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Self-similar branching Markov chains

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Abstract

The main purpose of this work is to study self-similar branching Markov chains. First we will construct such a process. Then we will establish certain Limit Theorems using the theory of self-similar Markov processes.

Key Words. Branching process, Self-similar Markov process, Tree of generations, Limit Theorems.

A.M.S. Classification. 60J80, 60G18, 60F25, 60J27.

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1 Introduction.

This work is a contribution to the study of a special type of branching Markov chains. We will construct a continuous time branching chain X which has a self-similar property and which takes its values in the space of finite point measures. This type of process is a generalisation of a self-similar fragmentation (see [4]), which applies to cases where the size modelises non additive quantities as e.g. surface energy in aerosols. We will focus on the case where the index of self-similarity α is non-negative, which means that the bigger individuals will reproduce faster than the smaller ones. There is of course no loss of generality by considering this model, as $1/X$ is then a self-similar process where the index of self-similarity equal to $-\alpha$ (which mean that the smaller individuals will reproduce faster than the bigger ones.)

We extend the method used in [4] to deal with more general processes where we allow an individual to have a mass bigger than that of its parent. First we will recall some background on trees and more specifically on marked trees. Then, we will construct such a process, indexed by generations (it is simply a random mark on the tree of generation, see Section 2). Thanks to a martingale which is associated to the latter and the theory of random stopping lines on a tree of generation, we will define the process indexed by time. After having constructed the process, we will study the evolution of the randomly chosen branch of the chain, from which we shall deduce some Limit Theorems, relying on the theory of self-similar Markov processes.

2 The marked tree.

Let $\alpha \geq 0$ be an index of self-similarity and ν be some **probability measure** on

$$\mathcal{M}_p(\mathbb{R}_+^*) := \left\{ m_s = \sum_{i=1}^n \delta_{s_i} \text{ with } n \in \mathbb{N} \text{ and } s_i > 0 \text{ for all } 1 \leq i \leq n \right\},$$

the space of finite point measures on \mathbb{R}_+^* . We also define for $f : \mathbb{R}_+^* \rightarrow \mathbb{R}$ measurable function and $m_s \in \mathcal{M}_p(\mathbb{R}_+^*)$

$$\langle f, m_s \rangle := \sum f(s_i),$$

by taking the sum over the atoms of m_s repeated according to their multiplicity and we will sometimes use the slight abuse of notation

$$\langle f(x), m_s \rangle := \sum f(s_i)$$

when f is defined as a function depending on the variable x . We endow the space $\mathcal{M}_p(\mathbb{R}_+^*)$ with the topology of weak convergence, which means that m_n converge to m if and only if $\langle f, m_n \rangle$ converge to $\langle f, m \rangle$ for all continuous bounded functions f .

The aim of this work is to construct a branching Markov chain $X = ((\sum \delta_{X_i(t)})_{t \geq 0})$ with values in $\mathcal{M}_p(\mathbb{R}_+^*)$, which is self-similar with index α and has reproduction law ν . The index of self-similarity will play a part in the rate in which an individual will reproduce and the reproduction law ν will specify the distribution of the offspring. We stress that our setting includes the case when

$$\nu(s_1 > 1) > 0, \tag{1}$$

which means that with a positive probability the size of a child can exceed that of his mother.

In order to do that, exactly as described in Chapter 1 section 1.2.1 of [4], we will construct a marked tree.

First we will introduce some notions. We consider the infinite tree

$$\mathcal{U} := \cup_{n=0}^{\infty} \mathbb{N}^n,$$

with the notation $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}^0 = \{\emptyset\}$. In the sequel the elements of \mathcal{U} are called nodes (or sometimes also individuals) and the distinguished node \emptyset the root. For each $u = (u_1, \dots, u_n) \in \mathcal{U}$, we call n the *generation* of u and write $g(u) = n$, with the obvious convention $g(\emptyset) = 0$. When $n \geq 0$ $u = (u_1, \dots, u_n) \in \mathbb{N}^n$ and $i \in \mathbb{N}$, we write $ui = (u_1, \dots, u_n, i) \in \mathbb{N}^{n+1}$ for the i -th child of u . We also define for $u = (u_1, \dots, u_n)$ with $n \geq 2$,

$$mu = (u_1, \dots, u_{n-1})$$

the *mother* of u , $mu = \emptyset$ if $u \in \mathbb{N}$. If $v = m^n u$ for some $n \geq 0$ we write $v \prec u$ and say that u *stems* from v . Additionally for M a set of \mathcal{U} , $M \prec v$ means that $u \prec v$ for some $u \in M$. Generally we write $M \prec L$ if for all $x \in L$ stem from M .

Here it will be convenient to identify the finite sequence $s = (s_1, \dots, s_n)$ of positive real numbers with the infinite sequence $(s_1, \dots, s_n, 0, \dots)$ obtained by aggregation of infinitely many 0's.

In particular we say that an infinite sequence $(\xi_i, i \in \mathbb{N})$ has the law ν , if there is a (random) index n such that $\xi_i = 0 \Leftrightarrow i > n$ and the finite point measure $\sum_{i=1}^n \delta_{\xi_i}$ has the law ν .

Definition 1. *Let two independent families of i.i.d. variables be indexed by the nodes of the tree, $(\tilde{\xi}_u, u \in \mathcal{U})$ and $(\mathbf{e}_u, u \in \mathcal{U})$, where for each $u \in \mathcal{U}$ $(\tilde{\xi}_{ui})_{i \in \mathbb{N}}$ is distributed according to the law ν , and $(\mathbf{e}_{ui})_{i \in \mathbb{N}}$ is a sequence of i.i.d. exponential variables with parameter 1. We define for some fixed $x > 0$*

$$\xi_\emptyset := x, \quad a_\emptyset := 0, \quad \zeta_\emptyset := x^{-\alpha} \mathbf{e}_\emptyset,$$

and for $u \in \mathcal{U}$ and $i \in \mathbb{N}$:

$$\xi_{ui} := \tilde{\xi}_{ui} \xi_u, \quad a_{ui} := a_u + \zeta_u, \quad \zeta_{ui} := \xi_{ui}^{-\alpha} \mathbf{e}_{ui}.$$

To each node u of the tree \mathcal{U} , we associate the mark (ξ_u, a_u, ζ_u) where ξ_u is the size, a_u the birth-time and ζ_u the lifetime of the individual with label u . We call

$$T_x = ((\xi_u, a_u, \zeta_u)_{u \in \mathcal{U}})$$

a marked tree with root of size x , and the law associated is denoted by \mathbb{P}_x . Let $\bar{\Omega}$ be the set of all the possible marked trees.

