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APPROXIMATION OF QUASISTATIC SIGNORINI PROBLEMS WITH LOCAL FRICTION BY A MIXED METHOD

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Abstract The approximation of the quasistatic two-body unilateral contact problem with local Coulomb friction by a mixed finite element method is studied. Continuous and discrete variational formulations are stated. Using a regularity result for the normal component of the stress vector for an auxiliary problem with given friction and some error estimates, convergence to the continuous quasistatic solution is proved when the discretization parameters tend to zero.

1. Introduction

This paper deals with the numerical analysis of the quasistatic problem of unilateral contact with local Coulomb friction between two elastic bodies. The approach we use here follows that of Rocca and Cocou (2001b), where the approximation of the contact problem for an elastic body by a mixed finite element method is studied.

Contact problems with local friction law were first considered in the static case by Nečas, Jarušek and Haslinger (1980). They proved an existence result in the case of a strip by a shifting technique when the friction coefficient is sufficiently small. This result was extended by Jarušek (1983) for a domain with smooth contact zone and for two elastic bodies.

Using a mixed finite element method, Haslinger (1983) proved the existence of discrete solutions converging towards a solution of the static unilateral contact problem with local friction.

Existence results for the quasistatic unilateral contact with local friction have been obtained by Andersson (2000) and by the authors, see Rocca (1999), Rocca and Cocou (2001a), who have proved that there exists a solution if the friction coefficient is sufficiently small and smooth. An extension to a quasistatic problem coupling unilateral contact, local friction and adhesion has been also studied by Cocou and Rocca (2000).

This paper is organized as follows. In section 2, we present the continuous variational formulations for the initial problem.

In section 3, we present the discrete variational formulations and their approximation.

In section 4 we prove the existence of a saddle point for the incremental regularized discrete problem. Using some error estimates we prove that the Lagrangian multiplier is bounded in $H^{-1/2+\beta}$ independently of the spatial discretization parameter provided that the friction coefficient is small enough. Section 5 is concerned with convergence results.

2. Classical and variational formulations

Let us consider two linear elastic bodies which occupy the bounded domains Ω^β , $\beta = 1, 2$, of \mathbf{R}^d , $d = 2, 3$. Let Γ_1^β , Γ_2^β and Γ_3 be three open disjoint parts of $\Gamma^\beta = \partial\Omega^\beta$ such that $\Gamma^\beta = \bar{\Gamma}_1^\beta \cup \bar{\Gamma}_2^\beta \cup \bar{\Gamma}_3$ and $\text{mes}(\Gamma_1^\beta) > 0$. We suppose that the two solids are initially in contact with local friction on Γ_3 , which is considered as being a bounded zone of contact. We denote by \mathbf{n}^β the outward normal unit vector to Γ^β , $\mathbf{u}^\beta = (u_1^\beta, \dots, u_d^\beta)$ the displacement field, $\boldsymbol{\epsilon}^\beta = (\epsilon_{ij}(\mathbf{u}^\beta))$ the strain tensor and by $\boldsymbol{\sigma}^\beta = (\sigma_{ij}^\beta(\mathbf{u}^\beta)) = (a_{ijkl}^\beta \epsilon_{kl}(\mathbf{u}^\beta))$ the stress tensor. We adopt the notations $u_N^\beta, \mathbf{u}_T^\beta, \sigma_N^\beta, \boldsymbol{\sigma}_T^\beta$ for the normal and tangential components of the displacement vector and of the stress vector with respect to \mathbf{n}^1 . The problem consists in finding the displacement fields solution of the equilibrium equations and boundary conditions. We suppose that a_{ijkl}^β are of class $C^{0,1/2+\delta}$ with $1/2 > \delta > 0$ in Ω^β , $1 \leq i, j, k, l \leq d$, with the usual conditions of symmetry and ellipticity.

Let us denote by $\boldsymbol{\phi} = (\boldsymbol{\phi}^1, \boldsymbol{\phi}^2)$ and $\boldsymbol{\psi} = (\boldsymbol{\psi}^1, \boldsymbol{\psi}^2)$ the given body forces and tractions. The initial displacements of the bodies are denoted by $\mathbf{u}^{0,\beta}$ and a displacement $\mathbf{U}^\beta = \mathbf{0}$ is prescribed on Γ_1^β , $\beta = 1, 2$. We suppose that the friction coefficient μ belongs to $L^\infty(\Gamma^1)$ and to the set of the multipliers of $H^{1/2}(\Gamma^1)$ denoted by $\mathcal{M}(H^{1/2}(\Gamma^1))$.

