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The loop-quantum-gravity vertex-amplitude

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Spinfoam theories are hoped to provide the dynamics of non-perturbative loop quantum gravity. But a number of their features remain elusive. The best studied one —the euclidean Barrett-Crane model— does not have the boundary state space needed for this, and there are recent indications that, consequently, it may fail to yield the correct low-energy n-point functions. These difficulties can be traced to the $SO(4) \to SU(2)$ gauge fixing and the way certain second class constraints are imposed, arguably incorrectly, *strongly*. We present an alternative model, that can be derived as a *bona fide* quantization of a Regge discretization of euclidean general relativity, and where the constraints are imposed *weakly*. Its state space is a natural subspace of the $SO(4)$ spin-network space and matches the $SO(3)$ hamiltonian spin network space. The model provides a long sought $SO(4)$-covariant vertex amplitude for loop quantum gravity.

The *kinematics* of loop quantum gravity (LQG) provides a well understood background-independent language for a quantum theory of physical space [1, 2, 3]. The *dynamics* of the theory is not understood as cleanly. Dynamics is studied along two lines: hamiltonian (as in the Schrödinger equation) [4] or covariant (as in Feynman's covariant quantum field theory). We focus on the second. The key object that defines the dynamics in this language is the vertex amplitude, like the vertex $e\gamma^\mu$ that defines the dynamics of perturbative QED. What is the vertex of LQG?

The spinfoam formalism [5] is viewed as a possible tool for answering this question. It can be derived in a remarkable number of distinct ways, which converge to the definition of transition amplitudes as a Feynman sum over spinfoams. A spinfoam is a two-complex (union of faces, edges and vertices) colored with quantum numbers (spins associated to faces and intertwiners associated to edges); it can be loosely interpreted as a history of a spin network (a colored graph). Its amplitude contains the product of the amplitudes of each vertex, and thus the vertices play a role similar to the vertices of Feynman's covariant QFT [6, 7]. This picture is nicely implemented in three dimensions (3d) by the Ponzano-Regge model [8], where the vertex amplitude is given by the 6j Wigner symbol, which can be obtained as a matrix element of the hamiltonian of 3d gravity [9].

Compelling and popular as it is, however, this picture has never been fully implemented in 4d. The best studied model in the 4d euclidean context is the Barrett-Crane (BC) model [10]. This is simple and elegant, has remarkable finiteness properties [11], and can be considered a modification of a topological BF quantum field theory, by means of constraints —called simplicity constraints— whose classical limit yields precisely the constraints that change BF theory into general relativity (GR). Furthermore, in the low-energy limit some of its n-point functions appear to agree with those computed from perturbative quantum GR [12]. However, the suspicion that something is wrong with the BC model has long been agitated. Its boundary state space is similar, but does not exactly match, that of loop quantum gravity; in particular the volume operator is ill-defined. Worse, recent calculations appear to indicate that some n-point functions fail to yield the correct low-energy limit [13]. All these problems are related to the way the *intertwiner* quantum numbers (associated to the operators measuring angles between the faces bounding the elementary quanta of space) are treated: These quantum numbers are fully constrained in the BC model by imposing the simplicity constraints as strong operator equations ($C_n\psi = 0$). But these constraints are second class and imposing such constraints strongly may lead to the incorrect elimination of physical degrees of freedom [14].

It is therefore natural to try to implement in 4d the general picture discussed above by correcting the BC model [7, 15]. In this letter we show that this is possible, by properly imposing some of the constraints weakly ($\langle \phi\, C_n\, \psi\rangle = 0$), and that the resulting theory has remarkable features. First,

its boundary quantum state space matches *exactly* the one of $SO(3)$ loop quantum gravity: no degrees of freedom are lost. Second, as the degrees of freedom missing in BC are recovered, the vertex may yield the correct low-energy n-point functions. Third, the vertex can be seen as a vertex over $SO(3)$ spin networks or $SO(4)$ spin networks, and is both $SO(3)$ and $SO(4)$ covariant. Finally, the theory can be obtained as a bona fide quantization of a discretization of euclidean GR on a Regge triangulation. Here we give the definition of the theory, we illustrate its main aspects and we give only a rapid sketch of its derivation from Regge GR. Details will be given elsewhere.

