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Derived Semidistributive Lattices

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Abstract

For L a finite lattice, let $\mathbb{C}(L) \subseteq L^2$ denote the set of pairs $\gamma = (\gamma_0, \gamma_1)$ such that $\gamma_0 \prec \gamma_1$ and order it as follows: $\gamma \leq \delta$ iff $\gamma_0 \leq \delta_0$, $\gamma_1 \not\leq \delta_0$, and $\gamma_1 \leq \delta_1$. Let $\mathbb{C}(L, \gamma)$ denote the connected component of γ in this poset. Our main result states that $\mathbb{C}(L, \gamma)$ is a semidistributive lattice if L is semidistributive, and that $\mathbb{C}(L, \gamma)$ is a bounded lattice if L is bounded.

Let \mathcal{S}_n be the permutohedron on n letters and \mathcal{T}_n be the associahedron on $n + 1$ letters. Explicit computations show that $\mathbb{C}(\mathcal{S}_n, \alpha) = \mathcal{S}_{n-1}$ and $\mathbb{C}(\mathcal{T}_n, \alpha) = \mathcal{T}_{n-1}$, up to isomorphism, whenever α is an atom.

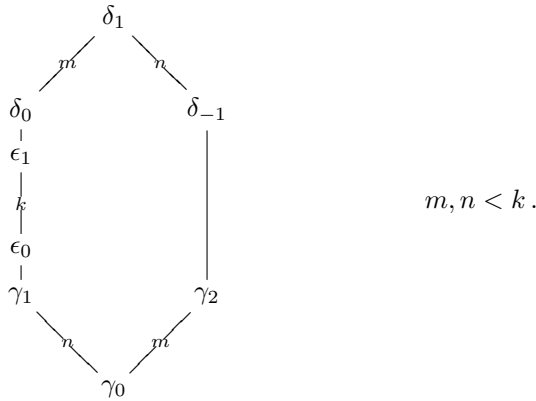
These results are consequences of new characterizations of finite join semidistributive and finite lower bounded lattices: (i) a finite lattice is join semidistributive if and only if the projection sending $\gamma \in \mathbb{C}(L)$ to $\gamma_0 \in L$ creates pullbacks, (ii) a finite join semidistributive lattice is lower bounded if and only if it has a strict facet labelling. Strict facet labellings, as defined here, are generalization of the tools used by Barbut et al. [4] to prove that lattices of Coxeter groups are bounded.

1 Introduction

The set of covers of a finite lattice has a natural ordering given by transposition of intervals. This is the main object investigated in this paper. The main result we shall present is that $\mathbb{C}(L)$, the poset of covers of a finite semidistributive lattice L , is the disjoint union of connected components each of which is again a semidistributive lattice; moreover, if L is bounded, then each such component is a bounded lattice as well. If γ is a cover of L , then we denote by $\mathbb{C}(L, \gamma)$ the connected component of γ in $\mathbb{C}(L)$ and call it the lattice derived from L by means of (or in the direction of) γ . Thus, if L is semidistributive, this process of constructing derivatives may be iterated.

These results are consequences of new characterizations of finite join semidistributive lattices and finite lower bounded lattices that strengthen well known facts. On the side of join semidistributivity, it is well known that a lattice has this property if and only if to each cover $\gamma_0 \prec \gamma_1$ there corresponds a unique

cover $j_* \prec j$ such that j is join irreducible and the two covers are perspective. Following a suggestion of [4, Theorem 1], we observe that existence of such a unique cover is due to the confluence property of a particular rewrite system. Studying further this system we arrive to the following observation: *a finite lattice is join semidistributive if and only if the poset of covers has pullbacks*. On the side of lower bounded lattices, we build on ideas used in [4] to prove that lattices arising from Cayley graphs of Coxeter groups are bounded. With respect to that work, we move from a sufficient condition to a complete characterization, and from boundedness to the weaker notion of being lower bounded. The statement sounds as follows: *a finite lattice is lower bounded if and only if it is join semidistributive and has a strict facet labelling*. A strict facet labelling is a labelling of covers by natural numbers which is constant on perspective covers. Moreover, such a labelling should be strictly increasing at the interior of a minimal pentagon (i.e. a facet). The last condition may be exemplified by the following picture, representing a pentagon where $(\delta_0, \delta_1), (\delta_{-1}, \delta_1), (\gamma_0, \gamma_1), (\gamma_0, \gamma_2), (\epsilon_0, \epsilon_1)$ are all covers.



In labelling covers of such a pentagon, the label of the interior cover (ϵ_0, ϵ_1) should be strictly greater than the labels of (δ_0, δ_1) and (γ_0, γ_1) .

We shall do some explicit computations: we prove that for \mathcal{S}_n the lattice of permutation on n -letters – i.e. the weak Bruhat order – and \mathcal{T}_n the Tamari lattice on $n + 1$ -letters the relations

$$\mathbb{C}(\mathcal{S}_n, \alpha) = \mathcal{S}_{n-1}, \quad \mathbb{C}(\mathcal{T}_n, \alpha) = \mathcal{T}_{n-1},$$

hold up to isomorphism, whenever α is of the form (\perp, α_1) with α_1 an atom. It is also trivial to verify that, under the same conditions on α , the relation

$$\mathbb{C}(\mathcal{B}_n, \alpha) = \mathcal{B}_{n-1}$$

holds, where \mathcal{B}_n is the Boolean algebra on n atoms. Therefore, these lattices share with Boolean algebras¹ a uniformity property: we call them *regular* meaning that the shape of $\mathbb{C}(L, \alpha)$ does not depend on the choice of the atom α_1 .

¹Recall that finite Boolean algebras geometrically are cubes.

This use of terminology from combinatorial geometry is on purpose. Indeed, with this work, we aimed at giving an algebraic and axiomatic ground to some geometric intuitions. For example, an edge of a permutohedron (or associahedron) can be seen as a cover, a facet, i.e. a two dimensional simplex, may be seen as a cover between covers, and so on. We refrain from giving now a complete account to these geometric intuitions, we are confident that the algebra developed in this paper may help to shade some light on this topic in some future.

Let us also stress that ideas and results presented here have their birth at the intersection between order theory and the theory of rewriting systems, in the spirit of [15] and [2]. We are not going to emphasize this point of view, however we feel worth to give a rough idea. It was suggested in [4] that the transposition relation between covers is generated by a sort of rewrite system. In section 3 we make explicit such a rewrite system, namely, we define the pushdown relation \rightarrow between covers. We also explicitly introduce the class of pushdown lattices, as our proofs make heavy use its properties. Briefly, a lattice is pushdown if the transposition order on covers is generated by the pushdown relation, thus allowing a sort of local reasoning on the global structure of the covers.

The paper is structured as follows. We recall first the basic concepts concerning finite lattices in Section 2 and we introduce pushdown lattices in Section 3. We give in Section 4 our characterization of finite join semidistributive lattices, and in Section 5 our characterization of lower bounded lattices. We define in Section 6 lattices of the form $\mathbb{C}(L, \gamma)$, that is lattices derived from semidistributive lattices, and prove then that properties such semidistributivity and being lower bounded lift from L to $\mathbb{C}(L, \gamma)$. Finally, in Section 7, we exemplify construction of derived lattices on Newman lattices.

2 Preliminaries

We begin by introducing standard definitions and notation on finite lattices.

Let P be a poset. A *cover* in P is an ordered pair $(\gamma_0, \gamma_1) \in P^2$ such that $\gamma_0 \leq \gamma_1$ and the closed interval $\{x \in L \mid \gamma_0 \leq x \leq \gamma_1\}$ is a two element set $\{\gamma_0, \gamma_1\}$ – in particular $\gamma_0 \neq \gamma_1$. As usual, we shall write $\gamma_0 \prec \gamma_1$ if (γ_0, γ_1) is a cover and say that γ_0 is a lower cover of γ_1 and that γ_1 is an upper cover of γ_0 . We shall denote by $\mathbb{C}(P)$ the set of covers of P and use Greek letters $\gamma, \delta \dots$ to range on covers.

If L is a finite lattice, an element is *join* (resp. *meet*) *irreducible* iff it has a unique lower (resp. upper) cover. We denote by $J(L)$ (resp. $M(L)$) the set of join (resp. meet) irreducible elements of L . If $j \in J(L)$ then j_* denotes the unique element of L such that $j_* \prec j$. If $m \in M(L)$, then m^* denotes the unique element of L such that $m \prec m^*$. Let us introduce the standard arrow relations between join and meet irreducible elements. For $j \in J(L)$ and $m \in M(L)$, we write $j \nearrow m$ iff $j \leq m^*$ and $j \not\leq m$, and $m \searrow j$ if and only if $j_* \leq m$ and $j \not\leq m$; we write $j \nearrow\!\!\! \nearrow m$ (or $m \searrow\!\!\! \searrow j$) iff $j \nearrow m$ and $m \searrow j$.

We finally recall that a lattice is *join semidistributive* if it satisfies the Horn

sentence

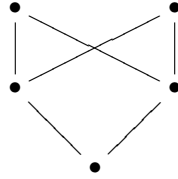
$$x \vee y = x \vee z \Rightarrow x \vee (y \wedge z) = x \vee y. \quad (SD_{\vee})$$

A lattice is *meet semidistributive* if it satisfies the Horn sentence dual of (SD_{\vee}) . It is *semidistributive* iff it is both meet and join semidistributive.

Posets with pullbacks. These posets will play a central role in our development. We introduce them now together with their elementary properties. Recall from [4] that a *hat* in a finite poset P is a triple (u, v, w) such that $u \prec v$, $w \prec v$, and $u \neq w$. An *antihat* in P is defined dually. A *cospan* in P is a triple of elements (u, v, w) such that $u \leq v$ and $w \leq v$; in particular a hat is a particular kind of a cospan. We say that a cospan (u, v, w) has a pullback if the meet $u \wedge w$ exists.

Definition 2.1. We say that a finite poset P has pullbacks if every cospan in P has a pullback. We say that $f : P \longrightarrow Q$ preserves pullbacks iff whenever the pullback $u \wedge w$ of the cospan (u, v, w) exists, then $f(u \wedge w)$ is the pullback of the cospan $(f(u), f(v), f(w))$.

Clearly, a poset has pullbacks iff every finite non empty set admitting an upper bound has a meet. Notice that every meet semilattice is a poset with pullbacks. The following diagram exhibits a poset with pullbacks which is not a meet semilattice.



The following Proposition is almost a reformulation of Definition 2.1.

Proposition 2.2. *A poset P has pullbacks iff every principal ideal of P is a lattice.*

We state next, without proofs, some facts illustrating the specific role of pullbacks of hats among all the pullbacks.

Lemma 2.3. *A finite poset P has pullbacks iff every hat has a pullback.*

Lemma 2.4. *Let P, Q be finite posets with pullbacks. If $f : P \longrightarrow Q$ preserves pullbacks of hats, then it preserves pullbacks.*

The following observations will turn to be more interesting for our goals.