The classical problem is as follows : *find* $\mathbf{u} = \mathbf{u}(t, \mathbf{x}) = (\mathbf{u}^1, \mathbf{u}^2)$ such that $\mathbf{u}(0) = (\mathbf{u}^{0,1}, \mathbf{u}^{0,2})$ in $\Omega^1 \times \Omega^2$ and for all $t \in]0, T[$, $\beta = 1, 2$,

$$\operatorname{div} \sigma^\beta(\mathbf{u}^\beta) = -\phi^\beta, \quad \sigma_{ij}^\beta(\mathbf{u}^\beta) = a_{ijkl}^\beta \epsilon_{kl}(\mathbf{u}^\beta) \text{ in } \Omega^\beta, \quad (1)$$

$$\mathbf{u}^\beta = \mathbf{0} \text{ on } \Gamma_1^\beta, \quad \sigma^\beta \mathbf{n}^\beta = \psi^\beta \text{ on } \Gamma_2^\beta, \quad (2)$$

$$\mathbf{u}_N^1 - \mathbf{u}_N^2 \leq 0, \quad \sigma_N^1(\mathbf{u}^1) \leq 0, \quad (\mathbf{u}_N^1 - \mathbf{u}_N^2) \sigma_N^1(\mathbf{u}^1) = 0 \text{ on } \Gamma_3, \quad (3)$$

$$\sigma^1 \mathbf{n}^1 = -\sigma^2 \mathbf{n}^2, \quad |\sigma_T^1| \leq -\mu \sigma_N^1 \text{ on } \Gamma_3 \text{ and} \quad (4)$$

$$\begin{cases} |\sigma_T^1| < -\mu \sigma_N^1 \Rightarrow \dot{\mathbf{u}}_T^1 - \dot{\mathbf{u}}_T^2 = \mathbf{0}, \\ |\sigma_T^1| = -\mu \sigma_N^1 \Rightarrow \exists \lambda \geq 0, \quad \dot{\mathbf{u}}_T^1 - \dot{\mathbf{u}}_T^2 = -\lambda \sigma_T^1. \end{cases} \quad (5)$$

We define $V = V^1 \times V^2$, where $V^\beta = \{\mathbf{v}^\beta \in [H^1(\Omega^\beta)]^d; \mathbf{v}^\beta = \mathbf{0} \text{ a.e. on } \Gamma_1^\beta\}$, $\beta = 1, 2$, and $K = \{\mathbf{v} \in V; \mathbf{v}_N^1 - \mathbf{v}_N^2 \leq 0 \text{ a.e. on } \Gamma_3\}$. The duality pairings on $H^{1/2}(\Gamma^\beta)$, $H^{-1/2}(\Gamma^\beta)$ and on $[H^{1/2}(\Gamma^\beta)]^d$, $[H^{-1/2}(\Gamma^\beta)]^d$ are denoted by $\langle \cdot, \cdot \rangle$. We suppose that $\mathbf{u}^0 = (\mathbf{u}^{0,1}, \mathbf{u}^{0,2}) \in K$, $\phi^\beta \in W^{1,2}(0, T; [L^2(\Omega^\beta)]^d)$ and that $\psi^\beta \in W^{1,2}(0, T; [L^2(\Gamma_2^\beta)]^d)$ such that its support is strictly included in Γ_2^β . Let us introduce the cut-off function $\theta \in C_0^\infty(\mathbf{R}^d)$, $0 \leq \theta \leq 1$, such that $\theta = 1$ on $\bar{\Gamma}_3$ and $\theta = 0$ on $\bar{\Gamma}_2^{\beta,0}$, where $\Gamma_2^{\beta,0}$ is an open subset having the property that for all $t \in [0, T]$, $\operatorname{supp} \psi^\beta(t) \subset \Gamma_2^{\beta,0} \subset \bar{\Gamma}_2^{\beta,0} \subset \Gamma_2^\beta$.

Using Green's formula we obtain a variational formulation of the quasi-static unilateral contact problem with local friction as follows.

