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A note on the Risk Management of CDOs

Jean-Paul LAURENT¹

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Abstract

The purpose of this note is to describe a risk management procedure applicable to options on large credit portfolios such as CDO tranches on iTraxx or CDX. Credit spread risk is dynamically hedged using single name defaultable claims such as CDS while default risk is kept under control thanks to diversification. The proposed risk management approach mixes ideas from finance and insurance and departs from standard approaches used in incomplete markets such as mean-variance hedging or expected utility maximisation. In order to ease the analysis and the exposure, default dates follow a multivariate Cox process.

JEL Classification: G 13

Key words: CDOs, default risk, credit spread risk, dynamic hedging, diversification, large portfolios, incomplete markets, Cox process, doubly stochastic Poisson process.

Introduction

The hedging of defaultable claims is an involved topic (see BLANCHET-SCALLIET & JEANBLANC [2004], BIELECKI et al [2004, 2006a], ELOUERKHAOUI [2006]), especially in a multivariate setting (see BIELECKI et al [2006b] for some discussion). To list only a few issues at hand, we can mention the possibility of simultaneous defaults, contagion effects (leading to jumps in credit spreads at default times), random recoveries, the occurrence of exogenous jumps in the default intensities. Thus, we are likely to be in an incomplete market framework. When considering the risk management of a CDO tranche, we must moreover deal with numerical issues related to the high number of names involved.

Though this is not yet well documented in the academic literature, a widely used approach amongst credit derivatives trading desks is to build some hedging portfolios based upon single name credit default swaps. The hedge ratios are computed as sensitivities to marginal credit curves in a copula framework (GREENBERG et al [2004], GREGORY & LAURENT [2003]). As a consequence, the main focus is put upon the credit spread hedging leaving aside the default risk. This departs from the academic approach: a copula model is associated

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with contagion effects² while default intensities are deterministic between two default times. That simple dynamics of the default intensities and a martingale representation theorem with respect to the natural filtration of default times leads to a hedging strategy concentrated upon the risk management of default risk (BIELECKI et al [2006b]).

Given the rather large number of names in iTraxx on CDX indices, default of a single name has a small effect on the aggregate running loss. In other words, default risk is already partly diversified when considering large portfolios. The theory of such *infinitely granular* portfolio is already well-developed in the static case (see VASICEK [1991], SCHÖNBUCHER [2002], GORDY [2003]). FREY & BACKHAUS [2004], JARROW, LANDO & YU [2005] consider similar issues in a more dynamical setting.

The purpose of that paper is to deal with such ideas with respect to dynamic hedging. Loosely speaking, we could think of dealing with the default risk management through diversification or insurance techniques while credit spread risk is dealt with through dynamic replication techniques. The core idea of the paper is to project the defaultable price process onto the filtration generated by the default intensities. In a second step, we consider the dynamic hedging of the associated smoothed payoff that only involves credit spread risks. The main result of the paper is that using that dynamic hedging strategy with the actual defaultable price processes allows to control the hedging error (with respect to the number of names).

There is now a large body of literature dedicated to large financial markets and completeness (see for instance JARROW & BÄTTIG [1999]). Let us emphasize that while some of our results might be extended to an infinite number of names, this paper remains in a *small market* or finite sample framework. Unlike BJÖRK & NÄSLUND [1998] or DE DONNO [2004] for example, hedging strategies are based on a finite and fixed number of assets and we do not need the use of *well diversified portfolios* such as the *infinitely granular* portfolio which are not readily tradable in the market. Using a finite number of assets simplifies the mathematical exposition. We do not require either that the asymptotic market should be complete.

To keep things simple, we have assumed that default times were modelled through a multivariate Cox process, thus leaving aside possible contagion effects (section 1). Sections 1 and 2 deal with defaultable price processes and their projections onto the filtration of credit spreads. The material there is rather standard and expository. Section 3 contains the main results related to the risk management of CDOs.