The model we discuss is defined by a standard spinfoam partition function

$$Z_{\rm GR} = \sum_{j_f, i_e} \prod_f (\dim \tfrac{j_f}{2})^2 \prod_v A(j_f, i_e) \qquad (1)$$

where the amplitude is given by

$$\begin{aligned} A(j_f, i_e) &= 15j_{SO(4)}\big((\tfrac{j_f}{2}, \tfrac{j_f}{2}), f(i_e)\big) \\ &= \sum_{i_e^+, i_e^-} 15j_{SO(4)}\big((\tfrac{j_f}{2}, \tfrac{j_f}{2}), i_e^+, i_e^-\big) \prod_{e\in v} f^{i_e}_{i_e^+ i_e^-} \end{aligned} \qquad (2)$$

Notation is as follows. The model is defined on a fixed 4d triangulation Δ. We do not discuss here the issue of the recovery of triangulation independence (see [2, 10, 16]). We denote by f, e, v respectively the faces, tetrahedra and 4-simplices of Δ. The choice of letters is motivated by the fact that it is convenient to think in terms of the cellular complex dual to Δ (whose 2-skeleton defines the spinfoam): triangles are dual to faces (f), tetrahedra to edges (e), and 4-simplices to vertices (v). The sum in (1) is over an assignment of an *integer* spin j_f (that is, an irreducible representation of $SO(3)$) to each face f, and over an assignment of an element i_e of a basis in the space of intertwiners to each edge e. We recall that an intertwiner is an element of the $SO(3)$ invariant subspace of the tensor product of the four Hilbert spaces carrying the four representations associated to the four f's adjacent to a given e. We use the usual basis given by the spin of the virtual link, under a fixed pairing of the four faces. $\dim j = 2j+1$ is the dimension of the representation j. $15j_{SO(4)}$ is the Wigner 15j symbol of the group $SO(4)$. It is a function of 15 $SO(4)$ irreducible representations. A representation of $SO(4)$ can be written as a pair of representations of $SU(2)$, in the form (j^+, j^-), and the $SO(4)$ 15j symbol is simply the product of two conventional Wigner $SU(2)$ 15j symbols

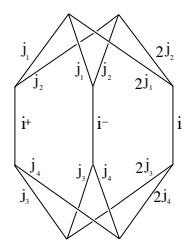
$$15j_{SO(4)}(j_f^+, j_f^-, i_e^+, i_e^-) = 15j(j_f^+, i_e^+)\ 15j(j_f^-, i_e^-). \qquad (3)$$

The last object to define, and the key ingredient of our construction, is the linear map f appearing in the first line of (2). This is a map from the space of the $SO(3)$ intertwiners between the representations $2j_1, ..., 2j_4$, to the space of the $SO(4)$ intertwiners between the representations $(j_1, j_1,), ..., (j_4, j_4)$. The second line of (2) simply reexpresses this map in terms of its linear coefficients in the basis chosen

$$f|i\rangle = \sum_{i^+, i^-} f^i_{i^+ i^-} |i^+, i^-\rangle. \qquad (4)$$

These coefficients are defined as the evaluation of the spin network

$$f^i_{i^+ i^-} = \qquad (5)$$



on the trivial connection. The amplitude can also be written in the form

$$A(j_f, i_e) = \int_{SU(2)^5} dV_e\ \langle \bigotimes_{ee'} \overset{j_f/2}{D}(V_e) \otimes \overset{j_f/2}{D}(V_{e'}^{-1}), \bigotimes_e i_e \rangle \qquad (6)$$

where index contraction is dictated by the standard 4-simplex graph and the j_f indices of the intertwiners are contracted with the $\frac{j_f}{2} \otimes \frac{j_f}{2}$ indices of the representation matrices D. This concludes the definition of the model (for information on the general formalism, and more details on notation see [2]). Let us now comment on its features.

First, the boundary states of the theory are spanned by trivalent graphs colored with $SO(3)$ spins and intertwiners. Second, the model is a simple modification of the BC model as follows. The BC model is given by

$$Z_{\rm BC} = \sum_{j_f} \prod_f (\dim j_f)^2 \prod_v A_{BC}(j_f) \qquad (7)$$

where here the sum is over half-integer spins and the amplitude is given by

$$A_{BC}(j_f) \;=\; 15 j_{SO(4)}\left((j_f, j_f), i_{BC}\right). \qquad (8)$$

The difference between the two theories is therefore in the intertwiner state space. The relevant (unconstrained) intertwiner space is here the $SO(4)$ intertwiner space between four simple representations

$$H_e = \mathrm{Inv}(H_{(j_1,j_1)} \otimes ... \otimes H_{(j_4,j_4)}). \qquad (9)$$

The Barret-Crane intertwiner

$$|i_{BC}\rangle = \sum_j (2j+1)|j,j\rangle \qquad (10)$$

is a vector in this space. The Barrett-Crane theory therefore constrains entirely the intertwiner degrees of freedom. In the model (1), instead, intertwiner degrees of freedom remain free. More precisely, the states (4) span a subspace K_e of H_e. The step from the single intertwiner i_{BC} to the space K_e is therefore the essential modification made with respect to the BC model. Why this step?