Problem P_0 : Find $\mathbf{u} \in W^{1,2}(0, T; V)$ such that $\mathbf{u}(0) = (\mathbf{u}^{0,1}, \mathbf{u}^{0,2})$ and for almost all $t \in]0, T[$ $\mathbf{u}(t) \in K$ and

$$a(\mathbf{u}, \mathbf{v} - \dot{\mathbf{u}}) + j_0(\mathbf{u}, \mathbf{v}) - j_0(\mathbf{u}, \dot{\mathbf{u}}) \geq \langle \phi, \mathbf{v} - \dot{\mathbf{u}} \rangle + \langle \psi, \mathbf{v} - \dot{\mathbf{u}} \rangle + \langle \theta \sigma_N^1(\mathbf{u}^1), [v_N] - [\dot{u}_N] \rangle \quad \forall \mathbf{v} \in V, \quad (6)$$

$$\langle \theta \sigma_N^1(\mathbf{u}^1), [z_N] - [u_N] \rangle \geq 0 \quad \forall \mathbf{z} \in K, \quad (7)$$

where $a(\mathbf{u}, \mathbf{v}) = \sum_\beta a^\beta(\mathbf{u}^\beta, \mathbf{v}^\beta)$, $a^\beta(\mathbf{u}^\beta, \mathbf{v}^\beta) = \int_{\Omega^\beta} a_{ijkl}^\beta \epsilon_{ij}(\mathbf{u}^\beta) \epsilon_{kl}(\mathbf{v}^\beta)$, $j_0(\mathbf{u}, \mathbf{v}) = -\langle \mu \sigma_N^1(\mathbf{u}^1), |[v_T]| \rangle$, $[\eta] = \eta^1 - \eta^2$, $\langle \phi, \mathbf{v} \rangle = \sum_\beta \langle \phi^\beta, \mathbf{v}^\beta \rangle$, $\langle \psi, \mathbf{v} \rangle = \int_{\Omega^\beta} \phi^\beta \cdot \mathbf{v}^\beta$, $\langle \psi, \mathbf{v} \rangle = \sum_\beta \langle \psi^\beta, \mathbf{v}^\beta \rangle = \sum_\beta \int_{\Gamma_2^\beta} \psi^\beta \cdot \mathbf{v}^\beta$.

Problem P_0 is equivalent to the following mixed variational formulation.

Problem P : Find two mappings $t \mapsto \mathbf{u}(t)$ and $t \mapsto \lambda(t)$ such that $\mathbf{u} \in W^{1,2}(0, T; V)$, $\lambda \in W^{1,2}(0, T; H^{-1/2}(\Gamma^1))$, $\mathbf{u}(0) = \mathbf{u}^0$, $\lambda(0) = \lambda^0 = \theta \sigma_N^1(\mathbf{u}^{0,1})$ and for almost all $t \in]0, T[$ $\lambda(t) \in C^{-}$ and*

$$a(\mathbf{u}, \mathbf{v} - \dot{\mathbf{u}}) + j(\lambda, \mathbf{v}) - j(\lambda, \dot{\mathbf{u}}) \geq \langle \phi, \mathbf{v} - \dot{\mathbf{u}} \rangle + \langle \psi, \mathbf{v} - \dot{\mathbf{u}} \rangle + \langle \lambda, [v_N] - [\dot{u}_N] \rangle \quad \forall \mathbf{v} \in V, \quad (8)$$

$$\langle \pi - \lambda, [u_N] \rangle \geq 0 \quad \forall \pi \in C^{*-}, \quad (9)$$

where $C^{*-} = \{\pi \in H^{-1/2}(\Gamma^1); \text{supp } \pi \subset \bar{\Gamma}_3\} \cap \mathcal{B}_-(\Gamma^1)$, $\mathcal{B}_-(\Gamma^1)$ is the set of bounded negative measures on Γ^1 and $j(\lambda, \mathbf{v}) = -\langle \mu\lambda, |[\mathbf{v}_T]| \rangle$. The Lagrange multiplier λ satisfies $\lambda = \theta\sigma_N^1(\mathbf{u}^1)$.

3. Discrete formulations

In order to prove the existence of discrete solutions converging towards \mathbf{u} , we consider the following finite element approximations.