1 Modelling of default times

1.1 probabilistic framework

Let us consider some filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ with a fixed time horizon $T \in \mathbb{R}^+$ and some random variables τ_1, \dots, τ_n that denote the default times of n obligors. For any $t \in [0, T]$, $N_i(t) = 1_{\{\tau_i \leq t\}}$, $i = 1, \dots, n$ denote the default indicators, $\mathcal{H}_{i,t} = \sigma(N_i(s), s \leq t)$, $\mathcal{H}_t = \mathcal{H}_{1,t} \vee \dots \vee \mathcal{H}_{n,t}$ and $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. The background filtration \mathcal{F} is typically associated with credit spread risks. The enlarged filtration \mathcal{G} corresponds to the actual information of market participants.

²See SCHÖNBUCHER & SCHUBERT [2001] for an analysis of the dynamics of the default intensities.

We now assume the existence of an arbitrage-free financial market where a savings account and defaultable T maturity zero-coupon bonds are being traded. A T maturity defaultable bond on name i is associated with a payment of $1_{\{\tau_i > T\}}$ ³. For simplicity, we thereafter assume that the default-free interest rates are equal to zero. We denote by $l^i(t, T)$ the time t price of an asset with a time T payoff $N_i(T) = 1_{\{\tau_i \leq T\}}$ ⁴, $i = 1, \dots, n$.

Assumption 1 *There exists a probability Q equivalent to P such that:*

1. for $i = 1, \dots, n$, the price processes of defaultable claims $l^i(\cdot, T)$ are (Q, \mathcal{G}) martingales:

$$l^i(t, T) = E^Q[N_i(T) \mid \mathcal{G}_t], \quad (1.1)$$

for $0 \leq t \leq T$.

2. the default times follow a multivariate Cox process:

$$\tau_i = \inf\{t \in \mathbb{R}^+, U_i \geq \exp(-\Lambda_{i,t})\}, \quad i = 1, \dots, n \quad (1.2)$$

where Λ_i are \mathcal{F} -predictable, absolutely continuous increasing processes such that $\Lambda_{i,0} = 0$, $\lim_{t \rightarrow \infty} \Lambda_{i,t} = \infty$, U_1, \dots, U_n are independent random variables uniformly distributed on $[0, 1]$ under Q and \mathcal{F} and $\sigma(U_1, \dots, U_n)$ are independent under Q .

3. $E^Q \left[\left(\frac{dP}{dQ} \right)^2 \right] < \infty$ ⁵.

The Cox process framework is now standard in finance (see LANDO [1994], the books by BIELECKI & RUTKOWSKI [2002], DUFFIE & SINGLETON [2003], LANDO [2004] and the references therein). More precisely, our setting corresponds to the conditionally independent default framework (see chapter 9 of BIELECKI & RUTKOWSKI [2002]). As a consequence, $t \in \mathbb{R}^+ \rightarrow \Lambda_{i,t \wedge \tau_i}$ is the (Q, \mathcal{G}) compensator of τ_i , i.e. the processes $N_i(t) - \Lambda_{i,t \wedge \tau_i}$, $i = 1, \dots, n$ are (Q, \mathcal{G}) martingales.

The multivariate Cox process framework is convenient since the so-called (H) hypothesis or martingale invariance property holds:

Lemma 1.1 *Every (Q, \mathcal{F}) square integrable martingale is also a (Q, \mathcal{G}) square integrable martingale.*