Proposition 2.5. *If a finite poset P has pullbacks, then each connected component of P has a least element.*

Proof. Since a principal ideal is a finite lattice, then it has a least element. Let therefore \perp_x denote the least element of the principal ideal of x . We argue that if x, y belong to the same connected component, then $\perp_x = \perp_y$. To this goal it suffices to establish that $\perp_x = \perp_y$ whenever $z \leq x, y$ for some z . We have $\perp_x \leq z \leq y$, and hence $\perp_x \leq \perp_y$. By symmetry, $\perp_x \leq \perp_y$. \square

Let us say that P has pushouts iff its dual poset P^{op} has pullbacks.

Corollary 2.6. *If a finite poset P has pullbacks and pushouts, then each connected component of P is a lattice.*

Proof. Since P has pushouts each connected component has a maximal element, that is, it is a principal ideal. Since P has pullbacks, such an ideal is a lattice. \square

3 Pushdown Lattices

In the following L will denote a fixed finite lattice. Let us introduce an order relation on the set $\mathbb{C}(L)$ of covers of L : for $\gamma, \delta \in \mathbb{C}(L)$, let $\gamma \leq \delta$ if and only if $\gamma_0 \leq \delta_0$, $\gamma_1 \not\leq \delta_0$, and $\gamma_1 \leq \delta_1$, i.e. if the two covers are perspective and δ transposes down to γ . This is an ordering on $\mathbb{C}(L)$ since L is a lattice: if $\gamma \leq \delta \leq \epsilon$ then clearly $\gamma_i \leq \epsilon_i$ for $i = 0, 1$, and if $\gamma_1 \leq \epsilon_0$, then also $\gamma_1 \leq \delta_1 \wedge \epsilon_0 = \delta_0$, a contradiction. Observe that a sufficient condition for such a relation to be an ordering is that L has pullbacks. When referring to the poset $\mathbb{C}(L)$ we shall mean the pair $\langle \mathbb{C}(L), \leq \rangle$. By $\mathbb{C}(L, \gamma)$ we shall denote the connected component of γ in $\mathbb{C}(L)$. The two projections

$$(\cdot)_i : \mathbb{C}(L) \longrightarrow L, \quad i = 0, 1,$$

sending γ to γ_i , are order preserving. They will play a key role in the rest of the paper.

An order preserving map $\pi : P \longrightarrow Q$ is *conservative* if it strictly preserves the order, i.e. $x \leq y$ and $\pi(x) = \pi(y)$ imply $x = y$, or, equivalently $x < y$ implies $\pi(x) < \pi(y)$. Our first remark is the following:

Lemma 3.1. *In any lattice L the projections $(\cdot)_i$, $i = 0, 1$, are conservative.*

Proof. If $\gamma \leq \delta$ and $\gamma_0 = \delta_0$, then the relations $\delta_0 = \gamma_0 < \gamma_1 \leq \delta_1$ and $\delta_0 < \delta_1$ imply $\delta_1 = \gamma_1$. \square

An order preserving function $\pi : Q \longrightarrow P$ is a *Grothendieck fibration* if for each $\delta \in Q$ the restriction of π to the principal ideal generated by δ is an embedding. Spelled out, this means that $\gamma, \epsilon \leq \delta$ and $\pi(\gamma) \leq \pi(\epsilon)$ implies $\delta \leq \epsilon$.

Definition 3.2. Let us say that a lattice L is a *pushdown* lattice if the projection $(\cdot)_0 : \mathbb{C}(L) \longrightarrow L$ is a Grothendieck fibration.

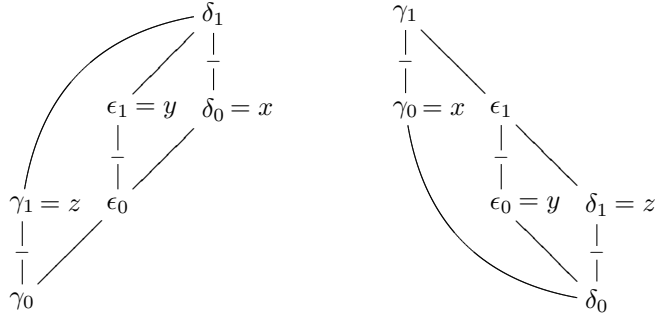
We shall say that L is a *pushup* lattice if the dual L^{op} is pushdown. Spelled out, L is pushdown iff $\gamma, \epsilon \leq \delta$ and $\gamma_0 \leq \epsilon_0$ implies $\gamma \leq \epsilon$, and L is pushup iff

$\delta \leq \gamma, \epsilon$ and $\epsilon_1 \leq \gamma_1$ implies $\epsilon \leq \gamma$. These properties may be simplified even more, for example L is pushdown iff $\gamma, \epsilon \leq \delta$ and $\gamma_0 \leq \epsilon_0$ implies $\gamma_1 \leq \epsilon_1$.

Examples of lattices having these properties arise from some form of semidistributivity.

Lemma 3.3. *If L is join semidistributive lattice, then it is both a pushdown and a pushup lattice.*

Proof. The proof is sketched in the two diagrams below:



Let x, y, z as above on the left: if $\gamma_1 \not\leq \epsilon_1$, then $z \wedge y = \gamma_0$ and consequently $x \vee y = x \vee z$ but $x \vee (y \wedge z) = x \vee \epsilon_0 = x < x \vee y$.

Let x, y, z as above on the right: if $\gamma_0 \not\leq \epsilon_0$, then $z \vee y = \gamma_1$ and consequently $x \vee y = x \vee z$ but $x \vee (y \wedge z) = x \vee \delta_0 = x < \gamma_1 = x \vee y$. \square

By duality, it follows that meet semidistributive lattices are both pushdown and pushup lattices.

To understand the relation of pushdown lattices to hats and antihats as defined in Section 2, let us introduce the relation \rightarrow on $\mathbb{C}(L)$ as follows. Let us write $\delta \xrightarrow{u} \gamma$ if $u \neq \delta_0$, $\gamma_0 = u \wedge \delta_0$, and $\gamma_1 \leq u < \delta_1$. Let us write $\delta \rightarrow \gamma$ if $\delta \xrightarrow{u} \gamma$ for some $u \in L$. Observe that if $\delta \xrightarrow{u} \gamma$, then (u, δ_1, δ_0) is a hat and γ_0 is its pullback. The next Lemma states that in a pushdown lattice there corresponds to a given hat a unique antihat.

Lemma 3.4. *In a pushdown lattice, for each $\delta \in \mathbb{C}(L)$ and $u \in L$ such that $u < \delta_1$ and $u \neq \delta_0$, there exists a unique γ such that $\delta \xrightarrow{u} \gamma$.*

Proof. We must show that $\delta \xrightarrow{u} \gamma, \epsilon$ implies $\gamma = \epsilon$. This is an immediate consequence of $\gamma_0 = u \wedge \delta_0 = \epsilon_0$ and $\gamma, \epsilon \leq \delta$, so that $\gamma \leq \epsilon$ and $\epsilon \leq \gamma$. \square

Remark now that $\delta \rightarrow \gamma$ implies $\gamma < \delta$, in any lattice.

Proposition 3.5. *In a pushdown lattice L the order of $\mathbb{C}(L)$ is the reflexive transitive closure of the relation converse of \rightarrow .*

Proof. By the previous remark, if $\delta \rightarrow^* \gamma$, then $\gamma \leq \delta$, in any lattice. Therefore we shall focus on the converse implication, that is, if $\gamma \leq \delta$, then we can find a

path of the \rightarrow relation from δ to γ . The proof is by induction on the height of the interval $[\gamma_0, \delta_0]$. If $\gamma_0 = \delta_0$, then $\gamma \leq \delta$ implies $\gamma = \delta$.

Let us suppose that $\gamma_0 < \delta_0$, so that $\gamma_1 < \delta_1$ as well. Pick u such that $\gamma_1 \leq u < \delta_1$ with $u \neq \delta_0$. Such a lower cover exists, since $\gamma_1 \not\leq \delta_0$. Let ϵ be determined by the property that $\delta \xrightarrow{u} \epsilon$. Then $\gamma_0 \leq u \wedge \delta_0 = \epsilon_0$, and hence, from the pushdown property, $\gamma \leq \epsilon$. Moreover, $[\gamma_0, \epsilon_0] \subset [\gamma_0, \delta_0]$ and, by the induction hypothesis, $\epsilon \xrightarrow{*} \gamma$. It follows that $\delta \xrightarrow{*} \gamma$. \square

We end this section with a characterization of pushdown lattices in terms of the relation \rightarrow . This was actually our original definition of pushdown lattices: if $\delta \xrightarrow{u} \gamma$, then the cover δ has been lowered to the cover γ by means of the push of u . This intuition is already present in [4].

Proposition 3.6. *A lattice L is a pushdown lattice if and only if $\gamma \leq \delta \xrightarrow{u} \epsilon$ with $\gamma_0 \leq u$ implies $\gamma \leq \epsilon$.*

Proof. Clearly, if a lattice is pushdown and $\gamma \leq \delta \xrightarrow{u} \epsilon$ and $\gamma_0 \leq u$, then $\gamma_0 \leq u \wedge \delta_0 = \epsilon_0$, so that $\gamma \leq \epsilon$.

Conversely, let us suppose that $\gamma, \epsilon \leq \delta$ with $\gamma_0 \leq \epsilon_0$ and that L has the property stated in the Proposition. Observe that this property allows to construct a path $\delta = \theta^0 \xrightarrow{u_1} \theta^1 \dots \theta^{n-1} \xrightarrow{u_n} \theta^n = \epsilon$, as in the proof of Proposition 3.5. Therefore we can inductively observe that if $\gamma \leq \theta^{i-1}$ and $\gamma_0 \leq \epsilon_0 \leq u^i$, then $\gamma \leq \theta^i$. Therefore $\gamma \leq \theta^n = \epsilon$. \square

4 Join Semidistributive lattices

Many characterizations of finite join semidistributive lattices are already available, see for example [1]. In this section we introduce one more characterization. If we look more closely, we are refining a standard characterization of these lattices [8, §2.56], stating that a finite lattice is join semidistributive if for each meet irreducible element m there exists a unique join irreducible element j such that $m \not\leq j$. This characterization may be rephrased in terms of the poset $\mathbb{C}(L)$. The correspondence sending a meet irreducible element m to the cover (m, m^*) establishes a bijection between $M(L)$ and maximal elements of $\mathbb{C}(L)$. A similar bijection may be defined between $J(L)$ and minimal elements of $\mathbb{C}(L)$. With these bijections at hand, we can state the previous characterization as follows: *a finite lattice is join semidistributive iff each maximal element of $\mathbb{C}(L)$ has a least element below it.* We shall see next that, whenever L is join semidistributive, the poset $\mathbb{C}(L)$ has a stronger property, concerning the existence of pullbacks.

An order preserving function $\pi : P \longrightarrow Q$ *creates pullbacks* if and only if whenever $x, y, z \in P$ are such that $x, y \leq z$ and $\pi(x) \wedge \pi(y)$ exists in Q , then there exists a unique $u \leq x, y$ such that $\pi(u) = \pi(x) \wedge \pi(y)$; moreover $u = x \wedge y$. For conservative order preserving maps this condition splits as the conjunction of two conditions, as in the following Lemma.