Let $(\mathcal{T}_h^\beta)_h$ be a set of regular triangulations of $\bar{\Omega}^\beta$, $\beta = 1, 2$, see Ciarlet (1978). Each triangulation is a collection of elements (Ω_i^β) such that $\bar{\Omega}^\beta = \bigcup_{i \in \mathcal{I}_h^\beta} \bar{\Omega}_i^\beta$ with $\Omega_k^\beta \cap \Omega_l^\beta = \emptyset$ for all $k, l \in \mathcal{I}_h^\beta$, $k \neq l$. We assume that each triangulation is compatible with the partition of Γ^β , that is, each point where the boundary condition changes is a node of a set $\bar{\Omega}_i^\beta$ and we require also that the nodes lying on Γ_3 belong to both triangulations. We shall use the notation $\mathcal{T}_h = \{\mathcal{T}_h^1, \mathcal{T}_h^2\}$. Let $(\mathcal{T}_H)_H$ be a set of triangulations of $\bar{\Gamma}_3 = \bigcup_{i \in \mathcal{J}_H} \Gamma_{3,i}$. We suppose that the elements $\Gamma_{3,i}$, $i \in \mathcal{J}_H$, are segments for $d = 2$ and rectangles for $d = 3$. We set $H^i = \text{length}(\Gamma_{3,i})$ for $d = 2$, and $H^i = (H_1^i; H_2^i)$, where H_1^i, H_2^i are the lengths of the edges of $\Gamma_{3,i}$ for $d = 3$. We assume that $(\mathcal{T}_H)_H$ are regular that is there exists a constant $\gamma > 0$ such that $\min_{i \in \mathcal{J}_H} (H^i)/H \geq \gamma$, where $H = \max_{i \in \mathcal{J}_H} (H^i)$ for $d = 2$, and $\min_{i \in \mathcal{J}_H} (H_1^i, H_2^i)/H \geq \gamma$, where $H = \max_{i \in \mathcal{J}_H} (H_1^i, H_2^i)$ for $d = 3$.

We introduce the following sets:

$$\begin{aligned} V_h &= \left\{ \mathbf{v}_h = (\mathbf{v}_h^1, \mathbf{v}_h^2) \in [C^0(\bar{\Omega}^1)]^d \times [C^0(\bar{\Omega}^2)]^d; \mathbf{v}_{h|\Omega_i^\beta}^\beta \in [P_1(\Omega_i^\beta)]^d \right. \\ &\quad \left. \forall i \in \mathcal{I}_h^\beta, \mathbf{v}_h^\beta = \mathbf{0} \text{ on } \Gamma_1^\beta, \beta = 1, 2 \right\}, \\ \tilde{V}_h &= \left\{ \mathbf{v}_h \in V_h; \mathbf{v}_{hT}^\beta = \mathbf{0} \text{ on } \Gamma^\beta, v_{hN}^\beta = 0 \text{ on } \Gamma^\beta \setminus \Gamma_3, \beta = 1, 2 \right\}, \\ L_H &= \left\{ \pi_H \in L^2(\Gamma^1); \pi_H = 0 \text{ on } \Gamma^1 \setminus \bar{\Gamma}_3, \pi_{H|\Gamma_{3,i}} \in P_0(\Gamma_{3,i}) \forall i \in \mathcal{J}_H \right\}, \\ \Pi_H &= \left\{ \pi_H \in L_H; \pi_H \leq 0 \text{ on } \Gamma_{3,i} \forall i \in \mathcal{J}_H \right\}, \\ K_{hH} &= \left\{ \mathbf{v}_h \in V_h; \langle \pi_H, [v_{hN}] \rangle \geq 0, \forall \pi_H \in \Pi_H \right\}, \end{aligned}$$

where $P_k(\omega)$ denotes the space of polynomials of degree lower or equal to k on ω .

We consider the following semi-discrete variational problem.

Problem P_{hH} : Find two mappings $t \rightarrow \mathbf{u}_h(t)$ and $t \rightarrow \lambda_H(t)$ such that $\mathbf{u}_h \in W^{1,2}(0, T; V_h)$, $\lambda_H \in W^{1,2}(0, T; L_H)$, $\mathbf{u}_h(0) = \mathbf{u}_h^0$, $\lambda_H(0) = \lambda_H^0$ and for almost all $t \in]0, T[$ $\lambda_H(t) \in \Pi_H$ and

$$\begin{aligned}
a(\mathbf{u}_h, \mathbf{v}_h - \dot{\mathbf{u}}_h) + j(\lambda_H, \mathbf{v}_h) - j(\lambda_H, \dot{\mathbf{u}}_h) &\geq (\phi, \mathbf{v}_h - \dot{\mathbf{u}}_h) \\
&+ \langle \psi, \mathbf{v}_h - \dot{\mathbf{u}}_h \rangle + \langle \lambda_H, [v_{hN}] - [\dot{u}_{hN}] \rangle \quad \forall \mathbf{v}_h \in V_h, \quad (10) \\
\langle \pi_H - \lambda_H, [u_{hN}] \rangle &\geq 0 \quad \forall \pi_H \in \Pi_H. \quad (11)
\end{aligned}$$

We adopt the following time discretization. For all $n \in \mathbf{N}^*$ we use the notation $\phi^i = \phi(i\Delta t)$ and $\psi^i = \psi(i\Delta t)$ for $i = 0, \dots, n$, where $\Delta t = T/n$. We consider the following incremental problem and its regularization for $i = 0, \dots, n-1$.