The reduction of the intertwiner space to the sole i_{BC} vector is commonly motivated by the imposition of the off-diagonal simplicity constraints. For each couple of faces f, f' adjacent to e, consider the pseudoscalar $SO(4)$ Casimir operator

$$C_{ff'} = \epsilon_{IJKL} B_f^{IJ} B_{f'}^{KL} \qquad (11)$$

on the representation $(H_{(j_f,j_f)} \otimes H_{(j_{f'},j_{f'})})$. ($\epsilon_{IJKL}$ is the fully antisymmetric object and summation over repeated indices is understood.) Here $f \neq f'$ and B_f^{IJ} with $I, J = 1, ..., 4$ are the generators of $SO(4)$ in $H_{(j_f,j_f)}$. In the context of the BC theory, these generators are the quantum operators corresponding to the classical bivector associated to the face f. $C_{ff'}$ vanishes in the classical theory because the bivectors of the faces a single tetrahedron span a 3d space and therefore their external products (11) are clearly zero. These are the off-diagonal simplicity constraints. (The diagonal simplicity constraint $C_{ff} = 0$ constrains the representations associated to each f to be simple.) In BC theory, the constraints $C_{ff'} = 0$ are imposed strongly on H_e, and the only solution of these constraint equations is i_{BC} [17]. But these constraints do not commute with one another, and are therefore second class. Imposing second class constraints strongly is a well-known way of erroneously killing physical degrees of freedom in a theory. An alternative way to rewrite the off-diagonal simplicity constraints is the following. As noted, these constraints impose the faces of the tetrahedron to lie on a common 3d subspace of 4d spacetime. Iff they are satisfied, there is a direction n^I orthogonal to all the faces: the direction normal to the tetrahedron. The B_f have vanishing components in this direction. Choose coordinates in which $n^I = (0,0,0,1)$ and let i, j be indices that run over the first 3 coordinates only. Then we have $2C_4 \equiv B_f^{IJ} B_f^{IJ} = B_f^{ij} B_f^{ij} \equiv C_3$. The off-diagonal simplicity constraints can be written as the requirement that there is a common direction n such that $C = 2C_4 - C_3 = 0$ for all the faces of the tetrahedron. In the quantum context, C_4 is the quadratic Casimir of $SO(4)$, with eigenvalues $j^+(j^+{+}1)\hbar^2 + j^-(j^-{+}1)\hbar^2$; while C_3 is the quadratic Casimir of the $SO(3)$ subgroup of $SO(4)$ that leaves n^I invariant, with eigenvalues $j(j+1)\hbar^2$, where we have momentarily restored $\hbar \neq 1$ units for clarity. Can the constraint $C = 2C_4 - C_3 = 0$ be imposed quantum mechanically on H_e? A simple $SO(4)$ representation (j, j) transforms under the $SO(3)$ subgroup in the representation $j \otimes j = 0 \oplus ... \oplus 2j$. Precisely in the $2j$ component, namely in the highest $SO(3)$ irreducible, this constraint (with suitable ordering:

$$C = \sqrt{C_3 + \frac{\hbar^2}{4}} - \sqrt{2C_4 + \hbar^2} + \frac{\hbar}{2}\ ) \qquad (12)$$

is solved. Thus imposing the constraints on each face selects from $(H_{(j_{f_1},j_{f_1})} \otimes ... \otimes H_{(j_{f_4},j_{f_4})})$ the space formed by the tensor product of the highest $SO(3)$ irreducibles. So far this depends on *which* $SO(3)$ subgroup we have chosen; but if we project to the $SO(4)$ invariant-tensor space, then the dependence drops out because all $SO(3)$ subgroups in $SO(4)$ are conjugate to one another. In fact, what we obtain is precisely K_e. Finally, it is easy to check that the off-diagonal simplicity constraints are all weakly zero in this space: this follows from the fact that they are antisymmetric in the i^+, i^- indices, while the states (4) are symmetric.