PROOF : an equivalent statement of the (H) hypothesis is the following: for any $t \in \mathbb{R}^+$, for any $s \in [t, T]$ and any bounded \mathcal{F}_s -measurable random variable ξ , we have: $E^Q[\xi \mid \mathcal{G}_t] = E^Q[\xi \mid \mathcal{F}_t]$. To show this, let us denote by $\mathcal{G}_{i,s} = \mathcal{F}_s \vee \mathcal{H}_{i,s}$ for some $i = 1, \dots, n$ (say $i = 2$) and by $\mathcal{H}_{(-i),t} = \mathcal{H}_{1,t} \vee \dots \vee \mathcal{H}_{i-1,t} \vee \mathcal{H}_{i+1,t} \vee \dots \vee \mathcal{H}_{n,t}$. The σ -fields $\mathcal{G}_{i,s}$ and $\mathcal{H}_{(-i),t}$ are conditionally independent given $\mathcal{G}_{i,t}$. Consequently, since ξ is $\mathcal{G}_{i,s}$ -measurable, we have $E^Q[\xi \mid \mathcal{G}_t] = E^Q[\xi \mid \mathcal{G}_{i,t} \vee \mathcal{H}_{(-i),t}] = E^Q[\xi \mid \mathcal{G}_{i,t}]$. Now, we are back in a univariate Cox process framework and it is well known that $E^Q[\xi \mid \mathcal{G}_{i,t}] = E^Q[\xi \mid \mathcal{F}_t]$.

³For simplicity, the recovery rates are equal to zero.

⁴This payoff corresponds to a long position in a default-free T maturity bond and a short position in a defaultable T maturity bond.

⁵Let us remark that the equivalent martingale measure Q is somehow independent of the number of names. This is related to the absence of *asymptotic free lunch* (see KLEIN [2000]).

Let us remark that there are no contagions effects under Q . The absence of contagion under Q will further provide a simple split between default and credit spread risks for large portfolios. Let us also remark that while (τ_1, \dots, τ_n) is a multivariate Cox process under Q , it may not be a Cox process under P . For instance, we may have some contagion effects under P (see KUSUOKA [1999]).

The joint survival function is such that $S(t_1, \dots, t_n) = E^Q [\prod_{i=1}^n \exp(-\Lambda_{i,t_i})]$ for $t_1, \dots, t_n \in \mathbb{R}^+$. Since it is continuous, we must have $Q(\tau_i = \tau_j) = 0$ for $i \neq j$ which precludes simultaneous defaults.

$E^Q \left[\left(\frac{dP}{dQ} \right)^2 \right] < \infty$ states that the historical measure P and the risk-neutral one Q do not depart too much from one to another. On economic grounds, it constrains the magnitude of default risk premia (see JARROW, LANDO & YU [2005] for a discussion).

From the absolute continuity of the Λ_i 's, there exist non-negative cadlag \mathcal{F} -adapted processes, $\lambda_1, \dots, \lambda_n$ such that:

$$\Lambda_{i,t} = \int_0^t \lambda_{i,u} du, \quad (1.3)$$

for $t \geq 0$, $i = 1, \dots, n$. λ_i is the (Q, \mathcal{G}) -intensity of the counting process N_i .

We can now state the dynamics of the defaultable claims:

Lemma 1.2 defaultable claim price dynamics

$$l^i(t, T) = (1 - N_i(t)) (1 - E^Q [\exp(\Lambda_{i,t} - \Lambda_{i,T}) | \mathcal{F}_t]) + N_i(t), \quad (1.4)$$

for $0 \leq t \leq T$ and $i = 1, \dots, n$.

PROOF: the σ -fields $\mathcal{G}_{i,T}$ and $\mathcal{H}_{(-i),t}$ are conditionally independent given $\mathcal{G}_{i,t}$. Consequently, $Q(\tau_i > T | \mathcal{G}_t) = Q(\tau_i > T | \mathcal{G}_{i,t})$. Since τ_i can be seen as a univariate Cox process, $Q(\tau_i > T | \mathcal{G}_{i,t}) = 1_{\{\tau_i > t\}} E^Q [\exp(\Lambda_{i,t} - \Lambda_{i,T}) | \mathcal{F}_t]$. We conclude by using $l^i(t, T) = E^Q [N_i(T) | \mathcal{G}_t] = 1 - Q(\tau_i > T | \mathcal{G}_t)$.

From the monotonicity of conditional expectations, $0 \leq l^i(t, T) \leq 1$, for $i = 1, \dots, n$ and $0 \leq t \leq T$. Thus, $l^i(\cdot, T)$, $i = 1, \dots, n$ are (Q, \mathcal{G}) square integrable martingales, with a jump at τ_i .