The size of the individuals $(\xi_u, u \in \mathcal{U})$ define a so-called multiplicative cascade (see the references in Section 3 of [5]). But the latter would not be enough to construct the process X , in fact we also need the information given by $((a_u, \zeta_u), u \in \mathcal{U})$.

Another useful concept is that of *stopping line*, or, for short, *line*. $L \subset \mathcal{U}$ is a (stopping) line if $u, v \in L$, $u \prec v \Rightarrow u = v$. The *pre-L-sigma algebra* is

$$\mathcal{H}_L := \sigma(\tilde{\xi}_u, \mathbf{e}_u; \exists l \in L : u \prec l).$$

A random set of individuals

$$\mathcal{J} : \bar{\Omega} \rightarrow \mathcal{P}(\mathcal{U})$$

is *optional* if $\{\mathcal{J} \prec L\} \in \mathcal{H}_L$ for all $L \subset \mathcal{U}$, where $\mathcal{P}(\mathcal{U})$ is the power set of \mathcal{U} . An *optional line* is a random stopping line which is optional. For any optional set \mathcal{J} we define the pre- \mathcal{J} -algebra by:

$$A \in \mathcal{H}_{\mathcal{J}} \Leftrightarrow \forall L \text{ line } \subset \mathcal{U} : A \cap \{\mathcal{J} \prec L\} \in \mathcal{H}_L.$$

See also the article of Chauvin [9] for additional information about this concept.

Lemma 1. *The marked tree constructed in Definition 1 verifies the strong Markov branching property: for \mathcal{J} an optional stopping line and $\varphi_u : \bar{\Omega} \rightarrow [0, 1]$, $u \in \mathcal{U}$, measurable functions, we get that,*

$$\mathbb{E}_1 \left(\prod_{u \in \mathcal{J}} \varphi_u \circ T^{(\xi_u)} \middle| \mathcal{H}_{\mathcal{J}} \right) = \prod_{u \in \mathcal{J}} \mathbb{E}_{\xi_u}(\varphi_u),$$

where $T^{(\xi_u)}$ is the marked tree extracted from T_1 at the node (ξ_u, a_u, ζ_u) . More precisely

$$T^{(\xi_u)} = ((\xi_{uv}, a_{uv} - a_u, \zeta_{uv})_{v \in \mathcal{U}}).$$

Proof. Thanks to the i.i.d properties of the random variables $(\tilde{\xi}_u, u \in \mathcal{U})$ and $(\mathbf{e}_u, u \in \mathcal{U})$, the Markov property for stopping lines is of course easily checked. In order to get the result for a more general optional stopping line, we use Theorem 4.14 of [11]. Indeed, the tree we have constructed is a special case of the tree constructed by Jagers in [11]. In our case the type ρ_u of $u \in \mathcal{U}$, is the mass of u : ξ_u, τ_u is here equal to ζ_u (because here the mother dies when she gives birth to her children whose birth times are all the same), and the birth time σ_u is a_u . \square

3 Malthusian hypotheses and the intrinsic martingale.

We start by introducing some notation:

$$\underline{p} := \inf \left\{ p \in \mathbb{R} : \int_{\mathcal{M}_p(\mathbb{R}_+^*)} \langle x^p, m_s \rangle \nu(dm_s) < \infty \right\},$$

and

$$\bar{p} := \inf \left\{ p > \underline{p} : \int_{\mathcal{M}_p(\mathbb{R}_+^*)} \langle x^p, m_s \rangle \nu(dm_s) = \infty \right\}$$

(with the convention $\inf \emptyset = \infty$) and then for every $p \in]\underline{p}, \bar{p}[$:

$$\kappa(p) := \int_{\mathcal{M}_p(\mathbb{R}_+^*)} (1 - \langle x^p, m_s \rangle) \nu(dm_s).$$

Note that κ is a continuous and concave function (but not necessarily a strictly increasing function) on $]\underline{p}, \bar{p}[$, as $p \rightarrow \int_{\mathcal{M}_p(\mathbb{R}_+^*)} \langle x^p, m_s \rangle \nu(dm_s)$ is a convex application. By concavity, the equation $\kappa(p) = 0$ has at most two solutions on $]\underline{p}, \bar{p}[$. When a solution exists, we denote by $p_0 := \inf\{p \in]\underline{p}, \bar{p}[: \kappa(p) = 0\}$ the smallest, and call p_0 the Malthusian exponent.

We now make the fundamental:

Malthusian Hypotheses. *We suppose that the Malthusian exponent p_0 exists, that $p_0 > 0$, and that*

$$\kappa(p) > 0 \text{ for some } p > p_0.$$

Furthermore we suppose that the integral

$$\int_{\mathcal{M}_p(\mathbb{R}_+^*)} (\langle x^{p_0}, m_s \rangle)^p \nu(dm_s)$$

is finite for some $p > 1$.

Throughout the rest of this article, these hypotheses will always be taken for granted.

In this article we will call *extinction* the event that for some $n \in \mathbb{N}$, all nodes u at the n -th generation have zero size, and *non-extinction* the complementary event. We see that the probability of extinction is always strictly positive whenever $\nu(s_1 = 0) > 0$, and equals zero if and only if $\nu(s_1 = 0) = 0$.