We close by sketching the derivation of this

model as a quantization of a discretization of GR (see [18]). Fix an oriented triangulation and restrict the metric to be a Regge metric on this triangulation; that is, a metric which is flat within each 4-simplex, and where curvature is concentrated on the triangles. In order to describe this metric, we choose as variables a co-tetrad one-form $e^I(t)$ for each tetrahedron of the triangulation, and a co-tetrad one-form $e^I(v)$ for each simplex. The two will be related by an $SO(4)$ group element $V_{vt} \equiv V_{tv}^{-1}$. For each face in each tetrahedron, we define $B_f(t) = \int_f \star(e(t) \wedge e(t))$, where the star is Hodge duality in R^4. $B_f(t)$ and $B_f(t')$ are related by $B_f(t)U_{tt'} = U_{tt'}B_f(t')$, where $U_{tt'} = V_{tv}V_{vt''}...V_{v^n t'}$ is the product of the group elements around the oriented link of f, from t to t'. The bulk action can be written as

$$S_{bulk}[e] = \sum_f Tr[B_f(t)U_f(t)] \tag{13}$$

where $U_f(t)$ is the product of the group elements $V_{tv}V_{vt'}$ around the link of f. The boundary terms of the action can be written as

$$S_{boundary}[e] = \sum_f Tr[B_f(t)U_{tt'}] \tag{14}$$

where $U_{tt'}$ is the product of the group elements of the sole part of the link which is in the triangulation. We take $B_f(t)$ and V_{tv} as basic variables, and take into account the constraints on B_f. These are the closure constraint

$$\sum_{f\in t} B_f(t) = 0 \tag{15}$$

and the simplicity constraints (11), for all f, f' (possibly equal) in t. (The constraints relating triangles that meet only at one point, which appear in other formulations, are automatically solved by our choice of variables.)

On the boundary of the triangulation, the boundary coordinates are the $B_f(t)$ for the boundary triangles f. These have only two adjacent tetrahedra t, t' on the boundary. The conjugate momentum (as can be seen from (14)) is a group element for each f. Therefore the canonical boundary variables are precisely the same as those of $SO(4)$ lattice gauge theory. We can thus choose the Hilbert space of $SO(4)$ lattice gauge theory as our unconstrained Hilbert space. This space can be represented as the L^2 space on the product of one $SO(4)$ per triangle. The two B_f variables at each f are represented by the left and right invariant vector fields on the group element at f, which are related to one another in the same manner as the corresponding classical quantities. The closure constraint (15) gives gauge-invariance at each tetrahedron, and reduces the space of states to the space of the $SO(4)$ spin networks on the graph dual to the boundary triangulation. The simplicity constraints (11), as seen above, reduce each $SO(4)$ link representation to a simple one, and the intertwiners spaces to K_e. The resulting space of states is not only mathematical isomorphic to the corresponding one of $SO(3)$ loop quantum gravity, but it can also be physically identified with it, because we have an explicit identification of the quantum operators on the two spaces with the same classical analogues, such as the area of the faces.

Finally, coming to the dynamics, we can evaluate the amplitude of a single 4-simplex v. Fixing the ten $B_{tt'} \equiv B_f(t)$ variables on the boundary, this can be formally written as

$$A[B_{tt'}] = \int dV_{vt}\ e^{i\sum Tr[B_{tt'}V_{tv}V_{vt'}]}. \tag{16}$$

Transforming to the conjugate variables gives

$$\begin{aligned} A[U_{tt'}] &= \int dB_{tt'} e^{-i\sum Tr[B_{tt'}U_{tt'}]}\ A[B_{tt'}] \\ &= \int dV_{vt} \prod_{tt'} \delta(U_{tt'}V_{t'v}V_{vt}). \end{aligned} \tag{17}$$

This is the amplitude. We can now transform back to the spin network basis, using the $SO(4)$ spin network functions $\Psi_{j^\pm_{tt'}, i^\pm_t}(U_{tt'})$

$$\begin{aligned} A[j^\pm_{tt'}, i^\pm_t] &= \int dU_{tt'} \Psi_{j^\pm_{tt'}, i^\pm_t}(U_{tt'})\ A[U_{tt'}] \\ &= \int dV_{vt}\ \Psi_{j^\pm_{tt'}, i^\pm_t}(V_{tv}V_{vt'}) \end{aligned} \tag{18}$$

Performing the integral gives

$$A[j^\pm_{tt'}, i^\pm_t] \ = \ 15 j_{SO(4)}(j^+_{tt'}, j^-_{tt'}, i^+_t, i^-_t). \tag{19}$$

Combining this $15j_{SO(4)}$ amplitude with the constraints discussed above, gives the model (1)-(2).

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