Lemma 4.1. *A conservative order preserving function $\pi : P \longrightarrow Q$ creates pullbacks if and only if (i) it is a Grothendieck fibration and (ii) if $x, y \leq z \in P$*

and $\pi(x) \wedge \pi(y)$ exists in Q , then there exists $u \leq x, y$ such that $\pi(u) = \pi(x) \wedge \pi(y)$.

Proof. Let us suppose that π creates pullbacks, so that (ii) certainly holds. If $x, y \leq z$ and $\pi(x) \leq \pi(y)$, then $\pi(x) = \pi(x) \wedge \pi(y)$, so that the meet of $\pi(x), \pi(y)$ exists in Q . Let $u \leq x, y$ be such that $\pi(u) = \pi(x) \wedge \pi(y) = \pi(x)$. Since π is conservative, then $u = x$, so that $x \leq y$, exhibiting π as a Grothendieck fibration.

Conversely, let us suppose that (i) and (ii) hold. Let $x, y \leq z$ be such that $\pi(x) \wedge \pi(y)$ exists, and let $u, u' \leq x, y$ be two preimages of such meet. Since π is an embedding when restricted to the principal ideal of z and $\pi(u) = \pi(u')$, then $u = u'$ as well. Let us show that $u = x \wedge y$: if $w \leq x, y$, then $\pi(w) \leq \pi(x), \pi(y)$ hence $\pi(w) \leq \pi(x) \wedge \pi(y) = \pi(u)$. Since $w, u \leq z$, we deduce $w \leq u$ by the pushdown property. \square

An obvious remark, worth recalling at this point, is that if π creates pullbacks and Q has pullbacks, then P has pullbacks as well which are preserved by π . We state next the main result of this section.

Theorem 4.2. *A finite lattice is join semidistributive if and only if $(\cdot)_0 : \mathbb{C}(L) \longrightarrow L$ creates pullbacks.*

Proof. If $(\cdot)_0$ creates pullbacks, then $\mathbb{C}(L)$ has pullbacks. If $m \in \mathbb{C}(L)$ is maximal, then the ideal it generates is a finite meet semilattice and hence it has a least element. Therefore L is join semidistributive.

Conversely, let us suppose that L is join semidistributive. Since $(\cdot)_0$ is conservative and a Grothendieck fibration by Lemma 3.3, it is enough by Lemma 4.1 to show that if $\gamma, \delta \leq \epsilon$, then we can find $\beta \leq \gamma, \delta$ such that $\beta_0 = \gamma_0 \wedge \delta_0$.

Observe that $\gamma_0 \wedge \delta_0 < \gamma_1 \wedge \delta_1$, since $\gamma_1 \vee \epsilon_0 = \delta_1 \vee \epsilon_0 = (\gamma_1 \wedge \delta_1) \vee \epsilon_0 = \epsilon_1$. Let $\beta_0 = \gamma_0 \wedge \delta_0$ and choose β_1 such that $\beta_0 < \beta_1 \leq \gamma_1 \wedge \delta_1$. We claim that $\beta_1 \not\leq \gamma_0$, otherwise $\beta_1 \leq \epsilon_0 \wedge \delta_1 = \delta_0$ and $\beta_1 \leq \gamma_0 \wedge \delta_0 = \beta_0$. It follows that $\beta \leq \gamma$ and, similarly, $\beta \leq \delta$. \square

The rest of this section is devoted to characterizing join semidistributive lattices among pushdown lattices.

Lemma 4.3. *In a pushdown lattice $\gamma < \delta$ implies $\delta \rightarrow \gamma$. In a join semidistributive lattice $\delta \rightarrow \gamma$ implies $\gamma < \delta$.*

Proof. By the pushdown property, if $\gamma < \delta$, then $\gamma_0 < \delta_0$ and $\gamma_1 < \delta_1$.

Since $\gamma_1 \not\leq \delta_0$, then we can find u such $\gamma_1 \leq u < \delta_1$, and therefore iff $\delta \xrightarrow{u} \epsilon$, then $\gamma \leq \epsilon$. Since $\epsilon < \delta$ and $\gamma < \delta$, then $\gamma = \epsilon$ and $\delta \xrightarrow{u} \gamma$.

Let us suppose that the underlying lattice is a join semidistributive lattice, that $\gamma < \epsilon < \delta$, and that $\delta \xrightarrow{v} \gamma$. It follows that $\gamma_1 < \epsilon_1 < \delta_1$ and consequently $\delta_1 \not\leq v$ (otherwise $\delta \leq \gamma$ by the pushdown property). We have $\epsilon_1 \vee v = \epsilon_1 \vee \delta_0 = \delta_1$ and therefore $\delta_1 = \epsilon_1 \vee (v \wedge \delta_0) = \epsilon_1 \vee \gamma_0 = \epsilon_1$, giving a contradiction. \square

Proposition 4.4. *For a pushdown lattice L the following are equivalent:*

1. If $\delta \rightarrow \gamma$ then $\gamma < \delta$,

2. If $v_1 \neq v_2$ and $\delta \xrightarrow{v_i} \gamma^i$ for $i = 1, 2$, then γ_0^1, γ_0^2 form an antichain.
3. Every hat in $\mathbb{C}(L)$ has a pullback.
4. $\mathbb{C}(L)$ has pullbacks.
5. L is join semidistributive.

Proof. (1) implies (2). Let us suppose that $v_1 \neq v_2$, $\delta \xrightarrow{v_i} \gamma^i$, $i = 1, 2$, and $\gamma_0^1 \leq \gamma_0^2$. Then $\gamma^i < \delta$ and $\gamma_0^1 \leq \gamma_0^2$ implies $\gamma^1 \leq \gamma^2$. Either $\gamma^1 < \delta$ is not a cover. Otherwise $\gamma^1 \prec \delta$, $\gamma^1 = \gamma^2$ and $\gamma_0^1 \leq v_1 \wedge v_2$. Let us write γ for $\gamma^1 = \gamma^2$. In this case, however, $(v_1, \delta_1) \xrightarrow{\delta_0} \psi$, where $\psi_0 = \gamma_0 = v_1 \wedge \delta_0$, and $(v_1, \delta_1) \xrightarrow{v_2} \xi$, where $\xi_0 = v_1 \wedge v_2 \geq \gamma_1 > \gamma_0 = \psi_0$. It follows from the pushdown property and $\psi_0 < \xi_0$ that $\psi < \xi$, thus implying that $\psi \leq (v_1, \delta_1)$ is not a cover.

(2) implies (3). Let us suppose that $\gamma, \epsilon \prec \delta$ with $\gamma \neq \epsilon$. As in every pushdown lattice, we can write $\delta \xrightarrow{v} \gamma$ and $\delta \xrightarrow{u} \epsilon$, where u, v, δ_0 are pairwise distinct. We claim that, due to property (2), $\gamma_0 \wedge \epsilon_0 < u \wedge v$. For if $u \wedge v \leq \gamma_0$ then $(u, \delta_1) \xrightarrow{\delta_0} \psi$, where $\psi_0 = u \wedge \delta_0 = \gamma_0$, and $(u, \delta_1) \xrightarrow{v} \xi$, where $\xi_0 = u \wedge v \leq \gamma_0 = \psi_0$. By our assumptions, this implies $v = \delta_0$, a contradiction.

Given that $\gamma_0 \wedge \epsilon_0 < u \wedge v$, let $\beta_0 = \gamma_0 \wedge \epsilon_0$ and pick β_1 such that $\beta_0 \prec \beta_1 < u \wedge v$. If $\beta_1 \leq \delta_0$ then $\beta_1 \leq \delta_0 \wedge u \wedge v = \gamma_0 \wedge \epsilon_0 = \beta_0$. Thus $\beta \leq \delta$ and $\beta_0 \leq \gamma_0$ imply $\beta \leq \gamma$. Similarly $\beta \leq \epsilon$. It is argued that $\beta = \gamma \wedge \epsilon$ as in Lemma 4.1.

(3) implies (4). By Proposition 2.3.

(4) implies (5). We claim that if $\mathbb{C}(L)$ has pullbacks, then they are preserved by $(\cdot)_0$. Consequently, using the fact that $(\cdot)_0$ is conservative Grothendieck fibration, $(\cdot)_0$ also creates pullbacks. If $\pi \leq \gamma, \epsilon \leq \delta$ and $\pi_0 < \gamma_0 \wedge \epsilon_0$, then $\pi_1 \not\leq \gamma_0 \wedge \epsilon_0$ and $\gamma_0 \wedge \epsilon_0 < \pi_1 \vee (\gamma_0 \wedge \delta_0) \leq \gamma_1, \epsilon_1$. Therefore $\gamma_0 \wedge \epsilon_0 < \gamma_1 \wedge \epsilon_1$ so that, as in the proof of Theorem 4.2, there exists $\beta \leq \gamma, \epsilon$ such that $\beta_0 = \gamma_0 \wedge \epsilon_0$.

(5) implies (1). By Lemma 4.3. \square

5 Lower Bounded Lattices

We recall that a finitely generated lattice L is said to be *lower bounded* if there exists a lattice morphism from a freely generated lattice $f : \mathcal{F}(X) \longrightarrow L$ such that, for each $y \in L$, the set $\{x \mid y \leq f(x)\}$ is either empty or has a least element [13]. Upper boundedness is the dual notion of lower boundedness, and a lattice is said to be bounded if it is both lower and upper bounded.

There are already many characterizations of finite lower bounded lattices [6, 7, 18] and this concept has also found applications within unexpected branches of lattice theory [9]. In this section we develop further the tools used in [4] to prove that lattices in the class \mathcal{HH} are bounded. In this way we shall obtain a new characterization of lower bounded lattices. Our starting point will be the following classical result [8, §2.39]:

Theorem 5.1 (Johnson, Nation). *A finite lattice is lower bounded if and only the join dependency relation between join irreducible elements contains no cycles.*

In order to use of this result, we must define the join dependency relation, hereby denoted D . The reader can find in [8, §2.3] its standard definition. Again, we shall use, as a definition, the characterization of the relation D in terms of the arrows relations [8, §11.10]:

Definition 5.2. For $j, k \in J(L)$ we let jDk iff $j \neq k$ and, for some $m \in M(L)$, $j \nearrow m$ and $m \searrow k$.

Let us introduce some more relations between join irreducible elements of a finite lattice:

Definition 5.3. We let:

- jAk iff $j \neq k$ and, for some $m \in M(L)$, $j \nearrow m$ and $m \searrow\!\!\!\searrow k$,
- jBk iff $j \neq k$ and, for some $m \in M(L)$, $j \nearrow\!\!\!\nearrow m$ and $m \searrow k$.
- jCk iff either jAk or jBk .

Let us remark that these relations are already known for semidistributive lattices [8, §2.5]. However, the definition presented here makes sense in any finite lattice. The next Lemma shows that the D relation may be replaced by the C relation defined above.