Definition 1.1 predefault bond price dynamics

We denote by $B_i(t, T) = E^Q [\exp(\Lambda_{i,t} - \Lambda_{i,T}) | \mathcal{F}_t]$, for $0 \leq t \leq T$. $B_i(t, T)$ corresponds to the defaultable bond price⁶ at time t on the set $\{\tau_i > t\}$.

Hence, the dynamics of the defaultable claims simplifies to:

$$l^i(t, T) = (1 - N_i(t)) (1 - B_i(t, T)) + N_i(t).$$

⁶Associated with a payoff $1_{\{\tau_i > T\}}$ at time T

2 Portfolio dynamics

2.1 default-free processes

It will be convenient to consider the following *default-free running loss processes*:

Definition 2.1 *The default-free running loss process associated with name $i \in \{0, \dots, n\}$, denoted by $p^i(\cdot)$ is such that for $0 \leq t \leq T$:*

$$p^i(t) \triangleq E^Q[N_i(t) | \mathcal{F}_t] = Q(\tau_i \leq t | \mathcal{F}_t) = 1 - \exp(-\Lambda_{i,t}). \quad (2.1)$$

The last equality is a direct consequence of assumption (1). p^i is a \mathcal{F} -adapted increasing process that, unlike N_i , does not jump at default times⁷. We can also define the *default free forward loss processes* by:

Definition 2.2 *The default free T forward loss process associated with name $i \in \{0, \dots, n\}$, denoted by $p^i(\cdot, T)$ is such that for $0 \leq t \leq T$:*

$$p^i(t, T) \triangleq E^Q[p^i(T) | \mathcal{F}_t] = E^Q[N_i(T) | \mathcal{F}_t] = Q(\tau_i \leq T | \mathcal{F}_t). \quad (2.2)$$

The second equality is a direct consequence of the definition of $p^i(T)$ and the law of total expectation. From the monotonicity of conditional expectations, we readily see that $0 \leq p^i(t, T) \leq 1$, for $i = 1, \dots, n$ and $0 \leq t \leq T$. Thus, the $p^i(\cdot, T)$ are (Q, \mathcal{F}) square integrable martingales and thus (Q, \mathcal{G}) square integrable martingales thanks to the martingale invariance property.

From the definition of $p^i(\cdot, T)$, we readily have: $p^i(t, T) = E^Q[1 - \exp(-\Lambda_{i,T}) | \mathcal{F}_t]$.

Lemma 2.1 *$p^i(t, T)$, $i = 1, \dots, n$ are projections of the forward price processes $l^i(t, T)$ on \mathcal{F}_t :*

$$p^i(t, T) = E^Q[l^i(t, T) | \mathcal{F}_t], \quad (2.3)$$

for $i = 1, \dots, n$ and $0 \leq t \leq T$.

PROOF: since $l^i(t, T) = E^Q[N_i(T) | \mathcal{G}_t]$, we only need to check that $p^i(t, T) = E^Q[N_i(T) | \mathcal{F}_t]$. We conclude from $\tau_i \leq T \Leftrightarrow U_i \geq \exp(-\Lambda_{i,T})$ and the independence between U_i and \mathcal{F}_t .

Lemma 2.2

$$l^i(t, T) - p^i(t, T) = Z_i(t)B_i(t, T), \quad (2.4)$$

for $i = 1, \dots, n$ and $0 \leq t \leq T$, where $Z_i(t) = \exp\left(-\int_0^t \lambda_{i,u} du\right) - (1 - N_i(t))$, and $B_i(t, T)$ is the predefault bond price (see definition (1.1)).