Theorem 1. *The process*

$$M_n := \sum_{g(u)=n} \xi_u^{p_0}, \quad n \in \mathbb{N}$$

is a martingale which is bounded in $L^p(\mathbb{P})$ for some $p > 1$, and in particular is uniformly integrable.

Moreover, conditionally on non-extinction the terminal value M_∞ is strictly positive a.s.

Remark 1. • *As κ is concave the equation $\kappa(p) = 0$ may have a second root $p_+ := \inf\{p > p_0, \kappa(p) = 0\}$. This second root will not interest us: even though*

$$M_n^+ := \sum_{g(u)=n} \xi_u^{p_+}, \quad n \in \mathbb{N},$$

is also a martingale, it is easy to check that for all $p > 1$ the p -variation of M_n^+ is infinite.

We can notice that for all $p \in]p_0, p_+[$ $(M_n^{(p)})_{n \in \mathbb{N}} := (\sum_{g(u)=n} \xi_u^p)_{n \in \mathbb{N}}$ is a supermartingale.

• *Using the branching property we get the identity in law*

$$M_\infty \stackrel{(d)}{=} \sum_{j=1}^{\infty} \xi_j^{p_0} M_\infty^{(j)}$$

where $\xi = (\xi_j, j \in \mathbb{N})$ follows the law $\nu(\cdot)$, and $M_\infty^{(j)}$ are independent copies of M_∞ , also independent of ξ .

Proof. In the first point we will use the article of Jagers [11] to show the uniform integrability of $M(t)$. In the second point we will use the fact that the empirical measure of the logarithm of the sizes of fragments

$$Z^{(n)} := \sum_{g(u)=n} \delta_{\log \xi_u} \quad (2)$$

can be viewed as a branching random walk (see the article of Biggins [7]). In the last point we will show the last part of the theorem.

• The aim of the first point is to show that the condition of Malthusian population is fulfilled in order to apply Theorem 5.1 of [11]. We first introduce some notation. For $r \in \mathbb{R}_+$, let ϑ_r be the structural probability:

$$\vartheta_r(B) := \mathbb{E}_r(\#\{u \in \mathcal{U} : \xi_u \in B\}) \quad \text{for } B \subset \mathcal{B},$$

where \mathcal{B} is the Borel algebra on \mathbb{R}_+ . Let the reproduction measure μ be such that for every $r \geq 0$:

$$\mu(r, du \times ds) := r^\alpha e^{-r^\alpha u} du \vartheta_r(ds)$$

and for any $\lambda \in \mathbb{R}$

$$\mu_\lambda(r, du \times ds) := e^{-\lambda s} \mu(r, du \times ds).$$

The composition operation $*$ denotes the Markov transition on the size space \mathbb{R}_+ and convolution on the time space \mathbb{R}_+ , so that: for all $A \in \mathcal{B}$ and $B \in \mathcal{B}$,

$$\mu^{*2}(s, A \times B) = \mu * \mu(s, A \times B) = \int_{\mathbb{R}_+ \times \mathbb{R}_+} \mu(r, A \times (B - u)) \mu(s, dr \times du).$$

With the convention that the $*$ -power 0 is $\mathbb{1}_{\{A \times B\}}(s, 0)$ which gives all the mass to $(s, 0)$. We define the renewal measure as

$$\psi_\lambda := \sum_0^\infty \mu_\lambda^{*n}.$$

Let

$$\alpha' := \inf\{\lambda : \psi_\lambda(s, \mathbb{R}_+ \times \mathbb{R}_+) < \infty \text{ for some } s \in \mathbb{R}_+\}.$$

Moreover as

$$\mu_\lambda(r, \mathbb{R}_+ \times \mathbb{R}_+) = \begin{cases} r^\alpha / (r^\alpha + \lambda) & \text{if } \lambda > -r^\alpha \\ \infty & \text{else,} \end{cases}$$

thus we get $\alpha' = 0$. For $A \in \mathcal{B}$, let

$$\pi(A) := \lim_{n \rightarrow \infty} \mu^{*n}(1, A \times \mathbb{R}_+)$$

which is well defined as $\mu^{*n}(1, A \times \mathbb{R}_+)$ is a decreasing function in n and nonnegative. Let $h(s) := s^{\beta_0}$ for all $s \in \mathbb{R}_+$ and $\beta := 1$. These objects correspond to those defined in [11]. Thus

by Theorem 5.1 in [11] applied for $L = \{u \in \mathcal{U} : g(u) = n\}$ and $M = \{u \in \mathcal{U} : g(u) = m\}$ with $m \geq n \geq 0$, we get that M_n is a martingale.

Let $\bar{\xi} := \int_{\mathbb{R}_+ \times \mathbb{R}_+} h(s)r^\alpha e^{-tr^\alpha} dt \vartheta_1(ds) = \sum_{g(u)=1} \xi_u^{p_0}$ and \mathbb{E}_π be the expectation with respect to $\int_{\mathbb{R}_+} \mathbb{P}_s(dw)\pi(ds)$. Therefore,

$$\mathbb{E}_\pi(\bar{\xi} \log^+ \bar{\xi}) = \int_{\mathbb{R}_+} \mathbb{E}_s \left(\sum \xi_i^{p_0} \left(\log^+ \sum \xi_j^{p_0} \right) \right) \pi(ds),$$

and it follows readily from the Malthusian hypotheses and the fact that $\sum_{g(u)=n} \xi_u^{pp_0}$ is a supermartingale, that this quantity is finite. Therefore by Theorem 6.1 [11] we get the uniform \mathbb{P}_s -integrability of the martingale.