Problem P_{hH}^i : For $\mathbf{u}_h^i \in K_{hH}$, find $\mathbf{u}_h^{i+1} \in K_{hH}$, $\lambda_H^{i+1} \in \Pi_H$ such that

$$\begin{aligned}
a(\mathbf{u}_h^{i+1}, \mathbf{v}_h - \mathbf{u}_h^{i+1}) + j(\lambda_H^{i+1}, \mathbf{v}_h - \mathbf{u}_h^i) - j(\lambda_H^{i+1}, \mathbf{u}_h^{i+1} - \mathbf{u}_h^i) \\
\geq (\phi^{i+1}, \mathbf{v}_h - \mathbf{u}_h^{i+1}) + \langle \psi^{i+1}, \mathbf{v}_h - \mathbf{u}_h^{i+1} \rangle \quad (12) \\
+ \langle \lambda_H^{i+1}, [v_{hN}] - [u_{hN}^{i+1}] \rangle \quad \forall \mathbf{v}_h \in V_h, \\
\langle \pi_H - \lambda_H^{i+1}, [u_{hN}^{i+1}] \rangle \geq 0 \quad \forall \pi_H \in \Pi_H. \quad (13)
\end{aligned}$$

Problem $P_{hH}^{i,\nu}$: For $\mathbf{u}_h^i \in K_{hH}$, find $\mathbf{u}_{h\nu} \in K_{hH}$, $\lambda_{H\nu} \in \Pi_H$ such that

$$\begin{aligned}
a(\mathbf{u}_{h\nu}, \mathbf{v}_h - \mathbf{u}_{h\nu}) + j_\nu(\lambda_{H\nu}, \mathbf{v}_h - \mathbf{u}_h^i) - j_\nu(\lambda_{H\nu}, \mathbf{u}_{h\nu} - \mathbf{u}_h^i) \\
\geq (\phi^{i+1}, \mathbf{v}_h - \mathbf{u}_{h\nu}) + \langle \psi^{i+1}, \mathbf{v}_h - \mathbf{u}_{h\nu} \rangle \quad (14) \\
+ \langle \lambda_{H\nu}, [v_{hN}] - [u_{h\nu N}] \rangle \quad \forall \mathbf{v}_h \in V_h, \\
\langle \pi_H - \lambda_{H\nu}, [u_{h\nu N}] \rangle \geq 0 \quad \forall \pi_H \in \Pi_H, \quad (15)
\end{aligned}$$

where $j_\nu(\lambda, \mathbf{v}) = -\langle \mu\lambda, \eta_\nu(|[\mathbf{v}_T]|) \rangle$ with $(\eta_\nu)_\nu$ a family of functions such that $\forall \nu > 0$, η_ν is convex, $\eta_\nu \in C^1(\mathbf{R})$, $0 \leq \eta'_\nu \leq 1$, $\eta_\nu(0) = \eta'_\nu(0) = 0$ and $0 \leq s - \eta_\nu(s) \leq \nu$, $\forall s \in \mathbf{R}$.

4. Existence of incremental solution

In order to prove the existence of a solution for problem (12,13) we consider the following auxiliary problem associated to (14,15).

Problem $P_{hH, g_H}^{i,\nu}$: For $g_H \in \Pi_H$, find $\mathbf{u}_h = \mathbf{u}_h(g_H) \in K_{hH}$, $\lambda_H \in \Pi_H$ such that

$$\begin{aligned}
a(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) - \langle \mu g_H, \eta_\nu(|[\mathbf{v}_{hT} - \mathbf{u}_{hT}^i]|) \rangle - \eta_\nu(|[\mathbf{u}_{hT} - \mathbf{u}_{hT}^i]|) \rangle \quad (16) \\
\geq (\phi^{i+1}, \mathbf{v}_h - \mathbf{u}_h) + \langle \psi^{i+1}, \mathbf{v}_h - \mathbf{u}_h \rangle + \langle \lambda_H, [v_{hN}] - [u_{hN}] \rangle \quad \forall \mathbf{v}_h \in V_h, \\
\langle \pi_H - \lambda_H, [u_{hN}] \rangle \geq 0 \quad \forall \pi_H \in \Pi_H. \quad (17)
\end{aligned}$$