Lemma 5.4. *The D relation has a cycle if and only if the C relation has a cycle. Consequently a finite lattice is lower bounded if and only if the C relation contains no cycle.*

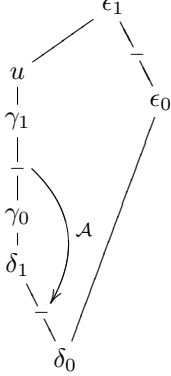
Proof. Let us suppose that jDk and let $m \in M(L)$ be such that $j \nearrow m$ and $m \searrow k$. Choose $l \in J(L)$ such that $m \searrow\!\!\!\searrow l$. If $l \in \{j, k\}$, then jAk or jBk . If $l \neq j, k$, then jAl and lBk . Therefore one step of the relation D may be replaced by at most two steps of the relation C . Conversely, since $C \subseteq D$, every C -cycle gives rise to a D -cycle. \square

Let us introduce a number of relations. We shall need first the dual of the relation \rightarrow : let us write $\delta \xrightarrow{u} \gamma$ if $\delta_0 \prec u \leq \gamma_0$ and $\gamma_1 = u \vee \delta_1$. Also let us write $\delta \rightarrow \gamma$ if $\delta \xrightarrow{u} \gamma$ for some $u \in L$. A remark is due now: if a lattice is join semidistributive, then $\delta \rightarrow \gamma$ implies $\gamma \rightarrow \delta$. Indeed, let us suppose that $\delta \xrightarrow{u} \gamma$ and let us choose $w \in L$ such that $\gamma_0 \prec w \leq \delta_0$. If $w \vee \gamma_1 < \delta_1$, then for $x = w \vee \gamma_1$, $y = u$ and $z = \delta_0$ the implication (SD_{\vee}) fails. In particular we have that in a semidistributive lattice $\delta \rightarrow \gamma$ is equivalent to $\gamma \rightarrow \delta$.

Definition 5.5. Let $\gamma, \delta \in \mathbb{C}(L)$. We let $\gamma \mathcal{A} \delta$ if and only if there exists $\epsilon \in \mathbb{C}(L)$ and $u \in L$ such that $\epsilon \xrightarrow{u} \delta$ and $\delta_1 \leq \gamma_0 \prec \gamma_1 \leq u$. The dual relation is defined as follows: $\gamma \mathcal{B} \delta$ if and only if there exists $\epsilon \in \mathbb{C}(L)$ and $u \in L$ such that $\epsilon \xrightarrow{u} \delta$ and $u \leq \gamma_0 \prec \gamma_1 \leq \delta_1$.

Intuitively, the \mathcal{A}, \mathcal{B} relations express the dependency relation of covers in terms of minimal pentagons or, as they are called in [4], facets. The following

picture illustrates this point with the \mathcal{A} relation:



The next Lemmas exemplify the connections between the relations A and \mathcal{A} . The Lemmas apply to arbitrary finite lattices.

Lemma 5.6. *Let $j, k \in J(L)$ such that for some $\gamma, \delta \in \mathbb{C}(L)$, $\gamma \mathcal{A} \delta$, $(j_*, j) \leq \gamma$ and $(k_*, k) \leq \delta$. Then $j \mathcal{A} k$.*

Proof. Let γ, δ such that $\gamma \mathcal{A} \delta$ so that $\epsilon \rightsquigarrow \delta$, for some $u \in L$ and $\epsilon \in \mathbb{C}(L)$. Observe first that $j \neq k$, since $k \leq \delta_1 \leq \gamma_0$ and $j \not\leq \gamma_0$. Let $\mu = (m, m^*)$ be maximal above ϵ , then $j \leq \gamma_1 \leq u \leq \epsilon_1 \leq m^*$. Let us suppose that $j \leq m$: then $j \leq u \wedge \epsilon_1 \wedge m = u \wedge \epsilon_0 = \delta_0 \leq \gamma_0$. This is a contradiction since $(j_*, j) \leq \gamma$. We have shown that $j \not\leq m$; since $(k_*, k) \leq \delta \leq \epsilon \leq (m, m^*)$, we have $m \not\leq k$, and therefore $j \mathcal{A} k$. \square

Lemma 5.7. *Let $j \in J(L)$ and $\delta \in \mathbb{C}(L)$ be such that $\delta_1 = j \vee \delta_0$. Then there exists $n \geq 0$ and a sequence (δ^i, γ^i) , $i = 0, \dots, n$, such that*

1. $\gamma^0 = (j_*, j)$ and $\delta^n = \delta$,
2. $\delta_1^i = j \vee \delta_0^i$ and $\delta^i \rightsquigarrow \gamma^i$, for $i = 0, \dots, n$,
3. $\delta^{i-1} \mathcal{A} \gamma^i$, for $i = 1, \dots, n$.

Proof. The proof is by induction on the height of the interval $[j, \delta_1]$. If $j = \delta_1$, then δ_1 is join irreducible so that $\delta_0 = j_*$ and the statement holds with $n = 0$ and $\gamma^0 = \delta^0 = \delta$.

Otherwise $j < \delta_1$: let $u \in L$ be such that $j \leq u < \delta_1$. Since $j \not\leq \delta_0$, we have $u \neq \delta_0$: let therefore $\gamma_0 = u \wedge \delta_0$ and recall in the following that $\gamma_0 < u, \delta_0$. Let $\delta'_1 = j \vee \gamma_0$ and observe that $\gamma_0 < \delta'_1$: if $\gamma_0 = \delta'_1 = j \vee \gamma_0$, then $j \leq \gamma_0 \leq \delta_0$, which is not the case. Let $\delta'_0 \in L$ such that $\gamma_0 \leq \delta'_0 < \delta'_1$. Since $\delta'_1 \leq u < \delta_1$, we have $[j, \delta'_1] \subset [j, \delta_1]$ and we use the inductive hypothesis to find a sequence (δ^i, γ^i) , $i = 0, \dots, m$, satisfying (ii) and (iii) and such that $\gamma^0 = (j_*, j)$ and $\delta^m = \delta'$.

We distinguish now two cases. Case 1. If $\gamma_0 = \delta'_0$ then $\delta \rightarrow \delta'$, so that we can let $n = m$ and append the pair (δ, γ^m) to the sequence (δ^i, γ^i) , $i = 0, \dots, m-1$ to obtain the desired sequence. Case 2. Otherwise $\gamma_0 < \delta_0$ and

we can chose γ_1 such that $\gamma_0 \prec \gamma_1 \leq \delta_0$. We have $\delta \rightarrow \gamma$ and $\delta' \mathcal{A} \gamma$. Therefore we let $n = m + 1$ and append the pair (δ, γ) to the sequence (γ^i, δ^i) to obtain a sequence satisfying (i), (ii) and (iii). \square

Following [4], we are ready to introduce strict facet labellings.

Definition 5.8. A *strict lower facet labelling* of a lattice L is a function $f : \mathbb{C}(L) \longrightarrow \mathbb{N}$ such that $f(\gamma) = f(\delta)$ if $\delta \rightarrow \gamma$ and $f(\gamma) < f(\delta)$ if $\delta \mathcal{A} \gamma$. A *strict upper facet labelling* is defined dually: it is a function $f : \mathbb{C}(L) \longrightarrow \mathbb{N}$ such that $f(\gamma) = f(\delta)$ if $\delta \rightarrow \gamma$ and $f(\gamma) < f(\delta)$ if $\delta \mathcal{B} \gamma$.

A *strict facet labelling* of a lattice L is a function $f : \mathbb{C}(L) \longrightarrow \mathbb{N}$ which is both a strict lower facet labelling and a strict upper facet labelling.

Lemma 5.9. *If L is a pushdown lattice, f is a strict lower facet labelling, and $\gamma \leq \delta$, then $f(\gamma) = f(\delta)$.*

Proof. If $\gamma \leq \delta$ then by Lemma 4.3 we can find a path of the relation \rightarrow from δ to γ . The statement follows since f is constant on the relation \rightarrow . \square

Lemma 5.10. *Let L be a lattice with a strict lower facet labelling f . Let $j \in J(L)$ and $\delta \in \mathbb{C}(L)$ be such that $\delta_1 = j \vee \delta_0$. Then $f(\delta) \leq f(j_*, j)$ and $f(\delta) = f(j_*, j)$ implies $(j_*, j) \leq \delta$.*

Proof. Let $n \geq 0$ and (δ^i, γ^i) , $i = 0, \dots, n$, be as in the statement of Lemma 5.7. We have $f(\delta^i) = f(\gamma^i)$, $i = 0, \dots, n$, and $f(\gamma^i) < f(\delta^{i-1})$ for $i = 1, \dots, n$. We deduce that $f(\delta) = f(\delta^n) \leq f(\gamma^0) = f(j_*, j)$ and, $f(\delta) < f(j_*, j)$ if $n \geq 1$. Therefore, if $f(\delta) = f(j_*, j)$, then $n = 0$, $\delta \overset{*}{\rightarrow} (j_*, j)$ and $(j_*, j) \leq \delta$. \square

By duality, we obtain the following Corollary.

Corollary 5.11. *Let L be a pushup lattice with an upper strict facet labelling f . Let $m \in M(L)$ and $\delta \in \mathbb{C}(L)$ such that $m \wedge \delta_1 = \delta_0$. Then $f(\delta) \leq f(m, m^*)$ and $f(\delta) = f(m, m^*)$ implies $\delta \leq (m, m^*)$.*

We are ready to achieve the first goal of this section, Proposition 5.14.

Lemma 5.12. *If L is a join semidistributive lattice with a strict lower facet labelling, then jAk implies $f(k_*, k) < f(j_*, j)$.*

Proof. If jAk then for some m , $j \nearrow m$ and $(k_*, k) \leq (m, m^*)$. It follows that $f(k_*, k) = f(m_*, m) \leq f(j_*, j)$. The latter is actually an inequality, since otherwise $j_* \leq m$ and, consequently, $j = k$ by the uniqueness of j such that $j \nearrow m$. \square

The next Lemma is not a mere consequence of duality.

Lemma 5.13. *If L is a join semidistributive lattice with a strict upper facet labelling f , then jBk implies $f(k_*, k) < f(j_*, j)$.*

Proof. Let $j, k \in J(L)$ be such that jBk , that is, $j \neq k$ and, for some $m \in M(L)$, $j \not\leq m$ and $m \searrow k$. Then Lemma 5.9 $f(j_*, j) = f(m, m^*)$, since L is a pushup lattice by Lemma 3.3. Lemma 5.11 ensures that $f(k_*, k) \leq f(m, m^*) = f(j_*, j)$ and $(k_*, k) \leq (m, m^*)$ if this is an equality. The latter case cannot happen, since otherwise $m \searrow k$ and $j = k$ by join semidistributivity. Hence $f(k_*, k) < f(m, m^*) = f(j_*, j)$. \square

Proposition 5.14. *A join semidistributive lattice with a strict facet labelling is lower bounded.*

Proof. It is enough to show that the relation C has no cycles, for which we shall argue that jCk implies $f(k_*, k) < f(j_*, j)$. If jAk then we use Lemma 5.12. If jBk then we can use Lemma 5.13. \square

Our next goal is to prove the converse of Proposition 5.14.