• We will now show that we actually have the stronger convergence by using Theorem 1 of [7]. In order to do that we first introduce some notation: recall (2) and for $\theta > \underline{p}$, we define

$$m(\theta) := \mathbb{E} \left(\int e^{\theta x} Z^{(1)}(dx) \right) = \mathbb{E} \left(\sum_{g(u)=1} \xi_u^\theta \right) = 1 - \kappa(\theta)$$

and

$$W^{(n)}(\theta) := m(\theta)^{-n} \int e^{\theta x} Z^{(n)}(dx) = (1 - \kappa(p))^{-n} \sum_{g(u)=n} \xi_u^\theta.$$

We notice that $M_n = W^{(n)}(p_0)$. Therefore in order to apply Theorem 1 of [7] and to get the convergence almost surely and in p th mean for some $p > 1$, it is enough to show that

$$\mathbb{E}(W^{(1)}(p_0)^\gamma) < \infty$$

for some $\gamma \in (1, 2]$ and

$$m(pp_0)/|m(p_0)|^p < 1$$

for some $p \in (1, \gamma]$. The first condition is a consequence of the Malthusian assumption. Moreover the second follows from the identities

$$m(pp_0)/|m(p_0)|^p = (1 - \kappa(pp_0))/|1 - \kappa(p_0)|^p = 1 - \kappa(pp_0)$$

which, by the definition of p_0 , is smaller than 1 for $p > 1$ well chosen. completes the proof of the first part of the statement.

• Finally, let us now check that $M_\infty > 0$ a.s. conditionally on non-extinction. Define $q = \mathbb{P}(M_\infty = 0)$, therefore as $\mathbb{E}(M_\infty) = 1$ we get that $q < 1$. Moreover, an application of the branching property yields

$$\mathbb{E}(q^{Z_n}) = q,$$

where Z_n is the number of individuals with positive size at the n -th generation. By the construction of the marked tree and as ν is a probability measure: $(Z_n, n \in \mathbb{N})$ is of course a Galton-Watson process and it follows that q is its probability of extinction. Since $M_\infty = 0$ conditionally on the extinction, the two events coincide a.s. \square

4 Evolution of the process in continuous time.

We define for $u \in \mathbb{R}_+$:

$$f(u) := \int_{\mathcal{M}_p(\mathbb{R}_+^*)} u^{\sharp m_s} \nu(dm_s)$$

where $\sharp m_s$ stands for the numbers of atoms of the point measure m_s . We notice that $f(u)$ is the generating function of the Galton-Watson process $(Z_n, n \geq 0) = (\sharp\{u \in \mathcal{U} : \xi_u > 0 \text{ and } g(u) = n\}, n \geq 0)$.

From now on, we will suppose that for every $\epsilon > 0$

$$\int_{1-\epsilon}^1 \frac{du}{f(u) - u} = \infty. \quad (3)$$

When an individual labelled by u has a positive size, $\xi_u > 0$, let $I_u := [a_u, a_u + \zeta_u)$ be the interval of times during which this individual labelled by u is alive. Otherwise we decide that $I_u = \emptyset$ when $\xi_u = 0$.

Definition 2. We define the process $X = (X(t), t \geq 0)$ with

$$X(t) = \sum_{u \in \mathcal{U}} \mathbb{1}_{\{t \in I_u\}} \delta_{\xi_u}, t \geq 0. \quad (4)$$

In particular we denote for $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ measurable function

$$\langle f, X(t) \rangle = \sum_{u \in \mathcal{U}} f(\xi_u) \mathbb{1}_{\{t \in I_u\}}.$$

For every $x \in \mathbb{R}_+^*$ let \mathbb{P}_x be the law of the process X starting from a single individual with size x . And for simplification, we denote \mathbb{P} for \mathbb{P}_1 , and let $(\mathcal{F}_t)_{t \geq 0}$ be the natural filtration of the process $(X(t), t \geq 0)$.

Theorem 2. The process X takes its values in the set $\mathcal{M}_p(\mathbb{R}_+^*)$, is a Markov chain, and more precisely, enjoys the branching property: for every $m_s = \sum \delta_{s_i} \in \mathcal{M}_p(\mathbb{R}_+^*)$ and every $t \geq 0$, the distribution of $X(t)$ given that $X(0) = m_s$ is the same as that of the sum $\sum X^{(i)}(t)$ of an independent random sequence where for each index i , $X^{(i)}(t)$ is distributed as $X(t)$ under \mathbb{P}_{s_i} .

The process X also has the scaling property, namely for every $c > 0$, the distribution of the rescaled process $(cX(c^\alpha t), t \geq 0)$ under \mathbb{P}_1 is \mathbb{P}_c .

For every measurable function $g : \mathbb{R}_+ \rightarrow (0, \infty)$, define a multiplicative functional such that for every $m_s = \sum \delta_{s_i} \in \mathcal{M}_p(\mathbb{R}_+^*)$:

$$\exp(-\langle g, m_s \rangle) = \exp(-\sum g(s_i)).$$

Then the generator G of the Markov process $X(t)$ fulfills for every $m_y = \sum \delta_{y_i} \in \mathcal{M}_p(\mathbb{R}_+^*)$:

$$G e^{-\langle g, m_y \rangle} = \sum y_i^\alpha e^{-\sum_{j \neq i} g(y_j)} \int_{\mathcal{M}_p(\mathbb{R}_+^*)} (e^{-\langle g(xy_i), m_s \rangle} - e^{-g(y_i)}) \nu(dm_s). \quad (5)$$

Proof. • First we will check that for all $t \geq 0$, $X(t)$ is a (random) finite point measure. By Theorem 1 and the Doob's L^p -inequality we get that for some $p > 1$:

$$\sup_{n \in \mathbb{N}} \sum_{g(u)=n} \xi_u^{p_0} \in L^p(\mathbb{P}).$$

As a consequence:

$$\sup_{u \in \mathcal{U}} \xi_u^{p_0} \in L^p(\mathbb{P})$$

and then by the definition of the process X , writing $X_1(t), \dots$ for the (possibly infinite) sequence of atoms of $X(t)$

$$\sup_i \sup_{t \in \mathbb{R}_+} X_i(t)^{p_0} \in L^p(\mathbb{P}).$$

Recall that $p_0 > 0$ by assumption. We fix some arbitrarily large $m > 0$. We now work conditionally on the event that the size of all individuals is bounded by m , and we will show that the number of the individuals alive at time t is finite for all $t \geq 0$.

