in the proof of Theorem 2.6.

In particular, one can easily establish a formula proving that $\det(Z'(n)) \in \pm 3^{\mathbb{N}}$ with $Z'(n)$ denoting the symmetric $n \times n$ submatrix consisting of the first n rows and columns of Z' .

Remark 6.1. *The case of the matrix (with coefficients in $\{0, 1\}$) obtained by reducing $\binom{s+t}{s}_{-1}$ modulo 2 yields nothing new since $\binom{s+t}{s}_{-1} \equiv \binom{s+t}{s}_1 = \binom{s+t}{s}$ (mod 2).*

6.3 Reduction modulo 2 and Beeblebrox reduction of $\binom{s+t}{s}_i$

Let M' be the symmetric matrix with coefficients

$$\psi\left(\binom{s+t}{s}_i\right) \in \{0, \pm 1, \pm x, \pm y\}, \quad 0 \leq s, t$$

where

$$\psi(\xi) = \begin{cases} \gamma(\xi) & \text{if } \xi \in \mathbb{N} \\ \gamma(a)x & \text{if } \xi = ai \in i\mathbb{N} \\ \gamma(a)y & \text{if } \xi = a(1+i) \in (1+i)\mathbb{N} \end{cases}$$

where $\gamma : \mathbb{N} \rightarrow \{0, \pm 1\}$ is either the reduction modulo 2 with values in $\{0, 1\}$ or the Beeblebrox reduction β .

As in section 6.2 we have $M' = L'D'(L')^t$ where

$$\begin{aligned} M' &= M \otimes \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & y & x & \\ 1 & x & & \\ 1 & & & \end{pmatrix}, \\ L' &= L \otimes \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & \frac{1-x}{1-y} & 1 & & \\ 1 & \frac{1}{1-y} & \frac{y-x}{x^2-2x+y} & 1 & \\ & & & & \end{pmatrix}, \\ D' &= D \otimes \begin{pmatrix} 1 & & & & \\ & y-1 & & & \\ & & \frac{x^2-2x+y}{1-y} & & \\ & & & & \frac{-x^2}{x^2-2x+y} \end{pmatrix}. \end{aligned}$$

M, L, D are given by the matrices P, L, D , respectively Z, L, D occurring in the proof of Theorem 2.1, respectively 2.6, if γ is the reduction modulo 2, respectively the Beeblebrox reduction.

It follows that the determinant $\det(M'(n))$ of the finite matrix $M'(n)$ consisting of the first n rows and columns of M' is of the form

$$\pm 3^{\mathbb{N}} x^{2\mathbb{N}} (y-1)^{\{0,1\}} (x^2 - 2x + y)^{\{0,1\}}$$

with powers of 3 only involved if γ is the Beeblebrox reduction. The factor $(y-1)$ appears if and only if $n \equiv 2 \pmod{4}$ and the factor $(x^2 - 2x + y)$ appears if and only if $n \equiv 3 \pmod{4}$.

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