Lemma 5.15. *Let L be a join semidistributive lattice. If $j, k \in J(L)$ are such that for some $\gamma, \delta \in \mathbb{C}(L)$, $\gamma \mathcal{B} \delta$, $(j_*, j) \leq \gamma$ and $(k_*, k) \leq \delta$, then jBk .*

Proof. Let γ, δ be such that $\gamma \mathcal{B} \delta$: for some $u \in L$ and $\epsilon \in \mathbb{C}(L)$, $\epsilon \rightarrow_u \delta$ and $\epsilon_0 \prec u \leq \gamma_0 \prec \gamma_1 \leq \delta_0$. Let j, k as in the statement, and observe that $j \neq k$, since $j \leq \gamma_1 \leq \delta_0$ and $k \not\leq \delta_0$. Let $\mu = (m, m^*)$ be maximal above γ , so that $j \not\leq m$, and let $l \in J(L)$ such $(l_*, l) \leq \epsilon$. We have $m \geq \gamma_0 \geq u \geq \epsilon_0 \geq l_*$. If $m \geq l$, then $m \geq u \vee \epsilon_0 \vee l = u \vee \epsilon_1 = \delta_1$; this in turn implies that $m \geq \delta_1 \geq \gamma_0$, a contradiction. Hence $m \not\geq l$ and $m \searrow l$. We have therefore jBl . It is now easy to see that $l = k$, since $(l_*, l) \leq \delta$ and $(k_*, k) \leq \delta$ imply $l = k$ by join semidistributivity. \square

Proposition 5.16. *A lower bounded lattice has a strict facet labelling.*

Proof. If L is bounded then it is join semidistributive and the join dependency relation D is acyclic. Let us denote by \trianglelefteq its reflexive and transitive closure, and let $g : J(L) \longrightarrow \mathbb{N}$ be an antilinear extension of the poset $\langle J(L), \trianglelefteq \rangle$. That is, g is such that $g(y) < g(x)$ whenever xDy . Define $f : \mathbb{C}(L) \longrightarrow \mathbb{N}$ as follows: $f(\delta) = g(j(\delta))$ where $j(\delta)$ is the unique join irreducible such that $(j_*(\delta), j(\delta)) \leq \delta$. Then f is a strict facet labelling. It is a lower facet labelling, since if $\delta \rightarrow \gamma$ then $j(\delta) = j(\gamma)$, and since $\delta \mathcal{A} \gamma$, then $j(\delta) \mathcal{A} j(\gamma)$, by Lemma 5.6, so that $f(\gamma) < f(\delta)$. It is an upper strict facet labelling, since $\delta \rightarrow_u \gamma$ implies $j(\gamma) = j(\delta)$ and $\delta \mathcal{B} \gamma$ implies $j(\delta) \mathcal{B} j(\gamma)$, by Lemma 5.15. Thus again $f(\gamma) < f(\delta)$. \square

We end this section collecting the observations presented so far into a main result:

Theorem 5.17. *A finite lattice is lower bounded if and only if it is join semidistributive and has a strict facet labelling.*

Finally, we observe that from Theorem 5.17 it is quite immediate to derive a standard result by Day, see [8, §2.64], stating that a semidistributive lower bounded finite lattice is bounded. Recall now that in a semidistributive lattice $\delta \rightarrow \gamma$ holds if and only if $\gamma \rightarrow \delta$ hold. Theorem (5.17) leads to a characterization of bounded lattices, that we shall rephrase in a language closer to [4]. Call a *facet* a quadruple of distinct covers $(\delta^0, \delta^1, \gamma^0, \gamma^1)$ such that

$$\delta_1^0 = \delta_1^1 \text{ and } \gamma_0^0 = \gamma_0^1, \quad \delta^0 \xrightarrow{\delta_0^1} \gamma_0 \text{ and } \delta^1 \xrightarrow{\delta_0^0} \gamma^1.$$

Say that ϵ is interior to the facet $(\delta^0, \delta^1, \gamma^0, \gamma^1)$ if $\gamma_1^0 \leq \epsilon_0 \prec \epsilon_1 \leq \delta_1^1$.

Theorem 5.18. *A lattice is bounded if and only if it is semidistributive and there exists a function $f : \mathbb{C}(L) \rightarrow \mathbb{N}$ such that for each facet $(\delta^0, \delta^1, \gamma^0, \gamma^1)$ $f(\delta^0) = f(\gamma^0)$ and $f(\delta^1) = f(\gamma^1)$, and moreover $f(\delta^1), f(\gamma^0) < f(\epsilon)$ whenever ϵ is interior to such a facet.*

6 Derived Semidistributive Lattices

The main result of this section is Theorem 6.5 stating that a poset of the form $\mathbb{C}(L, \gamma)$, $\gamma \in \mathbb{C}(L)$, is a semidistributive lattice whenever L is a finite semidistributive. Observe that if $\gamma = (j_*, j)$ with $j \in J(L)$ and L is join semidistributive, then $\mathbb{C}(L, \gamma)$ is the set $\{\delta \in \mathbb{C}(L) \mid \gamma \leq \delta\}$. With this in mind we observe:

Proposition 6.1. *If L is a semidistributive lattice, then $\mathbb{C}(L, \gamma)$ is a lattice, for each $\gamma \in \mathbb{C}(L)$.*

Proof. $\mathbb{C}(L)$ has pullbacks and pushouts, hence $\mathbb{C}(L, \gamma)$ is a lattice by Corollary 2.6. \square

We shall call $\mathbb{C}(L, \gamma)$ the semidistributive lattice *derived* from L and γ . We study next additional properties of the lattices of the form $\mathbb{C}(L, \gamma)$. We begin with the following Lemma.

Lemma 6.2. Let L be a join semidistributive lattice, and let $\gamma, \delta, \epsilon \in \mathbb{C}(L)$ such that $\gamma \leq \delta$, $\epsilon \prec \delta$, and $\gamma \not\leq \epsilon$. Then $\gamma_0 \vee \epsilon_1 = \delta_1$.

Proof. Clearly $\gamma_1 \vee \epsilon_1 \leq \delta_1$ and we pretend that this is an equality. If this is the case, then $\delta_1 = \gamma_1 \vee \epsilon_1 = \delta_0 \vee \epsilon_1$ implies that $\delta_1 = (\gamma_1 \wedge \delta_0) \vee \epsilon_1 = \gamma_0 \vee \epsilon_1$.

Since $\epsilon \prec \delta$ in $\mathbb{C}(L)$, there exists $u \in L$ such that $\delta \xrightarrow{u} \epsilon$; recall also from Proposition 4.4 that an u with this property is unique. Hence, if $\gamma_1 \vee \epsilon_1 < \delta_1$, then there exists a $w \in L$ such that $\gamma_1 \vee \epsilon_1 \leq w \prec \delta_1$, and from $\epsilon_1 \leq w \prec \delta_1$ we deduce $w = u$. Hence it also follows that $\gamma_0 \leq \gamma_1 \leq u$ and, by the equivalent pushdown property stated in Proposition 3.6, $\gamma \leq \epsilon$. \square

In the following, we shall use capital Greek letters to range on elements of $\mathbb{C}(\mathbb{C}(L, \gamma))$. Observe that the next Propositions make sense: if L is join semidistributive then $\mathbb{C}(L, \gamma)$ is a finite poset with pullbacks, which is enough to ensure that the relation \leq on $\mathbb{C}(\mathbb{C}(L, \gamma))$ is a partial ordering.

Proposition 6.3. *If L is a join semidistributive lattice then the projection*

$$(\cdot)_0 : \mathbb{C}(\mathbb{C}(L, \gamma)) \longrightarrow \mathbb{C}(L, \gamma)$$

is a Grothendieck fibration.

Proof. Let $\Gamma, \Psi, \Delta \in \mathbb{C}(\mathbb{C}(L, \gamma))$ be such that $\Gamma, \Psi \leq \Delta$ and $\Gamma_0 \leq \Psi_0$. We shall prove that $\Gamma_1 \leq \Psi_1$, thus implying that $\Gamma \leq \Psi$.

By definition, $\Delta_0 \prec \Delta_1$, $\Gamma_1 \leq \Delta_1$, $\Gamma_1 \not\leq \Delta_0$ and similarly $\Psi_1 \leq \Delta_1$ but $\Psi_1 \not\leq \Delta_0$. Let in the statement of Lemma 6.2 $\gamma = \Gamma_1$ ($\gamma = \Psi_1$), $\delta = \Delta_1$, $\epsilon = \Delta_0$, and deduce $\Gamma_{1,0} \vee \Delta_{0,1} = \Delta_{1,1} = \Psi_{1,0} \vee \Delta_{0,1}$. Hence $(\Gamma_{1,0} \wedge \Psi_{1,0}) \vee \Delta_{0,1} = \Delta_{1,1}$ and $(\Gamma_{1,0} \wedge \Psi_{1,0}) \not\leq \Delta_{0,1}$. Suppose on the other hand that $(\Gamma_1 \wedge \Psi_1)_0 = \Gamma_{1,0} \wedge \Psi_{1,0} < \Gamma_{1,0}$: it follows then that $\Gamma_0 \leq \Gamma_1 \wedge \Psi_1 < \Gamma_1$ $\Gamma_0 = \Gamma_1 \wedge \Psi_1$, since $\Gamma_0 \prec \Gamma_1$. In particular $(\Gamma_1 \wedge \Psi_1)_0 = \Gamma_{0,0} \leq \Delta_{0,1}$, a contradiction. We have therefore $\Gamma_{1,0} \wedge \Psi_{1,0} = \Gamma_{1,0}$, that is, $\Gamma_{1,0} \leq \Psi_{1,0}$. Considering that $\Gamma_1, \Psi_1 \leq \Delta_1$ we can use pushdown property (of L) to deduce $\Gamma_1 \leq \Psi_1$. \square

Proposition 6.4. *If L is join semidistributive, then the projection $(\cdot)_0$ creates pullbacks.*

Proof. Since the projection $(\cdot)_0$ is a Grothendieck fibration, by Lemma 4.1 it is enough to prove that if $\Gamma, \Psi \leq \Delta$, then there exists a $\Upsilon \leq \Gamma, \Psi$ such that $\Upsilon_0 = \Gamma_0 \wedge \Psi_0$.

To this goal, we observe first that $\Gamma_0 \wedge \Psi_0 < \Gamma_1 \wedge \Psi_1$: as in proof of the previous Proposition, we have $(\Gamma_1 \wedge \Psi_1)_0 \vee \Delta_{0,1} = \Delta_{1,1}$, $(\Gamma_1 \wedge \Psi_1)_0 \not\leq \Delta_{0,1}$, and consequently $\Gamma_1 \wedge \Psi_1 \not\leq \Delta_0$ since $(\cdot)_0$ is conservative.

