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# Lieb-Thirring inequalities with improved constants

Jean Dolbeault, Ari Laptev, and Michael Loss

## Abstract

Following Eden and Foias we obtain a matrix version of a generalised Sobolev inequality in one-dimension. This allows us to improve on the known estimates of best constants in Lieb-Thirring inequalities for the sum of the negative eigenvalues for multi-dimensional Schrödinger operators.

*Key-words:* Sobolev inequalities; Schrödinger operator; Lieb-Thirring inequalities.  
*MSC (2000):* Primary: 35P15; Secondary: 81Q10

## 1 Introduction

Let  $H$  be a Schrödinger operator in  $L^2(\mathbb{R}^d)$

$$H = -\Delta - V \quad (1)$$

For a real-valued potential  $V$  we consider Lieb-Thirring inequalities for the negative eigenvalues  $\{\lambda_n\}$  of the operator  $H$

$$\sum |\lambda_n|^\gamma \leq L_{d,\gamma} \int_{\mathbb{R}^d} V_+^{d/2+\gamma}(x) dx, \quad (2)$$

where  $V_+ = (|V| + V)/2$  is the positive part of  $V$ .

Eden and Foias have obtained in [3] a version of a one-dimensional generalised Sobolev inequality which gives best known estimates for the constants in the inequality (2) for  $1 \leq \gamma < 3/2$ . The aim of this short article is to extend the method from [3] to a class of matrix-valued potentials. By using ideas from [6] this automatically improves on the known estimates of best constants in (2) for multidimensional Schrödinger operators.

Lieb-Thirring inequalities for matrix-valued potentials for the value  $\gamma = 3/2$  were obtained in [6] and also in [2]. Here we state a result corresponding to  $\gamma = 1$ .

**Theorem 1.** *Let  $V \geq 0$  be a Hermitian  $m \times m$  matrix-function defined on  $\mathbb{R}$  and let  $\lambda_n$  be all negative eigenvalues of the operator (1). Then*

$$\sum |\lambda_n| \leq \frac{2}{3\sqrt{3}} \int_{\mathbb{R}} \text{Tr} [V^{3/2}(x)] dx. \quad (3)$$

**Remark 1.** The constant  $\frac{2}{3\sqrt{3}}$  should be compared with the Lieb-Thirring constant found in [7] for a class of single eigenvalue potentials and with the constant obtained in [5] which is twice as large as the semi-classical one

$$\frac{4}{3\sqrt{3}\pi} < \frac{2}{3\sqrt{3}} < 2 \times \frac{2}{3\pi} = 2 \times \frac{1}{2\pi} \int_{\mathbb{R}} (1 - \xi^2)_+ d\xi.$$

This is about  $0,2450 \dots < 0,3849 \dots < 0,4244 \dots$ .

**Remark 2.** Note that the values of the best constants for the range  $1/2 < \gamma < 3/2$  remain unknown.

Let  $\mathcal{A}(x) = (a_1(x), \dots, a_d(x))$  be a magnetic vector potential with real valued entries  $a_k \in L^2_{\text{loc}}(\mathbb{R}^d)$  and let

$$H(\mathcal{A}) = (i\nabla + \mathcal{A})^2 - V,$$

where  $V \geq 0$  is a real-valued function.

Denote the ratio of  $2/3\sqrt{3}$  and the semi-classical constant by

$$R := \frac{2}{3\sqrt{3}} \times \left(\frac{2}{3\pi}\right)^{-1} = 1.8138 \dots$$

By using the Aizenmann-Lieb argument [1], a ‘‘lifting’’ with respect to dimension [6], [5], and Theorem 1 we obtain the following result:

**Theorem 2.** For any  $\gamma \geq 1$  and any dimension  $d \geq 1$ , the negative eigenvalues of the operator  $H(\mathcal{A})$  satisfy inequalities

$$\sum |\lambda_n|^\gamma \leq L_{d,\gamma} \int_{\mathbb{R}^d} V^{d/2+\gamma}(x) dx,$$

where

$$L_{d,\gamma} \leq R \times L_{d,\gamma}^{\text{cl}} = R \times \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 - |\xi|)_+^\gamma d\xi.$$

**Remark 3.** Theorem 2 allows us to improve on the estimates of best constants in Lieb-Thirring inequalities for Schrödinger operators with complex-valued potentials recently obtained in [4].