PROOF: Since $p^i(t, T) = E^Q[N_i(T) | \mathcal{F}_t]$, $l^i(t, T) - p^i(t, T) = E^Q[N_i(T) | \mathcal{G}_t] - E^Q[N_i(T) | \mathcal{F}_t]$, which yields:

$$l^i(t, T) - p^i(t, T) = \left(\exp\left(-\int_0^t \lambda_{i,u} du\right) - (1 - N_i(t)) \right) \times E^Q \left[\exp\left(-\int_t^T \lambda_{i,u} du\right) | \mathcal{F}_t \right].$$

⁷ p^i can be given a simple financial interpretation. Let us subdivide name i into K names each with nominal $1/K$ and intensities equal to λ_i but with independent thresholds $U_{i,k}$. We denote by $\tau_{i,k}$ the corresponding default dates. $\frac{1}{K} \sum_{k=1}^K 1_{\tau_{i,k} \leq t}$ converges a.s. to $p^i(t)$ as $K \rightarrow \infty$. In other words, $p^i(t)$ corresponds to some aggregate running loss where diversification holds at the level of name i .

2.2 portfolio loss processes

Let us now consider portfolios based upon the previous individual processes:

Definition 2.3 aggregate running loss process *The aggregate loss at time t on a portfolio of n names is such that for $0 \leq t \leq T$:*

$$l_n(t) \triangleq \frac{1}{n} \sum_{i=1}^n N_i(t). \quad (2.5)$$

For simplicity, we have assumed that default exposures are equal to $\frac{1}{n}$ and that recovery rates are equal to zero. To emphasize the dependence upon the number of names n , we used the subscript in the running loss $l_n(t)$.

Definition 2.4 aggregate forward loss process *The T forward aggregate loss at time t is such that for $0 \leq t \leq T$:*

$$l_n(t, T) \triangleq E^Q [l_n(T) | \mathcal{G}_t]. \quad (2.6)$$

Since $0 \leq l_n(T) \leq 1$, we also have $0 \leq l_n(t, T) \leq 1$, for all $t \in [0, T]$ thanks to the monotonicity of conditional expectations. $l_n(\cdot, T)$ is thus a square integrable (Q, \mathcal{G}) martingale. We readily have:

$$l_n(t, T) = \frac{1}{n} \sum_{i=1}^n l^i(t, T), \quad (2.7)$$

which shows that $l_n(t, T)$ can be seen as a portfolio price process.

Definition 2.5 default-free aggregate running loss process *The default free aggregate running loss at time t is such that for $0 \leq t \leq T$:*

$$p_n(t) \triangleq \frac{1}{n} \sum_{i=1}^n p^i(t). \quad (2.8)$$

$p_n(t)$ is a \mathcal{F} - adapted increasing process. Unlike $l_n(t)$, $p_n(t)$ does not jump at default times. $p_n(t)$ corresponds to the aggregate loss of a portfolio where the risk has been (infinitely) diversified at the name level.

3 Option hedging

3.1 main result

We are now concerned by payoffs of the type $(l_n(T) - K)^+ = (\frac{1}{n} \sum_{i=1}^n N_i(T) - K)^+$, for some $K \in [0, 1]$ corresponding to so-called *zero-coupon CDOs*. Before proceeding further, let us state some technical lemmas:

Lemma 3.1

$$\| l_n(T) - p_n(T) \|_{2, Q}^2 = \frac{1}{n^2} \sum_{i=1}^n E^Q [(1 - \exp(-\Lambda_{i,T})) \exp(-\Lambda_{i,T})]. \quad (3.1)$$

PROOF: since the means of $l_n(T)$ and $p_n(T)$ are equal, we need to consider $\text{Var}^Q[l_n(T) - p_n(T)]$ which, thanks to the law of total variance, is equal to:

$$\text{Var}^Q [E^Q [l_n(T) - p_n(T) | \mathcal{F}_T]] + E^Q [\text{Var}^Q [l_n(T) - p_n(T) | \mathcal{F}_T]].$$

The first term is equal to zero and:

$$\text{Var}^Q [l_n(T) - p_n(T) | \mathcal{F}_T] = \text{Var}^Q [l_n(T) | \mathcal{F}_T] = \frac{1}{n^2} \sum_{i=1}^n \text{Var}^Q [N_i(T) | \mathcal{F}_T],$$

from the conditional independence of the $N_i(T)$ given \mathcal{F}_T . We conclude using:

$$\text{Var}^Q [N_i(T) | \mathcal{F}_T] = (1 - \exp(-\Lambda_{i,T})) \exp(-\Lambda_{i,T}).$$