Therefore we can choose (Υ_0, Υ_1) such that $\Gamma_0 \wedge \Psi_0 = \Upsilon_0 \prec \Upsilon_1 \leq \Gamma_1 \wedge \Psi_1$ and observe that $\Upsilon_1 \leq \Gamma_0$ iff $\Upsilon_1 \leq \Delta_0$ iff $\Upsilon_1 \leq \Psi_0$ iff $\Upsilon_1 \leq \Gamma_0 \wedge \Psi_0$, which is not the case. We have therefore $\Upsilon \leq \Gamma, \Delta$ with $\Upsilon_0 = \Gamma_0 \wedge \Delta_0$. \square

Using Proposition 6.4, also in its dual form, we arrive to the first achievement of this section.

Theorem 6.5. *If L is a semidistributive lattice, then $\mathbb{C}(L, \gamma)$ is a semidistributive lattice, for each $\gamma \in \mathbb{C}(L)$.*

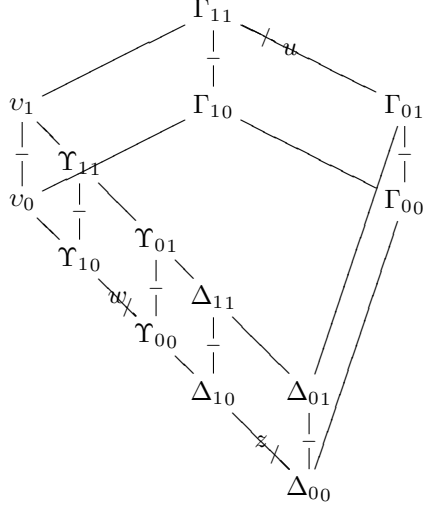
We shall use next the characterization of bounded lattice of Theorem 5.18 to obtain the second main result of this section.

Theorem 6.6. *If L is a finite bounded lattice, then so is $\mathbb{C}(L, \gamma)$, for each $\gamma \in \mathbb{C}(L)$.*

Proof. Let $f : \mathbb{C}(L) \longrightarrow \mathbb{N}$ be a strict facet labelling. Since a cover in $\mathbb{C}(L)$ is of the form (Γ_0, Γ_1) with $\Gamma_1 \xrightarrow{u} \Gamma_0$ for a unique u lower cover of $\Gamma_{1,1}$, we define a function $F : \mathbb{C}(\mathbb{C}(L, \gamma)) \longrightarrow \mathbb{N}$ by

$$F(\Gamma) = f(u, \Gamma_{1,1}).$$

Observe that if $\Gamma_0 \xrightarrow{w} \Gamma_1$, then $f(\Gamma_{0,0}, w) = f(u, \Gamma_{1,1}) = F(\Gamma)$. Let us prove that this is a strict facet labelling. To this goal, let us suppose that $\Gamma \xrightarrow{u} \Delta$ and $\Gamma_1 \leq \Upsilon_0 \prec \Upsilon_1 \leq v$, as sketched in the next diagram:



Let us also suppose that $\Gamma_1 \xrightarrow{u} \Gamma_0$, $\Delta_0 \xrightarrow{z} \Delta_1$, and $\Upsilon_0 \xrightarrow{w} \Upsilon_1$, so that $F(\Gamma) = f(u, \Gamma_{1,1})$, $F(\Delta) = f(\Delta_{0,0}, z)$, and $F(\Upsilon) = F(\Upsilon_{0,0}, w)$.

Recall that $\Delta_0 = v \wedge \Gamma_0$ and therefore $\Delta_{0,0} = v_0 \wedge \Gamma_{0,0} = v_0 \wedge \Gamma_{1,0} \wedge u = v_0 \wedge u$. Since $\Delta_0 \prec z \leq v_0$, then $z \not\leq u$. We have therefore $z \leq \Gamma_{1,1}$, $\Delta_{0,0} \leq u$, $z \not\leq u$, that is $(\Delta_{0,0}, z) \leq (u, \Gamma_{1,1})$. Consequently, $F(\Delta) = f(\Delta_{0,0}, z) = f(u, \Gamma_{1,1}) = F(\Gamma)$.

In order to show that $F(\Delta) < F(\Upsilon)$ it is enough to show that if $j, k \in J(L)$, $(j_*, j) \leq (\Upsilon_0, w)$, and $(k_*, k) \leq (u, \Gamma_{1,1})$, then jAk . It follows then, by Lemma 5.12, that

$$F(\Gamma) = f(u, \Gamma_{1,1}) = f(k_*, k) < f(j_*, j) \leq f(\Upsilon_0, w) = F(\Upsilon).$$

Let $m \in M(L)$ such that $(u, \Gamma_{1,1}) \leq (m, m^*)$. We have $j \leq w \leq \Gamma_{1,1} \leq m^*$, and if $j \leq m$, then $j \leq v_0 \wedge \Gamma_{1,1} \wedge m = v_0 \wedge u = \Delta_{0,0} \leq \Upsilon_{0,0}$, a contradiction.

By duality, it also follows that $F(v, \Gamma_1) < F(\Upsilon)$. \square

7 Derived Lattices of Newman Lattices

We refer the reader to [2, 3, 5, 17] for introductory readings on Newman lattices. In this section we explicitly compute derived lattices $\mathbb{C}(L, \alpha)$ when L is a permutohedron or an associahedron and α is an atom.²We shall see that these derived lattices are again permutohedra (respectively, associahedra) in one dimension less. We remark therefore a peculiar property of these lattices, they

²We say that $\alpha \in \mathbb{C}(L)$ is an atom if α_0 is the bottom of the lattice and α_1 is an atom in the usual sense.

are regular, meaning that $\mathbb{C}(L, \alpha)$ does not depend on the choice of the atom α . We shall exhibit later a semidistributive lattice – not complemented – that it not regular. Regularity is a property reminiscent of Boolean algebras: if \mathcal{B}_n is the Boolean algebra with n -atoms, α being one of them, then the equality $\mathbb{C}(\mathcal{B}_n, \alpha) = \mathcal{B}_{n-1}$ holds up to isomorphism. More generally:

Proposition 7.1. *If α is an atom of a distributive lattice L , then the projection $(\cdot)_0$ from $\mathbb{C}(L, \alpha)$ to the lower set $\{x \in L \mid \alpha_1 \not\leq x\}$ is an isomorphism.*

The Proposition depends on modularity, since if $\alpha_1 \not\leq x$, then $x \prec x \vee \alpha_1$.

In the proofs to follow we shall make intensive use of the category of finite ordinals and functions among them, a skeleton of the category of finite sets and functions. To this goal, let $[n]$ be the set $\{1, \dots, n\}$ and, for $i \in [n]$, denote by $\hat{i}_n : [n-1] \longrightarrow [n]$ the unique order preserving injection whose image is $[n] \setminus \{i\}$. For $k \in [n-1]$ denote by $N_n^k : [n] \longrightarrow [n-1]$ the unique order preserving surjection such that $N_n^k(k) = N_n^k(k+1)$. As the subscripts n will always be understood from the context, we shall omit them and write only \hat{i} and N^k .

Proposition 7.2. *Let \mathcal{S}_n be the permutohedron on n letters (i.e. the weak Bruhat order on permutations on n elements). If α is an atom of \mathcal{S}_n then $\mathbb{C}(\mathcal{S}_n, \alpha)$ is isomorphic to \mathcal{S}_{n-1} .*

Proof. As usual we represent a permutation $w \in \mathcal{S}_n$ as the word $w(1) \dots w(n) = w_1 \dots w_n$. An *increase* of w is an index $i \in \{1, \dots, n-1\}$ such that $w_i < w_{i+1}$. If i is an increase of w and σ^i denotes the exchange permutation $(i, i+1)$, then we represent the cover $w \prec w \circ \sigma^i$ of \mathcal{S}_n by the pair (w, i) . Every cover arises in this way.

Remark now that a cover (w, i) is perspective to the atom (\perp, σ^k) if and only if $w_i = k$ and $w_{i+1} = k+1$. If (w, i) is such a cover, then we define $\psi^k(w, i)$ as the composal

$$\begin{array}{ccc} [n] & \xrightarrow{w} & [n] \\ \widehat{i+1} \uparrow & & \downarrow N^{k+1} \\ [n-1] & \xrightarrow{\psi^k(w, i)} & [n-1] \end{array} \quad (1)$$

For example, $\psi^1(35124, 3)$ is the permutation 2413. Let us remark that $\psi^k(w, i)$ is injective, hence $\psi^k(w, i) \in \mathcal{S}_{n-1}$. If not, there exists $x, y \in [n]$ such that $x, y, i+1$ are pairwise distinct and $N^{k+1}(w_x) = N^{k+1}(w_y)$. But this may happen only if $\{w_x, w_y\} = \{k, k+1\}$ and, by the assumption on the permutation w , this happens exactly when $\{x, y\} = \{i, i+1\}$.

It is easily seen that ψ^k is a bijection from $\mathbb{C}(\mathcal{S}_n, (\perp, \sigma^k))$ to \mathcal{S}_{n-1} : if $u \in \mathcal{S}_{n-1}$ then (w, i) , where i is determined by $u_i = k$, and where

$$w_j = \begin{cases} k+1, & j = i+1, \\ \widehat{k+1}(u_{N^{i+1}(j)}), & \text{otherwise,} \end{cases}$$

is the unique cover of \mathcal{S}_n which is sent by ψ^k to u .

To prove that ψ^k is an order isomorphism, we prove that $(w, i) \prec (w', i')$ if and only if $\psi^k(w, i) \prec \psi^k(w', i')$. This equivalence is an immediate consequence of the following two claims.

Claim 7.3: For $j \in [n]$, $j \neq i$, j is an increase of w if and only if $N^{i+1}(j)$ is an increase of $\psi^k(w, i)$.

Define

$$(x, y) = \begin{cases} (i, j+1), & j = i+1, \\ (j, j+1), & \text{otherwise.} \end{cases}$$

Observe that the assumption on the index i , namely that $w_i = k < k+1 = w_{i+1}$, implies (i) $i+1 \notin \{x, y\}$, $\widehat{i+1}(N^{i+1}(j)) = x$, and $\widehat{i+1}(N^i(j)+1) = y$, (iii) $w_j < w_{j+1}$ if and only if $w_x < w_y$. By (i) and (ii) we have two commuting diagrams of order preserving maps

$$\begin{array}{ccc} [x, y] & \xrightarrow{\quad\quad\quad} & [n] \\ \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) & & \widehat{i+1} \uparrow \\ [N^{i+1}(j), N^{i+1}(j)+1] & \xrightarrow{\quad\quad\quad} & [n-1] \end{array}$$

By pasting along $\widehat{i+1}$ this diagram with diagram (1) defining $\psi^k(w, i)$, we obtain that $w_j < w_{j+1}$ iff $w_x < w_y$ iff $\psi^j(w, i)_{N^{i+1}(j)} < \psi^j(w, i)_{N^{i+1}(j)+1}$.
□ *Claim*

Claim 7.4: Given an increase $j \neq i$ of w , let w' and i' be determined by the the pushup relation $(w, i) \xrightarrow{w \circ \sigma^j} (w', i')$. Then

$$\psi^k(w', i') = \psi^k(w, i) \circ \sigma^{N^{i+1}(j)}.$$

Let us first compute w' and i' in the pushup relation $(w, i) \xrightarrow{w \circ \sigma^j} (w', i')$. If $|i-j| > 1$, then $w \circ \sigma^i \circ \sigma^j = w \circ \sigma^j \circ \sigma^i$, so that $w' = w \circ \sigma^j$ and $i' = i$. Otherwise $|i-j| = 1$, $w \circ \sigma^i \circ \sigma^j \circ \sigma^i = w \circ \sigma^j \circ \sigma^i \circ \sigma^j$ and therefore $w' = w \circ \sigma^j \circ \sigma^i$ and $i' = j$.