## 2 One-dimensional generalised Sobolev inequality for matrices

Let  $\{\phi_n\}_{n=0}^N$  be an ortho-normal system of vector-functions in  $L^2(\mathbb{R}, \mathbb{C}^M)$ ,  $M \in \mathbb{N}$ ,

$$(\phi_n, \phi_m) = (\phi_n, \phi_m)_{L^2(\mathbb{R}, \mathbb{C}^M)} = \sum_{j=1}^M \int_{\mathbb{R}} \phi_n(x, j) \overline{\phi_m(x, j)} dx = \delta_{nm},$$

where  $\delta_{nm}$  is the Kronecker symbol. Let us introduce an  $M \times M$  matrix  $U$  with entries

$$u_{j,k}(x, y) = \sum_{n=0}^N \phi_n(x, j) \overline{\phi_n(y, k)}.$$

Clearly

$$U^*(x, y) = U(y, x). \quad (4)$$

The fact that the functions  $\phi_n$  are orthonormal can be written in a compact form

$$\int_{\mathbb{R}} U(x, y) U(y, z) dy = U(x, z). \quad (5)$$

The latter two properties (4) and (5) prove that  $U(x, y)$  could be considered as the integral kernel of an orthogonal projection  $P$  in  $L^2(\mathbb{R}, \mathbb{C}^M)$  whose image is the subspace of vector-functions spanned by  $\{\phi_n\}_{n=1}^N$ .

**Theorem 3.** *Let us assume that the vector-function  $\phi_n$ ,  $n = 1, 2, \dots, N$ , are from the Sobolev class  $H^1(\mathbb{R}, \mathbb{C}^M)$ . Then*

$$\int_{\mathbb{R}} \text{Tr} [U(x, x)^3] dx \leq \sum_{n=1}^N \sum_{j=1}^M \int_{\mathbb{R}} |\phi'_n(x, j)|^2 dx.$$

*Proof.*

$$\begin{aligned} & \frac{d}{dy} \text{Tr} [U(x, y) U(y, x) U(x, x)] \\ = & \text{Tr} \left[ \left( \frac{d}{dy} U(x, y) \right) U(y, x) U(x, x) \right] + \text{Tr} \left[ U(x, y) \left( \frac{d}{dy} U(y, x) \right) U(x, x) \right] \end{aligned} \quad (6)$$

By integrating (6) and taking absolute values one obtains

$$\begin{aligned} & \frac{1}{2} \text{Tr} [U(x, z) U(z, x) U(x, x)] \\ & \leq \frac{1}{2} \int_{-\infty}^z \left| \text{Tr} \left[ \left( \frac{d}{dy} U(x, y) \right) U(y, x) U(x, x) \right] \right. \\ & \quad \left. + \text{Tr} \left[ U(x, y) \left( \frac{d}{dy} U(y, x) \right) U(x, x) \right] \right| dy \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{2} \operatorname{Tr} \left[ U(x, z) U(z, x) U(x, x) \right] \\ & \leq \frac{1}{2} \int_z^\infty \left| \operatorname{Tr} \left[ \left( \frac{d}{dy} U(x, y) \right) U(y, x) U(x, x) \right] \right. \\ & \quad \left. + \operatorname{Tr} \left[ U(x, y) \left( \frac{d}{dy} U(y, x) \right) U(x, x) \right] \right| dy. \end{aligned}$$

Taking absolute values and adding the two inequalities yields for any  $z \in \mathbb{R}$

$$\begin{aligned} & \left| \operatorname{Tr} \left[ U(x, z) U(z, x) U(x, x) \right] \right| \\ & \leq \frac{1}{2} \int_{\mathbb{R}} \left| \operatorname{Tr} \left[ \left( \frac{d}{dy} U(x, y) \right) U(y, x) U(x, x) \right] \right| dy \\ & \quad + \frac{1}{2} \int_{\mathbb{R}} \left| \operatorname{Tr} \left[ U(x, y) \left( \frac{d}{dy} U(y, x) \right) U(x, x) \right] \right| dy. \quad (7) \end{aligned}$$