As we are conditioning on the event $\{\sup_{u \in \mathcal{U}} \xi_u \leq m\}$, by the construction of the marked tree, we get that the life time of an individual can be stochastically bounded from below by an exponential variable of parameter m^α . Therefore we can bound the number of individuals present at time t by the number of individuals of a continuous time branching process denoted by GW in which each individual lives for a random time whose law is exponential of parameter m^α and the probability distribution of the offspring is the law of $\sharp s \vee 1$ under ν (we have taken the supremum with 1 in order to be sure that there is never any death). For the Markov branching process GW , we are in the temporally homogeneous case and, we notice that

$$\int_{\mathcal{M}_p(\mathbb{R}_+^*)} u^{(\sharp m_s) \vee 1} \nu(dm_s) = (f(u) - u)\nu(\sharp m_s \neq 0) + u,$$

therefore as we have supposed (3), we can use Theorem 1 p.105 of the book of Athreya and Ney [3] (proved in Theorem 9 p.107 of the book of Harris [10]) and get that we are in the non-explosive case for the GW . As the number of the individuals is bounded by that of GW we get that the number of individuals at time t is finite.

Therefore conditioning on the event $\{\sup_{u \in \mathcal{U}} \xi_u \leq m\}$, we have that for all $t \geq 0$, the number of individuals at time t is finite, i.e. $X(t)$ is a finite point measure.

• Second we will show the Markov property. Fix $s \in \mathbb{R}_+$. Let τ_s be equal to $\{u \in \mathcal{U} : s \in I_u\}$. We notice that τ_s is an optional line. In fact for all lines $L \subset \mathcal{U}$ we have that

$$\{\tau_s \prec L\} = \{s < a_u + \zeta_u \quad \forall u \in L\} \in \mathcal{H}_L.$$

By the definition of X , we have that the point measure

$$\sum \mathbb{1}_{\{X_j(t+s) > 0\}} \delta_{X_j(t+s)}$$

is equal to

$$\sum_{u \in \mathcal{U}} \mathbb{1}_{\{t+s \in I_u\}} \delta_{\xi_u}.$$

Let $X(s) = \sum_{i=1}^n \delta_{x_i} \in \mathcal{M}_p(\mathbb{R}_+^*)$ and consider the nodes $(\xi_{v_1}, \xi_{v_2}, \dots, \xi_{v_n})$ such that $X(s) = \sum_{i=1}^n \delta_{\xi_{v_i}}$. Define for all $i \leq n$

$$\tilde{T}_i := ((\xi_{v_i u}, a_{v_i u} - a_{v_i}, \zeta_{v_i u} - \mathbb{1}_{\{u=\emptyset\}}(s - a_{v_i}))_{u \in \mathcal{U}}) = ((\tilde{\xi}_u, \tilde{a}_u, \tilde{\zeta}_u)_{u \in \mathcal{U}}),$$

$\tilde{I}_u := [\tilde{a}_u, \tilde{a}_u + \tilde{\zeta}_u[$ and

$$X^{(i)}(t) = \sum_{u \in \mathcal{U}} \mathbb{1}_{\{t \in \tilde{I}_u\}} \delta_{\tilde{\xi}_u}.$$

Then

$$X(t+s) = \sum_{i=1}^n X^{(i)}(t).$$

By the lack of memory of the exponential variable, we have that for $u \in \mathcal{U}$, given $s \in I_u$ the law of the marked tree \tilde{T}_i is the same as that of

$$T^{(\xi_{v_i})} := ((\xi_{v_i u}, a_{v_i u} - a_{v_i}, \zeta_{v_i u})_{u \in \mathcal{U}}) = ((\xi_u^i, a_u^i, \zeta_u^i)_{u \in \mathcal{U}}).$$

Thus we have the equality in law:

$$\sum_{u \in \mathcal{U}} \mathbb{1}_{\{t \in \tilde{I}_u\}} \delta_{\tilde{\xi}_u} \stackrel{d}{=} \sum_{u \in \mathcal{U}} \mathbb{1}_{\{t \in I_u^i\}} \delta_{\xi_u^i},$$

with $I_u^i := [a_u^i, a_u^i + \zeta_u^i[$.

Let $\tau_s^i := \{v_i u \in \mathcal{U} : s \in I_u^i\}$. Moreover for all lines $L \in \mathcal{U}$ we have that

$$\{\tau_s^i \prec L\} = \{s < a_{v_i u} + \zeta_{v_i u} \quad \forall v_i u \in L\} \in \mathcal{H}_L.$$

Therefore τ_s^i is an optional line and by applying Lemma 1 for the optional stopping line τ_s^i , we have that the point measure

$$\sum_{u \in \mathcal{U}} \mathbb{1}_{\{t+s \in I_u^i\}} \delta_{\xi_u^i}$$

conditionally on $\mathcal{H}_{\tau_s^i}$ is distributed as $X(t)$ under \mathbb{P}_{x_i} . Therefore $(X^{(1)}, X^{(2)}, \dots, X^{(n)})$ is a sequence of independent random processes, where for each i $X^{(i)}(t)$ is distributed as $X(t)$ under \mathbb{P}_{x_i} . We then have proven the Markovian property and the branching property.

- The scaling property is an easily consequence of the definition of the tree T_x . \square

The intrinsic martingale M_n is indexed by the generations; it will also be convenient to consider its analogue in continuous time, i.e

$$M(t) := \langle x^{p_0}, X(t) \rangle = \sum_{u \in \mathcal{U}} \mathbb{1}_{\{t \in I_u\}} \xi_u^{p_0}.$$

It is straightforward to check that $(M(t), t \geq 0)$ is again a martingale in the natural filtration $(\mathcal{F}_t)_{t \geq 0}$ of the process $(X(t), t \geq 0)$; and more precisely, the argument Proposition 1.5 in [4] gives:

Corollary 3. *The process $(M(t), t \geq 0)$ is a martingale, and more precisely*

$$M(t) = \mathbb{E}(M_\infty | \mathcal{F}_t),$$

where M_∞ is the terminal value of the intrinsic martingale $(M_n, n \in \mathbb{N})$. In particular $M(t)$ converges in $L^p(\mathbb{P})$ to M_∞ for some $p > 1$.