Theorem 1 For all h, H there exists a solution $(\mathbf{u}_h, \lambda_H)$ of the problem $P_{hH, g_H}^{i,\nu}$ and \mathbf{u}_h is unique. Moreover, λ_H is unique if we assume that the following condition holds

$$\langle \lambda_H, [v_{hN}] \rangle = 0, \quad \forall \mathbf{v}_h \in \tilde{V}_h \implies \lambda_H = 0, \quad (18)$$

Proof. Let $J(\mathbf{v}) = \frac{1}{2}a(\mathbf{v}, \mathbf{v}) - \langle \mu g_H, \eta_\nu(|[\mathbf{v}_T - \mathbf{u}_{hT}^i]|) \rangle - (\phi^{i+1}, \mathbf{v}) - \langle \psi^{i+1}, \mathbf{v} \rangle$ and $\mathcal{L}(\mathbf{v}, \pi_H) = J(\mathbf{v}) - \langle \pi_H, [v_{hN}] \rangle$. The Lagrangian functional $\mathcal{L}(\cdot, \cdot)$ has a saddle point $(\mathbf{u}_h, \lambda_H)$ on $V_h \times \Pi_H$ (see for example Ekeland and Temam (1974)). From the coerciveness of $a(\cdot, \cdot)$ and relation (16) it follows that \mathbf{u}_h is unique. By condition (18), we obtain that λ_H is also unique.

Condition (18) is fulfilled if $2h \leq H$ and we will suppose that this relation holds in the following. Theorem 1 enables us to define the mapping Φ_H as follows : $\Pi_H \ni g_H \rightarrow \Phi_H(g_H) = \lambda_H \in \Pi_H$, where λ_H is the solution of (16,17). The end of the section is devoted to prove that Φ_H has a fixed point with an estimate independent of h, H , for the Lagrange multiplier λ_H .

Theorem 2 *Under the above hypotheses there exists a solution of problem $P_{hH}^{i,\nu}$ for any value of the friction coefficient.*

By Schauder's second fixed point theorem (see Zeidler (1993)) it follows that the mapping Φ_H has a fixed point which is a solution of (14,15).

Theorem 3 *Let $\mathcal{T}_h, \mathcal{T}_H$ be such that there exist $\tau_1 > 0, \tau_2 > 0$ independent of h and H with $\tau_1 \leq h/H \leq \tau_2$ and $\alpha \in]0, 1/2[$. Then there exists a value $\mu^* > 0$ such that the following relation holds*

$$\begin{aligned} \|\lambda_H\|_{H^{-1/2+\alpha}(\Gamma^1)} &\leq \|\mu\|_{L^\infty(\Gamma^1)} / \mu^* \|g_H\|_{H^{-1/2+\alpha}(\Gamma^1)} \\ &+ C_1 \left(\|\phi^{i+1,1}\|_{[L^2(\Omega^1)]^d} + \|\psi^{i+1,1}\|_{[L^2(\Gamma^1)]^d} + \|\mathbf{u}_h^{i,1}\|_{[H^1(\Omega^1)]^d} \right). \end{aligned} \quad (19)$$

Thus for μ such that $\|\mu\|_{L^\infty(\Gamma^1)} < \mu^*$, there exists $r_0 > 0$ such that the mapping Φ_H has a fixed point in $\{\pi_H \in \Pi_H; \|\pi_H\|_{H^{-1/2+\alpha}(\Gamma^1)} \leq r_0\}$. Consequently one obtains once again that problem $P_{hH}^{i,\nu}$ has a solution.

Theorem 4 *There exists a solution $(\mathbf{u}_h^{i+1}, \lambda_H^{i+1})$ to problem P_{hH}^i satisfying the following estimate*

$$\begin{aligned} \|\lambda_H^{i+1}\|_{H^{-1/2+\alpha}(\Gamma^1)} &\leq C_2 \left(\|\phi^{i+1,1}\|_{[L^2(\Omega^1)]^d}, \|\psi^{i+1,1}\|_{[L^2(\Gamma^1)]^d}, \right. \\ &\left. \|\phi^{i,1}\|_{[L^2(\Omega^1)]^d}, \|\psi^{i,1}\|_{[L^2(\Gamma^1)]^d} \right). \end{aligned} \quad (20)$$