Since $0 \leq (1 - \exp(-\Lambda_{i,T})) \exp(-\Lambda_{i,T}) \leq \frac{1}{2}$, we can also state:

$$\| l_n(T) - p_n(T) \|_{2,Q}^2 \leq \frac{1}{2n}.$$

Lemma (3.1) simply states that the accumulated losses $l_n(T)$ can be well approximated by the \mathcal{F} -adapted random variable $p_n(T)$. In other words, for large n , we can neglect default risks and concentrate on the credit spread risks embedded in $p_n(T)$.

Lemma 3.2 *Let $A(\cdot)$ be a finite variation \mathcal{F} -adapted process such that $A(0) = 0$ and $E^Q[A(T)] < \infty$. Let $\theta(\cdot)$ be a \mathcal{G} -adapted process such that $0 \leq \theta(t) \leq \bar{K}$, for $0 \leq t \leq T$ for some positive \bar{K} . Then, $E^Q \left[\int_0^T \theta(t) dA(t) \right] = E^Q \left[\int_0^T E^Q [\theta(t) | \mathcal{F}_t] dA(t) \right]$.*

PROOF : let us consider some partition $0 < \dots < t_{j-1} < t_j < \dots < T$ of $[0, T]$. Using linearity of expectations and the law of total expectation,

$$E^Q \left[\sum_j \theta(t_{j-1}) (A(t_j) - A(t_{j-1})) \right] = E^Q \left[\sum_j E^Q [\theta(t_{j-1}) | \mathcal{F}_{t_{j-1}}] (A(t_j) - A(t_{j-1})) \right].$$

As the mesh of the partition tends to zero, the two discrete sums converge (for any state of the world) to the Stieltjes integrals $\int_0^T \theta(t) dA(t)$ and $\int_0^T E^Q[\theta(t) | \mathcal{F}_t] dA(t)$. When A is increasing, we conclude using Lebesgue theorem. This extends to the finite variation case by linearity.

From lemma (3.1), we know that $p_n(T)$ is close to $l_n(T)$ for large n . The idea is thus to consider the hedging of the payoff $(p_n(T) - K)^+$. Since $p_n(T)$ only involves credit spread risks and not default risks, it is more likely that we can hedge the latter payoff. More formally, we make the following assumption:

Assumption 2 *There exists some bounded \mathcal{F} -predictable processes $\theta_1(\cdot), \dots, \theta_n(\cdot)$ such that:*

$$(p_n(T) - K)^+ = E^Q \left[(p_n(T) - K)^+ \right] + \frac{1}{n} \sum_{i=1}^n \int_0^T \theta_i(t) dp^i(t, T) + z_n, \quad (3.2)$$

where z_n is \mathcal{F}_T -measurable, of Q -mean zero and Q -strongly orthogonal to $p^1(\cdot, T), \dots, p^n(\cdot, T)$.

The previous equation is simply the Galtchouk - Kunita - Watanabe decomposition of $(p_n(T) - K)^+$ for (Q, \mathcal{F}) . $\theta_1(\cdot), \dots, \theta_n(\cdot)$ correspond to the optimal (Q, \mathcal{F}) mean-variance hedging strategy based upon the abstract forward price processes $p^1(\cdot, T), \dots, p^n(\cdot, T)$.

If the default intensities $\lambda_1, \dots, \lambda_n$ follow a multivariate Itô process, then $(p^1(\cdot, T), \dots, p^n(\cdot, T))$ also follows a multivariate Itô process. Assuming that the diffusion matrix is of rank n , then $z_n = 0$. This corresponds to some completeness of the credit spread market. In the case of jump-diffusion processes, the residual term z_n usually differs from zero.