Proof. We know that M_n converges in $L^p(\mathbb{P})$ to M_∞ as n tends to ∞ , so

$$\mathbb{E}(M_\infty | \mathcal{F}_t) = \lim_{n \rightarrow \infty} \mathbb{E}(M_n | \mathcal{F}_t).$$

By Theorem 1 as we have

$$\sup_{u \in \mathcal{U}} \xi_u^{p_0} \in L^p(\mathbb{P}),$$

we fix $m > 0$. We now work on the event $B_m := \{\sup_{u \in \mathcal{U}} \xi_u \leq m\}$.

By applying the Markov property at time t we easily get that

$$\mathbb{E}(M_n | \mathcal{F}_t) = \sum X_i^{p_0}(t) \mathbb{1}_{\{G(X_i(t)) \leq n\}} + \sum_{g(u)=n} \xi_u^{p_0} \mathbb{1}_{\{a_u + \zeta_u < t\}} \quad (6)$$

where $G(\xi_v)$ stands for the generation of the individual v (i.e. $G(\xi_v) = g(v)$), and $a_u + \zeta_u$ is the instant when the individual corresponding to the node u reproduces. We can rewrite the latter as

$$a_u + \zeta_u = \xi_{m^{g(u)-1}u}^{-\alpha} \mathbf{e}_0 + \xi_{m^{g(u)-2}u}^{-\alpha} \mathbf{e}_1 + \dots + \xi_u^{-\alpha} \mathbf{e}_{g(u)}$$

where \mathbf{e}_0, \dots is a sequence of independent exponential variables with parameter 1, which is also independent of ξ_u . We can remark that in the first term of sum (6) we sum over the size of the individuals which belong to the n -th generation and are alive at time t , and in the second term we sum over those belonging to the n -th generation and are dead at time t .

As α is non negative, and as we are working on the event B_m : $\xi_{m^{i_u}}^{-\alpha} \geq m^{-\alpha}$ we have that for each fixed node $u \in \mathcal{U}$, $a_u + \zeta_u$ is bounded from below by the sum of $g(u) + 1$ independent exponential variables with parameter m^α which are independent of ξ_u . Thus

$$\lim_{n \rightarrow \infty} \mathbb{E} \left(\sum_{g(u)=n} \xi_u^{p_0} \mathbb{1}_{\{a_u + \zeta_u < t\}} \mathbb{1}_{\{B_m\}} \right) = 0,$$

and therefore by (6) on the event $\{B_m\}$, we get that for all $m > 0$: $\mathbb{E}(M_\infty | \mathcal{F}_t) \mathbb{1}_{\{B_m\}} = M(t) \mathbb{1}_{\{B_m\}}$, and then by letting m tend to ∞ we get the result. \square

5 A randomly tagged leaf.

We will here (as in [4]) define what a tagged individual is by using a tagged leaf.

We call *leaf* of the marked tree \mathcal{U} an infinite sequence of integers $l = (u_1, \dots)$. For each n , we associate to l its ancestor $l_n := (u_1, \dots, u_n)$ at the generation n . We enrich the probabilistic structure by adding the information about a so called tagged leaf, chosen at random as follows. Let \mathbb{H}_n be the space of bounded functionals Φ which depend on the mark M and of the leaf l on the n -th first generation, i.e. such that $\Phi(M, l) = \Phi(M', l')$ if $l_n = l'_n$ and $M(u) = M'(u)$ whenever $g(u) \leq n$. For such functionals, we use the slightly abusing notation $\Phi(M, l) = \Phi(M, l_n)$. As in [4] for a pair (M, λ) where $M : \mathcal{U} \rightarrow [0, 1] \times \mathbb{R}_+ \times \mathbb{R}_+$ is a random mark on the marked tree and λ is a random leaf of \mathcal{U} , the joint distribution denoted by \mathbb{P}^* (and by \mathbb{P}_x^* if the size of the first mark is x instead of 1) can be defined unambiguously by

$$\mathbb{E}^*(\Phi(M, \lambda)) = \mathbb{E} \left(\sum_{g(u)=n} \Phi(M, u) \xi_u^{p_0} \right), \quad \Phi \in \mathbb{H}_n.$$

Moreover since the intrinsic martingale $(M_n, n \in \mathbb{Z}_+)$ is uniformly integrable (cf. Theorem 1), the first marginal of \mathbb{P}^* is absolutely continuous with respect to the law of the random mark M under \mathbb{P} , with density M_∞ .

Let λ_n be the node of the tagged leaf at the n -th generation. We denote $\chi_n := \xi_{\lambda_n}$ for the size of the individual corresponding to the node λ_n and $\chi(t)$ for the size of the tagged individual alive at time t , viz.

$$\chi(t) := \chi_n \quad \text{if } a_{\lambda_n} \leq t < a_{\lambda_n} + \zeta_{\lambda_n},$$

because in the case considered $\sup_{n \in \mathbb{N}} a_{\lambda_n} = \infty$. Exactly as in [4] Lemma 1.4 there becomes:

Lemma 2. *Let $k : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a measurable function such that $k(0) = 0$. Then we have for every $n \in \mathbb{N}$*

$$\mathbb{E}^*(k(\chi_n)) = \mathbb{E} \left(\sum_{g(u)=n} \xi_u^{p_0} k(\xi_u) \right),$$

and for every $t \geq 0$

$$\mathbb{E}^*(k(\chi(t))) = \mathbb{E}(\langle x^{p_0} k(x), X(t) \rangle).$$

Proposition 1.6 of [4] becomes:

Proposition 4. *Under \mathbb{P}^* ,*

$$S_n := \ln \chi_n, \quad n \in \mathbb{Z}_+$$

is a random walk on \mathbb{R} with step distribution

$$\mathbb{P}(\ln \chi_n - \ln \chi_{n+1} \in dy) = \tilde{\nu}(dy),$$

where the probability measure $\tilde{\nu}$ is defined by

$$\int_{]0, \infty[} k(y) \tilde{\nu}(dy) = \int_{\mathcal{M}_p(\mathbb{R}_+^*)} \langle x^{p_0} k(\ln(x)), m_s \rangle \nu(dm_s).$$