This enables us to select a subsequence still denoted by (ν) such that $\nu \rightarrow 0^+$, $\mathbf{u}_{h\nu} \rightarrow \mathbf{u}_h^{i+1}$ in V , $\lambda_{H\nu} \rightarrow \lambda_H^{i+1}$ in $H^{-1/2+\alpha}(\Gamma^1)$. It follows that

$$\begin{aligned} \lim_{\nu \rightarrow 0^+} a(\mathbf{u}_{h\nu}, \mathbf{u}_{h\nu}) &= a(\mathbf{u}_h^{i+1}, \mathbf{u}_h^{i+1}), \\ \lim_{\nu \rightarrow 0^+} j_\nu(\lambda_{H\nu}, \mathbf{u}_{h\nu} - \mathbf{u}_h^i) &= j(\lambda_H^{i+1}, \mathbf{u}_h^{i+1} - \mathbf{u}_h^i), \\ \lim_{\nu \rightarrow 0^+} \langle \lambda_{H\nu}, u_{h\nu N} \rangle &= \langle \lambda_H^{i+1}, u_{hN}^{i+1} \rangle. \end{aligned}$$

All the previous relations, as well as other similar ones concerning the limits with respect to the parameter ν enable us to pass to the limit in (14,15) and to obtain (12,13). Relation (20) is obtained from (19).

5. Convergence results

In this section we prove that there exists a sequence of discrete solutions of problems $(P_{hH}^i)_i$ which converges towards $(\mathbf{u}_h, \lambda_H)$, solution of problem P_{hH} . If $\|\mu\|_{\mathcal{M}(H^{1/2}(\Gamma))} < \tilde{\mu}$, then there exist two constants $\mathcal{C}_3 > 0$, $\mathcal{C}_4 > 0$ such that

$$\begin{aligned} \|\Delta \mathbf{u}_h^i\|_V &\leq \mathcal{C}_3 \left(\|\Delta \phi^i\|_V + \|\Delta \psi^i\|_{[H^{-1/2}(\Gamma^1)]^d \times [H^{-1/2}(\Gamma^2)]^d} \right), \\ \|\Delta \lambda_H^i\|_{H^{-1/2}(\Gamma^1)} &\leq \mathcal{C}_4 \left(\|\Delta \phi^i\|_V + \|\Delta \psi^i\|_{[H^{-1/2}(\Gamma^1)]^d \times [H^{-1/2}(\Gamma^2)]^d} \right). \end{aligned}$$

These two estimates allow us to construct some sequences of discrete solutions for the displacement fields and the Lagrange multipliers which converge towards some absolutely continuous mappings with respect to time, as follows. For $i = 0, \dots, n$ we define

$$\begin{aligned} \mathbf{u}_{hn}(t) &= \mathbf{u}_h^{i+1}, \quad \tilde{\mathbf{u}}_{hn}(t) = \mathbf{u}_h^i + (t - t_i)(\mathbf{u}_h^{i+1} - \mathbf{u}_h^i)/\Delta t, \\ \lambda_{Hn}(t) &= \lambda_H^{i+1}, \quad \tilde{\lambda}_{Hn}(t) = \lambda_H^i + (t - t_i)(\lambda_H^{i+1} - \lambda_H^i)/\Delta t, \\ \mathbf{u}_{hn}(0) &= \tilde{\mathbf{u}}_{hn}(0) = \mathbf{u}_h^0, \quad \lambda_{Hn}(0) = \tilde{\lambda}_{Hn}(0) = \lambda_H^0 \end{aligned}$$

for $t \in]t_i, t_{i+1}]$, $t_i = i\Delta t$. There exist two elements \mathbf{u}_h, λ_H and a subsequence $(n_k)_{k \in \mathbb{N}}$ such that

$$\begin{aligned} \tilde{\mathbf{u}}_{hn_k} &\rightarrow \mathbf{u}_h && \text{in } W^{1,2}(0, T; V), \\ \tilde{\lambda}_{Hn_k} &\rightarrow \lambda_H && \text{in } W^{1,2}(0, T; H^{-1/2}(\Gamma^1)), \\ \mathbf{u}_{hn_k}(t) &\rightarrow \mathbf{u}_h(t) && \text{in } V \quad \forall t \in [0, T], \\ \lambda_{Hn_k}(t) &\rightarrow \lambda_H(t) && \text{in } H^{-1/2}(\Gamma^1) \quad \forall t \in [0, T]. \end{aligned}$$