Note that we have reproved Agmon's inequality

$$|f(x)|^2 \leq \int_{\mathbb{R}} |f(y) f'(y)| dy$$

for traces of matrices. By using properties of traces, the Cauchy-Schwarz inequality for matrix-functions and also properties (4) and (5), we find that for all  $z \in \mathbb{R}$

$$\begin{aligned} & \left( \int_{\mathbb{R}} \left| \operatorname{Tr} \left[ \left( \frac{d}{dy} U(x, y) \right) U(y, x) U(x, x) \right] \right| dy \right)^2 \\ & \leq \int_{\mathbb{R}} \operatorname{Tr} \left[ \frac{d}{dy} U(x, y)^* \frac{d}{dy} U(x, y) \right] dy \int_{\mathbb{R}} \operatorname{Tr} \left[ U(x, y)^* U^2(x, x) U(x, y) \right] dy \\ & = \int_{\mathbb{R}} \operatorname{Tr} \left[ \frac{d}{dy} U(y, x) \frac{d}{dy} U(x, y) \right] dy \int_{\mathbb{R}} \operatorname{Tr} \left[ U^2(x, x) U(x, y) U(y, x) \right] dy \\ & = \int_{\mathbb{R}} \operatorname{Tr} \left[ \frac{d}{dy} U(x, y) \frac{d}{dy} U(y, x) \right] dy \operatorname{Tr} \left[ U(x, x)^3 \right], \end{aligned}$$

and similarly

$$\begin{aligned} & \left( \int_{\mathbb{R}} \left| \operatorname{Tr} \left[ U(x, y) \frac{d}{dy} U(y, x) U(x, x) \right] \right| dy \right)^2 \\ & \leq \int_{\mathbb{R}} \operatorname{Tr} \left[ \frac{d}{dy} U(x, y) \frac{d}{dy} U(y, x) \right] dy \operatorname{Tr} \left[ U(x, x)^3 \right]. \end{aligned}$$

Thus, using this, and setting  $x = z$  in (7), we arrive at

$$\left| \operatorname{Tr} \left[ U(x, x)^3 \right] \right| \leq \int_{\mathbb{R}} \operatorname{Tr} \left[ \frac{d}{dy} U(x, y) \frac{d}{dy} U(y, x) \right] dy.$$

Integrating with respect to  $x$  we finally obtain

$$\begin{aligned} \int_{\mathbb{R}} \left| \operatorname{Tr} \left[ U(x, x)^3 \right] \right| dx \\ \leq \sum_{n,k=1}^N \sum_{i,j=1}^M \int_{\mathbb{R}} \int_{\mathbb{R}} \phi_n(x, i) \overline{\phi_n(y, j)} \phi'_k(y, j) \overline{\phi_k(x, i)} dx dy \\ = \sum_{n=1}^N \sum_{j=1}^M \int_{\mathbb{R}} |\phi'_n(x, j)|^2 dx, \end{aligned}$$

which completes the proof.  $\square$

### 3 Lieb-Thirring inequalities for Schrödinger operators with matrix-valued potentials

Let us assume that  $V \in C_0^\infty(\mathbb{R})$ ,  $V \geq 0$ , be a  $M \times M$  Hermitian matrix-valued potential with entries  $\{v_{ij}\}_{i,j=1}^M$ . Then the negative spectrum of the Schrödinger operator  $H = -\frac{d^2}{dx^2} - V$  in  $L^2(\mathbb{R})$  is finite.

Denote by  $\{\phi_n\}$  the ortho-normal system of eigen-vector functions corresponding to the eigenvalues  $\{\lambda_n\}_{n=1}^N$

$$-\frac{d^2}{dx^2} \phi_n - V \phi_n = \lambda_n \phi_n.$$

Clearly,

$$\sum_n \lambda_n = \sum_{n,j} \int_{\mathbb{R}} |\phi'_n(x, j)|^2 dx - \operatorname{Tr} \left[ \int_{\mathbb{R}} V(x) U(x, x) dx \right]$$

and by Hölder's inequality for traces,

$$\int_{\mathbb{R}} \operatorname{Tr} [V(x) U(x, x)] dx \leq \left( \int_{\mathbb{R}} \operatorname{Tr} [V^{3/2}(x)] dx \right)^{\frac{2}{3}} \left( \int_{\mathbb{R}} \operatorname{Tr} [U(x, x)^3] dx \right)^{\frac{1}{3}},$$

so that using Theorem 3

$$\sum_n \lambda_n \geq X - \left( \int_{\mathbb{R}} \operatorname{Tr} [V^{3/2}(x)] dx \right)^{\frac{2}{3}} X^{\frac{1}{3}}$$

with  $X := \int_{\mathbb{R}} \text{Tr} [U(x, x)^3] dx$ . Minimising the right hand side with respect to  $X$  we finally complete the proof of Theorem 1

$$\sum_n \lambda_n \geq -\frac{2}{3\sqrt{3}} \int_{\mathbb{R}} \text{Tr} [V^{3/2}(x)] dx .$$

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