The key point in Assumption (2) is the boundedness of the θ_i 's (or credit deltas). Let us remark that the individual credit deltas are equal to $\frac{\theta_i(t)}{n}$ and thus decrease at the rate $\frac{1}{n}$. For simplicity, we will thereafter assume that $0 \leq \theta_i(\cdot) \leq 1$ for $i = 1, \dots, n$. This boundedness assumption is related to the propagation of convexity property. We refer to BERGENTHUM & RÜSCHENDORF [2004], EKSTRÖM & TYSK [2006] and the references therein for some discussion in a multivariate jump diffusion setting.

We now state another lemma related to the control of hedging errors:

Lemma 3.3 *Under assumptions (1) and (2), the following inequality holds:*

$$\left\| \frac{1}{n} \sum_{i=1}^n \int_0^T \theta_i(t) d(l^i(t, T) - p^i(t, T)) \right\|_{2, Q}^2 \leq \frac{1}{n^2} \sum_{i=1}^n (Q(\tau_i \leq T) + E^Q[B_i]_T). \quad (3.3)$$

PROOF: from Kunita and Watanabe, we have:

$$\left\| \sum_{i=1}^n \int_0^T \theta_i(t) d(l^i(t, T) - p^i(t, T)) \right\|_{2, Q}^2 = \sum_{i, j=1}^n E^Q \left[\int_0^T \theta_i(t) \theta_j(t) d[l^i(t, T) - p^i(t, T), l^j(t, T) - p^j(t, T)]_t \right],$$

which involves the quadratic covariations of the (Q, \mathcal{G}) square integrable bounded martingales $l^i(\cdot, T) - p^i(\cdot, T)$. Since $l^i(t, T) - p^i(t, T) = Z_i(t)B_i(t, T)$ (see lemma (2.2)), when $i \neq j$, the quadratic covariation $[l^i(\cdot, T) - p^i(\cdot, T), l^j(\cdot, T) - p^j(\cdot, T)]$ involves only the quadratic covariation of $B_i(t, T)$ and $B_j(t, T)$ ⁸:

$$[l^i(\cdot, T) - p^i(\cdot, T), l^j(\cdot, T) - p^j(\cdot, T)]_t = Z_i(t)Z_j(t)[B_i, B_j]_t, \quad i \neq j.$$

The quadratic variation of $l^i(t, T) - p^i(t, T)$ involves two terms, one associated with the quadratic variation of $Z_i(t)$ (or default risk) and one associated with the quadratic variation of $B_i(t, T)$ or credit spread risk. The quadratic variation associated with the default indicator part of $l^i(t, T) - p^i(t, T)$, $Z_i(t)$, is equal to $N_i(t)$.

We can write:

$$\sum_{i, j=1}^n \theta_i(t) \theta_j(t) d[l^i(\cdot, T) - p^i(\cdot, T), l^j(\cdot, T) - p^j(\cdot, T)]_t = \sum_{i, j=1}^n A_{i, j} d[B_i, B_j]_t + \sum_{i=1}^n D_i dN_i(t),$$

⁸This is the core of the Cox modelling: there are no simultaneous defaults; defaults are not contagious. This allows for diversification of default risk in large portfolios. This holds even if the predefault bond prices have common jump components.

where the $A_{i,j}, D_i$ are given by: $A_{i,j} = \theta_i(t)\theta_j(t)Z_i(t)Z_j(t)$ and $D_i = \theta_i^2(t)B_i^2(t, T)$.

We can thus write:

$$\left\| \sum_{i=1}^n \int_0^T \theta_i(t) d(l^i(t, T) - p^i(t, T)) \right\|_{2, Q}^2 = E^Q \left[\int_0^T \left(\sum_{i,j=1}^n A_{i,j} d[B_i, B_j]_t + \sum_{i=1}^n D_i dN_i(t) \right) \right].$$

Let us firstly consider the terms $E^Q \left[\int_0^T A_{i,j} d[B_i, B_j]_t \right]$. Thanks to lemma (3.2),

$$E^Q \left[\int_0^T A_{i,j} d[B_i, B_j]_t \right] = E^Q \left[\int_0^T E^Q[A_{i,j} | \mathcal{F}_t] d[B_i, B_j]_t \right].$$