Equivalently, the Laplace transform of the step distribution is given by

$$\mathbb{E}^*(e^{pS_1}) = \mathbb{E}^*(\chi_1^p) = 1 - \kappa(p + p_0), \quad p \geq 0.$$

Moreover, conditionally on $(\chi_n, n \in \mathbb{Z}_+)$ the sequence of the lifetimes $(\zeta_{\lambda_0}, \zeta_{\lambda_1}, \dots)$ along the tagged leaf is a sequence of independent exponential variables with respective parameters $\chi_0^\alpha, \chi_1^\alpha, \dots$

We now see that we can use this proposition to obtain the description of χ_n by a Lamperti transformation. Let

$$\eta_t := S \circ N_t, \quad t \geq 0,$$

with N a Poisson process with parameter 1 which is independent of the random walk S ; for probabilities and expectations related to η we use the notation P and E . The process $(\chi(t), t \geq 0)$ is Markovian and enjoys a scaling property. More precisely under \mathbb{P}_x^* we get that

$$\chi(t) = \exp(\eta_{\tau(tx^{-\alpha})}), \quad t \geq 0,$$

where η is the compound Poisson defined above and τ the time-change defined implicitly by

$$t = \int_0^{\tau(t)} \exp(\alpha \eta_s) ds, \quad t \geq 0.$$

6 Asymptotic behaviors.

6.1 The convergence of the size of a tagget individual.

Let

$$\kappa'(p_0) = - \int_{\mathcal{M}_p(\mathbb{R}_+^*)} \langle x^{p_0} \ln(x), m_s \rangle \nu(dm_s)$$

denote the derivative of κ at the Malthusian parameter p_0 .

Proposition 5. *Suppose that $\alpha > 0$, that the support of ν is not a discrete subgroup $r\mathbb{Z}$ for any $r > 0$ and that $0 < \kappa'(p_0) < \infty$. Then for every $y > 0$*

$$\mathbb{P}_y^*(t^{1/\alpha} \chi(t) \in \cdot) \Rightarrow \bar{\mathbb{P}}(\chi(1) \in \cdot),$$

as $t \rightarrow \infty$, where $\bar{\mathbb{P}}$ is a probability measure such that for every measurable function $k : \mathbb{R}_+ \rightarrow \mathbb{R}_+$

$$\bar{\mathbb{E}}(k(\chi(1)^\alpha)) = \frac{1}{\alpha m} E(k(I)I^{-1}),$$

with $I := \int_0^\infty \exp(\alpha \eta_s) ds$ and $m := E(\eta_1)$.

Proof. As $-\kappa'(p_0)$ is the mean of the step distribution of the random walk S_n (see Proposition 4), therefore $\kappa'(p_0) > 0$ imply that $E(-\eta_1) > 0$ thus the assumption of Theorem 1 in the article by Bertoin and Yor [6] is fulfilled by the self-similar Markov process $\chi(t)^{-1}$, which gives the result.

We could also see by direct calculation that $E(-\eta_1)$ is equal to $\kappa'(p_0)$. □

Remark 2. • *In the case $\kappa'(p_0) = 0$ we can extend this proposition. More precisely if $\int_{\mathcal{M}_p(\mathbb{R}_+^*)} \langle x^{p_0} | \ln(x) |, m_s \rangle \nu(dm_s) < \infty$,*

$$J := \int_1^\infty \frac{x \nu^-(]x, \infty[) dx}{1 + \int_0^x dy \int_y^\infty \nu^-(]-\infty, -z[) dz} < \infty,$$

(with ν^- is the image of $\tilde{\nu}$ by the map $u \rightarrow -u$ where $\tilde{\nu}$ is defined in Proposition 4) and $E\left(\log^+ \int_0^{T_1} e^{-\eta_s} ds\right) < \infty$ (with $T_z := \inf\{t : -\eta_t \geq z\}$) hold then

1. *For any $y > 0$*

$$\mathbb{P}_y^*(t^{1/\alpha} \chi(t) \in \cdot) \Rightarrow \mathbb{P}_0(\chi(1) \in \cdot),$$

as $t \rightarrow \infty$, where \mathbb{P}_0 is a probability measure.

2. *For any bounded and continuous function k and for $t > 0$:*

$$\mathbb{E}_0(k(\chi(t)^\alpha)) = \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} E(I_\lambda^{-1} k(I_\lambda/t)),$$

where $I_\lambda = \int_0^\infty \exp(\alpha\eta_s - \lambda s) ds$.

The proof is the same as the previous one using Theorem 1 and Theorem 2 from the article by Caballero and Chaumont [8] instead of [6].

• *We could also try to use the same method as the one used in [4] Proposition 1.7. But the problem is that in our case $\mathbb{E}(\langle x^p, X(t) \rangle)$ is not necessarily finite when p is large, and even if it was the case, its derivative is not completely monotone because κ is not necessarily positive when p is large. This explains why we have to use a different method.*

6.2 Convergence of the mean measure and L^p -convergence.

We encode the configuration of masses $X(t) = \{X_i(t)\}$ by the weighted empirical measure

$$\sigma_t := \sum X_i^{p_0}(t) \delta_{t^{1/\alpha} X_i(t)}$$

which has total mass $M(t)$.

The associated mean measure σ_t^* is defined by the formula

$$\int_0^\infty k(x) \sigma_t^*(dx) = \mathbb{E} \left(\int_0^\infty k(x) \sigma_t(dx) \right)$$

which is required to hold for all compactly supported continuous functions k . Since $M(t)$ is a martingale, σ_t^* is a probability measure.