The proof is similar to the ones given in Cocou and Rocca (2000) or Rocca and Cocou (2001) in semidiscrete cases. We still denote by $(\mathbf{u}_{hn}), (\lambda_{Hn}), (\tilde{\mathbf{u}}_{hn})$ and $(\tilde{\lambda}_{Hn})$, the corresponding convergent subsequences. Using the previous estimates, one can prove the

Theorem 5 *Let $(\mathcal{T}_{h_j})_{j \in \mathbb{N}}$ and $(\mathcal{T}_{H_j})_{j \in \mathbb{N}}$ be such that for all $j \in \mathbb{N}$ we have $\tau_1 \leq h_j/H_j \leq \tau_2$. Let $\mu \in \mathcal{M}(H^{1/2}(\Gamma^1))$ be such that $\|\mu\|_{L^\infty(\Gamma^1)} < \mu^*$ and $\|\mu\|_{\mathcal{M}(H^{1/2}(\Gamma^1))} < \tilde{\mu}$. Then there exists a subsequence $(j_k)_{k \in \mathbb{N}}$ such that $(\mathbf{u}_{h_{j_k}}, \lambda_{H_{j_k}})_{k \in \mathbb{N}}$, the solutions of problems $(P_{h_{j_k} H_{j_k}})$, converge weakly in $W^{1,2}(0, T; V) \times W^{1,2}(0, T; H^{-1/2}(\Gamma^1))$ towards $(\mathbf{u}, \lambda = \theta \sigma_N(\mathbf{u}))$ solution of problem P . Moreover, we have*

$$\|\mathbf{u}_{h_{i_k}} - \mathbf{u}\|_V \rightarrow 0, \quad \|\lambda_{H_{i_k}} - \lambda\|_{H^{-1/2}(\Gamma^1)} \rightarrow 0. \quad (21)$$

The result follows by using the previous estimates and by passing to the limit in (10,11).

6. Conclusions

In this paper we have considered the unilateral contact with local friction between two elastic bodies. An implicit Euler scheme with respect to time and a mixed finite element method for the space discretization have been adopted. The fully discrete problem was solved by a fixed point approach. This result and some estimates independent of the discretization parameters enable us to pass to the limit with respect to the mesh size and to time. It would be interesting to consider the numerical analysis of unilateral contact problems with local friction for curved contact zones. The problem of obtaining rates of convergence for the quasistatic contact problem is open.

References

- Andersson, L. E. (2000) Existence results for quasistatic contact problems with Coulomb friction, *Appl. Math. Opt.* **42**, 169 – 202.
- Ciarlet, P. G. (1978) *The finite element method for elliptic problems*, North-Holland, Amsterdam.
- Cocou, M., Pratt, E., and Raous, M. (1996) Formulation and approximation of quasistatic frictional contact, *Int. J. Engrg. Sci.* **34**, 783–798.
- Cocou, M., and Rocca, R. (2000) Existence results for unilateral quasistatic contact problems with friction and adhesion, *Math. Modelling and Num. Analysis* **34**, 981–1001.
- Ekeland, I., and Temam, R. (1974) *Analyse convexe et problèmes variationnels*, Dunod, Paris.
- Haslinger, J. (1983) Approximation of the Signorini problem with friction, obeying the Coulomb law, *Math. Meth. Appl. Sci.*, **5**, 422–437.
- Hlaváček, J., Haslinger, J., Nečas, J., and Lovisek, J. (1988) *Solution of variational inequalities in mechanics*, Springer-Verlag, New-York, Berlin, Heidelberg.
- Jarušek, J. (1983) Contact problems with bounded friction - coercive case, *Czechoslovak Math. J.* **33**, 237–261.
- Nečas, J., Jarušek, J., and Haslinger, J. (1980) On the solution of the variational inequality to the Signorini problem with small friction, *Bolletino U.M.I.* (5) **17-B**, 796–811.
- Rocca, R. (1999) Existence of a solution for a quasi static contact problem with local friction, *C. R. Acad. Sci. Paris* **328 Serie I**, 1253–1258.
- Rocca, R., and Cocu, M. (2001) Existence and approximation of a solution to quasistatic Signorini problem with local friction, *Int. J. Engrg. Sci.* **39**, 1233–1255.
- Rocca, R., and Cocou, M. (2001) Numerical analysis of quasistatic unilateral contact problems with local friction, *SIAM Journal of Numerical Analysis*, to appear.
- Zeidler, E. (1993) *Nonlinear functional analysis and its applications, Vol I*, Springer-Verlag, New-York.