Let us suppose that $|i-j| > 1$. We have then

$$\sigma^j \circ \widehat{i+1} = \widehat{i+1} \circ \sigma^{N^{i+1}(j)} \tag{2}$$

and therefore

$$\psi^k(w \circ \sigma^j, i) = \psi^k(w, i) \circ \sigma^{N^{i+1}(j)} \tag{3}$$

by virtue of the following commuting diagram:

$$\begin{array}{ccccc} [n] & \xrightarrow{\sigma^j} & [n] & \xrightarrow{w} & [n] \\ \widehat{i+1} \uparrow & & \widehat{i+1} \uparrow & & \downarrow N^{k+1} \\ [n-1] & \xrightarrow{\sigma^{N^{i+1}(j)}} & [n-1] & \xrightarrow{\psi(w, i)} & [n-1] \end{array} .$$

We leave verification of (2) to the reader.

Let us suppose otherwise that $|i - j| = 1$. In this case we claim that

$$\sigma^j \circ \sigma^i \circ \widehat{j+1} = \widehat{i+1} \circ \sigma^{N^{i+1}(j)} \quad (4)$$

so that

$$\psi^k(w \circ \sigma^j \circ \sigma^i, j) = \psi^k(w, i) \circ \sigma^{N^{i+1}(j)} \quad (5)$$

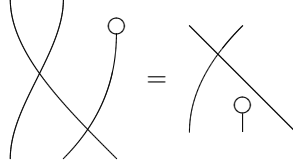
by the diagram

$$\begin{array}{ccccc} [n] & \xrightarrow{\sigma^j \circ \sigma^i} & [n] & \xrightarrow{w} & [n] \\ \widehat{j+1} \uparrow & & \widehat{i+1} \uparrow & & \downarrow N^{k+1} \\ [n-1] & \xrightarrow{\sigma^{N^{i+1}(j)}} & [n-1] & \xrightarrow{\psi(w, i)} & [n-1] \end{array} .$$

To prove equation (4) we use the standard properties of the calculus of strings, see [12, §2.3] for example. If $i < j$, then we have

$$\sigma^j \circ \sigma^i \circ \widehat{j+1} = \widehat{i+1} \circ \sigma^i$$

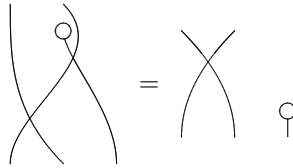
as witnessed by



Otherwise $j < i$ and

$$\sigma^j \circ \sigma^i \circ \widehat{j+1} = \widehat{i+1} \circ \sigma^j,$$

the latter relation being witnessed by



□ *Claim*

This also completes the proof of Proposition 7.2. □

We use Proposition 7.2 to argue that derived semidistributive lattices of the form $\mathbb{C}(L, \gamma)$ are not quotients of L in the most obvious way. It is a standard reasoning to argue that $\delta \in \mathbb{C}(L, \gamma)$ implies $(\delta_0, \delta_1) \in \theta(\gamma_0, \gamma_1)$, where $\theta(\gamma_0, \gamma_1)$ is the congruence generated by the pair (γ_0, γ_1) . It is reasonable to ask whether the lattice $\mathbb{C}(L, \gamma)$ is related to the specific quotient lattice $L/\theta(\gamma_0, \gamma_1)$. The following Proposition gives a first answer in the negative, showing that these two lattices are not in general isomorphic.

Proposition 7.5. For $k \in \{1, \dots, n-1\}$ the lattice $\mathcal{S}_n/\theta(\perp, \sigma^k)$ is isomorphic to the lattice $\mathcal{S}_k \times \mathcal{S}_{n-k}$.

Proof. We recall that for a finite lattice L and a congruence θ of L , each equivalence class $[x]_\theta$ has a least element $\mu_\theta(x)$, computed as follows:

$$\mu_\theta(x) = \bigvee \{j \in J(L) \mid j \leq x \text{ and } (j_*, j) \notin \theta\}.$$

The quotient L/θ is then isomorphic to the poset $\langle \{\mu_\theta(x) \mid x \in L\}, \leq \rangle$. We use this representation to give explicit form to $\mathcal{S}_n/\theta(\perp, \sigma^k)$.

Recall that a permutation is join irreducible iff it has a unique descent, i.e. a unique index $i \in \{1, \dots, n-1\}$ such that $w_i > w_{i+1}$. In [17] we called the pair (w_{i+1}, w_i) the principal plan of the join irreducible w . Let us denote by \trianglelefteq the reflexive transitive closure of the join dependency relation between join irreducible permutations. From the characterization given there of the join dependency relation, we have that $w \trianglelefteq \sigma^k$ iff the $[k, k+1] \subseteq [a, b]$, where a, b is the principal plan of w . Therefore, a join irreducible permutation is not congruent to its unique lower cover modulo $\theta(\perp, \sigma^k)$ if and only if its principal plan does not contain the interval $[k, k+1]$.

Remark that, for $a < b < c$, we have $[k, k+1] \subseteq [a, c]$ if and only if $[k, k+1] \subseteq [a, b]$ or $[k, k+1] \subseteq [b, c]$. Using this fact, we see that if $D = D(w)$ is the set of disagreements (or inversions) of some permutation w , then $D' = \{(a, b) \in D \mid [k, k+1] \not\subseteq [a, b]\}$ is also the set of disagreements of some permutation w' . To this goal, it is enough to verify that D' is closed – i.e. $(a, b), (b, c) \in D'$ implies $(a, c) \in D'$ – and open as well – i.e. $a < b < c$ and $(a, c) \in D'$ implies $(a, b) \in D'$ or $(b, c) \in D'$. Since $(a, b) \in D(w)$ if and only if there exists $j \in J(\mathcal{S}_n)$ such that (a, b) is the principal plan of j , then we deduce that the permutation w' is the least element in the congruence class of w .

Knowing that the order on \mathcal{S}_n is given by inclusion of disagreement sets, the relation

$$D(w') = \{(a, b) \in D(w) \mid [a, b] \subseteq [1, k]\} \uplus \{(a, b) \in D(w) \mid [a, b] \subseteq [k+1, n]\}$$

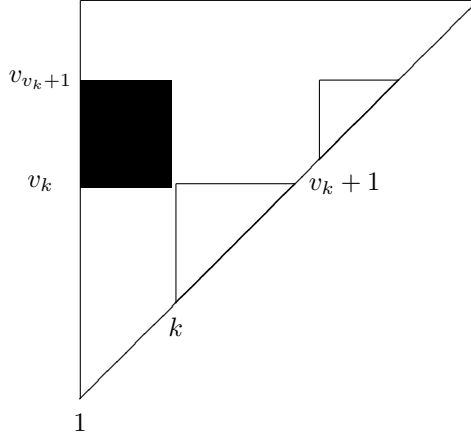
exhibits $\mathcal{S}_n/\theta(\perp, \sigma^k)$ as the product of \mathcal{S}_k and \mathcal{S}_{n-k} . □

Considering that finite semidistributive lattices and bounded lattices form pseudovarieties [11, 14], we leave it as an open problem whether derived lattices are constructible by means of standard operations such as homomorphic images, subalgebras and products.

We are ready to tackle computation of the lattices derived from associahedra by atoms. The computation we present here is a direct one. Considering however that associahedra are quotient of permutohedra, see [16, §9], we expect that the next Proposition may be derived from Proposition 7.2 in a more informative manner.

Proposition 7.6. Let \mathcal{T}_n be the associahedron on $n+1$ letters (i.e. the Tamari lattice). If α is an atom of \mathcal{T}_n then $\mathbb{C}(\mathcal{T}_n, \alpha)$ is isomorphic to \mathcal{T}_{n-1} .

To prove the propositions we shall review some facts about the explicit representation of the Tamari lattices as lattices of bracketing vectors with the pointwise order, see [10, 2, 5]. A *bracketing vector* is a vector $v \in \{1, \dots, n\}^n$ such that (i) $i \leq v_i$ and (ii) $i < j \leq v_i$ implies $v_j \leq v_i$. We are going to determine covers of the pointwise order. Let us say that $k \in \{1, \dots, n-1\}$ is a *split* of a bracketing vector v if $i < k \leq v_i$ implies $v_{v_k+1} \leq v_i$. The next diagram should help understanding the condition: if k is a split, then the black region is a forbidden area meaning that it does not contain points of the form (i, v_i) .



Lemma 7.7. *Let k be a split of a bracketing vector v and define the vector v^k by*

$$v_i^k = \begin{cases} v_{v_k+1}, & i = k, \\ v_i, & \text{otherwise.} \end{cases}$$

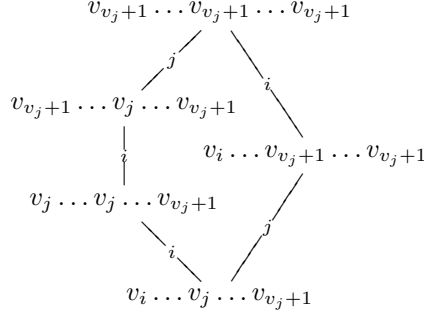
Then $v < v^k$ and moreover all the covers in \mathcal{T}_n arise in this way.

Proof. We observe first that v^k is again a bracketing vector. Condition (i) is satisfied: if $i \neq k$, then $i \leq v_i = v_i^k$, and otherwise $k \leq v_k < v_k + 1 \leq v_{v_k+1} = v_k^k$. Condition (ii) clearly holds if both i and j are distinct from k . Let us suppose that $j = k$, that is $i < k \leq v_i^k = v_i$. Since k is a split, then $v_k^k = v_{v_k+1} \leq v_i$. Let us suppose that $i = k$, that is, $k < j \leq v_k^k = v_{v_k+1}$. If $j \leq v_k$ then $v_j \leq v_k < v_k^k$. If $v_k < j$, then $v_k + 1 \leq j \leq v_{v_k+1}$ and $v_j \leq v_{v_k+1} = v_k^k$.