Therefore, with Proposition 5 and Lemma 2, we easily get:

Corollary 6. *With the assumptions of Proposition 5 we get:*

1. *The measures σ_t^* converge weakly, as $t \rightarrow \infty$, to the probability measure $\bar{\mathbb{P}}(\chi(1) \in \cdot)$ i.e. for any measurable function $k : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, we have:*

$$\mathbb{E}(\langle x^{p_0} k(t^{1/\alpha} x), X(t) \rangle) \xrightarrow{t \rightarrow \infty} \bar{\mathbb{E}}(k(\chi(1))).$$

2. *For all $p_+ > p > p_0$:*

$$t^{(p-p_0)/\alpha} \mathbb{E}(\langle x^p, X(t) \rangle) \xrightarrow{t \rightarrow \infty} \bar{\mathbb{E}}((\chi(1))^{p-p_0}).$$

We now formulate a more precise result concerning the convergence of the empirical measure:

Theorem 7. *Under the same assumptions as in Proposition 5 we get that for every bounded continuous function k :*

$$L^p - \lim_{t \rightarrow \infty} \int_0^\infty k(x) \sigma_t(dx) = M_\infty \bar{\mathbb{E}}(k(\chi(1))) = \frac{M_\infty}{\alpha m} E(k(I)I^{-1}),$$

for some $p > 1$.

Remark 3. *A slightly different version of Corollary 6 and Theorem 7 exists also under the assumptions in Remark 2.*

See also Asmussen and Kaplan [1] and [2] for a closely related result.

Proof. We use the same method as Section 1.4. in [4] and in this direction we use Lemma 1.5 there: for $(\lambda(t))_{t \geq 0} = (\lambda_i(t), i \in \mathbb{N})_{t \geq 0}$ a sequence of non-negative random variables such that for fixed $p > 1$

$$\sup_{t \geq 0} \mathbb{E} \left(\left(\sum_{i=1}^{\infty} \lambda_i(t) \right)^p \right) < \infty \quad \text{and} \quad \lim_{t \rightarrow \infty} \mathbb{E} \left(\sum_{i=1}^{\infty} \lambda_i(t) \right) = 0,$$

and for $(Y_i(t), i \in \mathbb{N})$ a sequence of random variables which are independent conditionally on $\lambda(t)$, we assume that there exists a sequence $(\bar{Y}_i, i \in \mathbb{N})$ of i.i.d variables in $L^p(\mathbb{P})$, which is independent of $\lambda(t)$ for each fixed t , and such that $|Y_i(t)| \leq \bar{Y}_i$ for all $i \in \mathbb{N}$ and $t \geq 0$.

Then we know from Lemma 1.5 in [4] that

$$\lim_{t \rightarrow \infty} \sum_{i=1}^{\infty} \lambda_i(t) (Y_i(t) - \mathbb{E}(Y_i(t) | \lambda(t))) = 0. \tag{7}$$

Now, let k be a continuous function bounded by 1 and let

$$A_t := \langle x^{p_0} k(t^{1/\alpha} x), X(t) \rangle.$$

By application of the Markov property at time t for A_{t+s} and the self-similarity property of the process X we can rewrite A_{t+s} as

$$\sum_{i=1}^{\infty} \lambda_i(t) Y_i(t, s)$$

where $\lambda_i(t) := X_i^{p_0}(t)$ and

$$Y_i(t, s) := \langle x^{p_0} k((t+s)^{1/\alpha} X_i(t)x), X_{i,\cdot}(s) \rangle,$$

with $X_{1,\cdot}, X_{2,\cdot}, \dots$ a sequence of i.i.d. copies of X which is independent of $X(t)$.

By Theorem 1 we get that

$$\sup_{t \geq 0} \mathbb{E} \left(\left(\sum_{i=1}^{\infty} \lambda_i(t) \right)^p \right) < \infty.$$

By the last corollary we also obtain that

$$\mathbb{E} \left(\sum_{i=1}^{\infty} \lambda_i^p(t) \right) \sim t^{-(p-1)p_0} \overline{\mathbb{E}}(\chi^{(p-1)p_0}(1)) \rightarrow 0,$$

as $t \rightarrow \infty$.

Moreover the variables $Y_i(t, s)$ are uniformly bounded by

$$Y_i = \sup_{s \geq 0} \langle x^{p_0}, X_{i,\cdot}(s) \rangle,$$

which are i.i.d. variables and also bounded in $L^p(\mathbb{P})$ thanks to Doob's inequality (as we have that $\langle x_{p_0}, X_{i,\cdot}(s) \rangle$ is a martingale bounded in $L^p(\mathbb{P})$).

Thus we may apply (7), which reduces the study to that of the asymptotic behaviour of:

$$\sum_{i=1}^{\infty} \lambda_i(t) \mathbb{E}(Y_i(t, s) | X(t)),$$

as t tends to ∞ . On the event $\{X_i(t) = y\}$, we get

$$\mathbb{E}(Y_i(t, s) | X(t)) = \mathbb{E}(\langle x^{p_0} k((t+s)^{1/\alpha} yx), X(s) \rangle).$$

Then by Lemma 2:

$$\mathbb{E}(\langle x^{p_0} k((t+s)^{1/\alpha} yx), X_{i,\cdot}(s) \rangle) = \mathbb{E}^*(k((t+s)^{1/\alpha} y\chi(s))).$$

With Proposition 5, we obtain

$$\lim_{t \rightarrow \infty} \mathbb{E}^* (k((t+s)^{1/\alpha} y \chi(s))) = \overline{\mathbb{E}}(k(\chi(1))).$$

Moreover recall from Corollary 3 that $\sum_{i=1}^{\infty} \lambda_i(t)$ converges to M_{∞} in $L^p(\mathbb{P})$. Therefore we finally get that when t goes to infinity:

$$\sum_{i=1}^{\infty} \lambda_i(t) \mathbb{E}(Y_i(t, s) | X(t)) \sim \overline{\mathbb{E}}(k(\chi(1))) \sum_{i=1}^{\infty} \lambda_i(t) \sim \overline{\mathbb{E}}(k(\chi(1))) M_{\infty}.$$

□

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