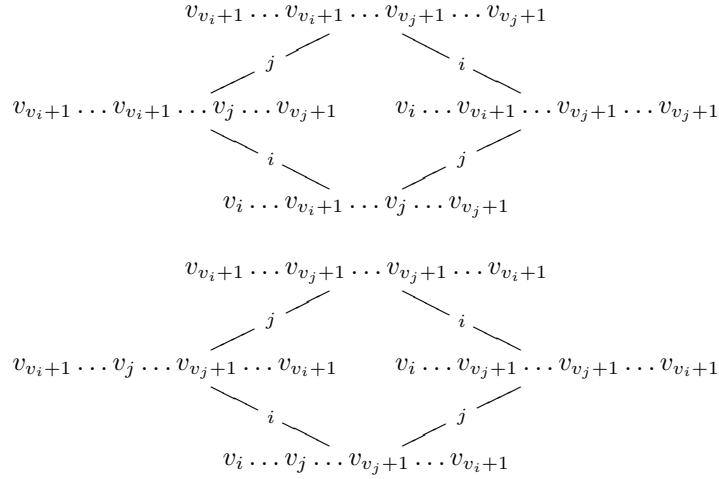
Let us suppose that $v < w$ and let k be the least index such that $v_k < w_k$. Observe first that $v_{v_k+1} \leq w_k$: from $v_k < w_k$ we can write $k < v_k + 1 \leq w_k$ and hence $v_{v_k+1} \leq w_{v_k+1} \leq w_k$. Also k is a split of v : if $i < k \leq v_i$ then $i < k \leq w_i$, $w_k \leq w_i$ so that $v_{v_k+1} \leq w_k \leq w_i = v_i$. This shows that $v < v^k$ and moreover that any upper cover of v is of the form v^k for some split of v . \square

Lemma 7.8. *Let v be a bracketing vector.*

1. If i, j are two splits of v with $j = v_i + 1$, then we have the following pentagon:



2. If i, j are two splits of v and $j \neq v_i + 1$, then we have one of the following diamonds:



We refer the reader to [5, Propositions 4 and 5] for a detailed proof of this Lemma.

If v is a bracketing vector and k is a split of v , then we denote the cover $v \prec v^k$ by the pair (v, k) . From the previous Lemma it immediately follows:

Corollary 7.9. *If $i \neq j$ and $(v, j) \xrightarrow{\text{push}} (w, k)$, then $k = j$ and*

$$w = \begin{cases} v^{ii}, & j = v_i + 1, \\ v^i, & \text{otherwise.} \end{cases}$$

Corollary 7.10. *A cover (v, j) is perspective to the atom (\perp, k) if and only if $j = k$ and $v_k = k$. In this case k is the unique index i such that $v_i = k$.*

Proof. The condition is necessary: the property holds for (v, k) and, by Lemma 7.8, is preserved under the operation of pushing up covers. The condition is

also sufficient, for which it is enough to remark that if $v_k = k$, then $\perp \leq v$, $\perp^k = k + 1 \not\leq k = v_k$, $\perp^k = k + 1 \leq v_k^k = v_{v_k+1}$. For the last statement, let us suppose that k is a split of v and that $v_k = k$. If $v_i = k$, then $i \leq k$. However $i < k$ contradicts i being a split. \square

We are ready to proof Proposition 7.6.

Proof of Proposition 7.6. From a cover $(v, k) \in \mathbb{C}(\mathcal{T}_n)$, perspective to an atom, define a bracketing vector $\psi(v, k) \in \mathcal{T}_{n-1}$ as the composal

$$\begin{array}{ccc} [n] & \xrightarrow{v} & [n] \\ \hat{k} \uparrow & & \downarrow N^k \\ [n-1] & \xrightarrow{\psi(v, k)} & [n-1] \end{array} .$$

Before carrying on with the proof, we collect first some remarks. Observe that $N^k(\hat{k}(x)) = x$, while $x \leq \hat{k}(N^k(x))$ and this is an equality if $x \neq k$. Therefore \hat{k} is right adjoint to N^k and moreover N^k is inverse to \hat{k} if restricted to $[n] \setminus \{k\}$. Also $N^k(x) + 1 = N^k(x + 1)$ if $x \neq k$. If (v, k) is perspective to an atom, so that $v_j = k$ implies $j = k$, an integer of the form $v_{\hat{k}(j)}$ is not equal to k , otherwise $k = \hat{k}(j)$, a contradiction. Consequently we shall use formulas such as $\hat{k}(N^k(v_{\hat{k}(j)})) = v_{\hat{k}(j)}$, and $N^k(v_{\hat{k}(j)}) + 1 = N^k(v_{\hat{k}(j)} + 1)$.

Let us verify that $\psi(v, k)$ is a bracketing vector. The relation $i \leq \psi(v, k)_i = N^k(v_{\hat{k}(i)})$ immediately follows from $\hat{k}(i) \leq v_{\hat{k}(i)}$. If $i < j \leq \psi(v, k)_i = N^k(v_{\hat{k}(i)})$, then $\hat{k}(i) < \hat{k}(j) \leq \hat{k}(N^k(v_{\hat{k}(i)})) = v_{\hat{k}(i)}$ and $v_{\hat{k}(j)} \leq v_{\hat{k}(i)}$ since v is a bracketing vector; the relation $\psi(v, k)_j \leq \psi(v, k)_i$ follows then by applying N^k .

The correspondence ψ is a bijection: given $w \in \mathcal{T}_{n-1}$ the vector $v \in \mathcal{T}_n$, defined by $v_i = k$ if $i = k$ and $v_i = \hat{k}(w_{N^k(i)})$ otherwise, is the unique bracketing vector such that (v, k) is a cover perspective to (\perp, k) and $\psi(v, k) = w$.

We are going to verify that (a) j is a split of $\psi(v, k)$ iff $\hat{k}(j)$ is a split of v , (b) if $(v, k) \xrightarrow{\psi} (w, k)$ then $\psi(w, k) = \psi(v, k)^{N^k(j)}$. From these properties it follows that ψ preserves and reflects the covering relation and therefore it is an order isomorphism.

(a) Let us suppose first that $\hat{k}(j)$ is a split of v and that $l < j \leq \psi(v, k)_l = N^k(v_{\hat{k}(l)})$. It follows that $\hat{k}(l) < \hat{k}(j) \leq \hat{k}(N^k(v_{\hat{k}(l)})) = v_{\hat{k}(l)}$ and therefore $v_{v_{\hat{k}(j)}+1} \leq v_{\hat{k}(l)}$. Hence

$$\begin{aligned} \psi(v, k)_{\psi(v, k)_j+1} &= N^k(v_{\hat{k}(N^k(v_{\hat{k}(j)}+1))}) = N^k(v_{\hat{k}(N^k(v_{\hat{k}(j)}+1))}) \\ &= N^k(v_{v_{\hat{k}(j)}+1}) \leq N^k(v_{\hat{k}(l)}) = \psi(v, k)_l . \end{aligned}$$

Let us suppose now that j is a split of $\psi(v, k)$ and that $l < \hat{k}(j) \leq v_l$. Observe that the relation $l < v_l$ implies that $l \neq k$. Since both l and $\hat{k}(j)$ are distinct

from k , the relation $l < \hat{k}(l)$ is strictly preserved by N^k and consequently

$$\begin{aligned} N^k(l) &< N^k(\hat{k}(j)) = j \\ &\leq N^k(v_l) = N^k(v_{\hat{k}(N^k(l))}) = \psi(v, k)_{N^k(l)}. \end{aligned}$$

We have therefore $\psi(v, k)_{\psi(v, k)_{j+1}} \leq \psi(v, k)_{N^k(l)}$ and

$$\begin{aligned} N^k(v_{v_{\hat{k}(j)}+1}) &= N^k(v_{\hat{k}(N^k(v_{\hat{k}(j)}+1))}) = N^k(v_{\hat{k}(N^k(v_{\hat{k}(j)}+1))}) \\ &= \psi(v, k)_{\psi(v, k)_{j+1}} \leq \psi(v, k)_{N^k(l)} = N^k(v_l). \end{aligned}$$

Transposing this relation and considering that $l \neq k$ we deduce

$$v_{v_{\hat{k}(j)}+1} \leq \hat{k}(N^k(v_l)) = v_l.$$

(b) Let us suppose that $(v, k) \xrightarrow{\psi} (w, k)$, so that $w = v^{jj}$ if $k = v_j + 1$ and $w = v^j$ otherwise. We want to prove that $\psi(w, k) = \psi(v, k)^{N^k(j)}$. Let us begin to show that these two vectors coincide in each component i such that $i \neq N^k(j)$ (or equivalently $j \neq \hat{k}(i)$):

$$\psi(w, k)_i = N^k(w_{\hat{k}(i)}) = N^k(v_{\hat{k}(i)}) = \psi(v, k)_i = \psi(v, k)_i^{N^k(j)}.$$

Therefore we are left to compare the values of the two vectors at the coordinate $i = N^k(j)$. On the one hand, we have

$$\begin{aligned} \psi(v, k)_{N^k(j)}^{N^k(j)} &= \psi(v, k)_i^i = \psi(v, k)_{\psi(v, k)_{i+1}} = N^k(v_{\hat{k}(N^k(v_{\hat{k}(i)}+1))}) \\ &= N^k(v_{\hat{k}(N^k(v_{\hat{k}(i)}+1))}) = N^k(v_{\hat{k}(N^k(v_j+1))}) \\ &= \begin{cases} N^k(v_{k+1}), & k = v_j + 1 \\ N^k(v_{v_j+1}), & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand, we have

$$\psi(w, k)_{N^k(j)} = \begin{cases} N^k(v_j^{jj}) = N^k(v_{v_j+1}^j) = N^k(v_{v_{v_j+1}+1}^j) \\ \quad = N^k(v_{v_k+1}^j) = N^k(v_{k+1}^j) \\ \quad = N^k(v_{k+1}), & k = v_j + 1, \\ N^k(v_j^j) = N^k(v_{v_j+1}), & \text{otherwise.} \end{cases}$$

This completes the proof of Proposition 7.6. \square

Let us say that a finite semidistributive lattice is *regular* if the lattices $\mathbb{C}(L, \alpha)$, α an atom of L , are all isomorphic. It is not the case that every semidistributive lattice is regular as witnessed by the multinomial lattice $\mathcal{L}(2, 2, 1)$,

the bottom of which is represented in figure 1. Let $\alpha = aaabbc \prec ababc$ and $\beta = aaabbc \prec aabcb$ be two atoms of this lattice, if we consider the bottoms of $\mathbb{C}(L, \alpha)$ and $\mathbb{C}(L, \beta)$ we observe these two lattices are not isomorphic. We remark that the lattice $\mathcal{L}(2, 2, 1)$ is not complemented, contrary to the Newman lattices considered in this section. It might be conjectured that complemented semidistributive lattices are regular. More generally it is an open problem to identify sufficient conditions that ensure that a semidistributive lattice is regular.

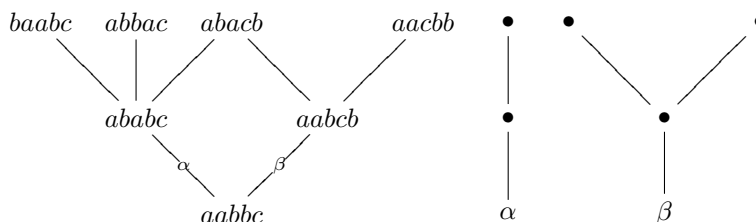


Figure 1: The bottom of the lattices $L = \mathcal{L}(2, 2, 1)$, $\mathbb{C}(L, \alpha)$ and $\mathbb{C}(L, \beta)$.

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