Since θ_i and θ_j are \mathcal{F} -adapted, $E^Q[A_{i,j} | \mathcal{F}_t] = \theta_i(t)\theta_j(t)E^Q[Z_i(t)Z_j(t) | \mathcal{F}_t]$. Thanks to the conditional independence of default times upon \mathcal{F} , $E^Q[Z_i(t)Z_j(t) | \mathcal{F}_t] = \text{Cov}^Q(N_i(t), N_j(t) | \mathcal{F}_t) = 0$ for $i \neq j$ and $0 \leq t \leq T$. As a consequence, $E^Q[A_{i,j} | \mathcal{F}_t] = 0$ for $i \neq j$ and $E^Q \left[\int_0^T \sum_{i,j=1}^n A_{i,j} d[B_i, B_j]_t \right] = E^Q \left[\int_0^T \sum_{i=1}^n \theta_i^2(t) Z_i^2(t) d[B_i]_t \right]$. Since $0 \leq \theta_i^2(t) Z_i^2(t) \leq 1$, $E^Q \left[\int_0^T \sum_{i=1}^n \theta_i^2(t) Z_i^2(t) d[B_i]_t \right] \leq \sum_{i=1}^n [B_i]_T$. Similarly, since $0 \leq D_i \leq 1$, $E^Q \left[\int_0^T (\sum_{i=1}^n D_i dN_i(t)) \right] \leq \sum_{i=1}^n Q(\tau_i \leq T)$.

We can now state our main result with respect to the hedging error:

Proposition 1 *Under assumptions (1) and (2), the hedging error ε_n defined as:*

$$\varepsilon_n = (l_n(T) - K)^+ - E^Q \left[(l_n(T) - K)^+ \right] - \frac{1}{n} \sum_{i=1}^n \int_0^T \theta_i(t) dl^i(t, T), \quad (3.4)$$

is such that $E^P[\|\varepsilon_n\|]$ is bounded by:

$$\frac{1}{\sqrt{2n}} \left(1 + \left(E^Q \left[\left(\frac{dP}{dQ} \right)^2 \right] \right)^{1/2} \right) + \frac{1}{n} \left(E^Q \left[\left(\frac{dP}{dQ} \right)^2 \right] \right)^{1/2} \left(\sum_{i=1}^n (Q(\tau_i \leq T) + E^Q[B_i]_T) \right)^{1/2} + E^P[\|z_n\|] \quad (3.5)$$

Let us proceed to the proof of the proposition. Using triangle inequalities, we readily bound $E^P[\|\varepsilon_n\|]$ by:

$$E^P[\|l_n(T) - p_n(T)\|] + E^Q[\|l_n(T) - p_n(T)\|] + \frac{1}{n} E^P \left[\left\| \sum_{i=1}^n \int_0^T \theta_i(t) d(l^i(t, T) - p^i(t, T)) \right\| \right] + E^P[\|z_n\|].$$

As for the term $E^P[\|l_n(T) - p_n(T)\|]$, we have: $E^P[\|l_n(T) - p_n(T)\|] \leq \left(E^Q \left[\left(\frac{dP}{dQ} \right)^2 \right] \right)^{1/2} \times \|l_n(T) - p_n(T)\|_{2, Q}$. Using lemma (3.1), we can thus bound $E^P[\|l_n(T) - p_n(T)\|] + E^Q[\|l_n(T) - p_n(T)\|]$ by:

$$\frac{1}{\sqrt{2n}} \left(1 + \left(E^Q \left[\left(\frac{dP}{dQ} \right)^2 \right] \right)^{1/2} \right).$$

Let us now concentrate upon the dynamic hedging strategy term. From Cauchy-Schwarz inequality, we get:

$$E^P \left[\left\| \sum_{i=1}^n \int_0^T \theta_i(t) d(l^i(t, T) - p^i(t, T)) \right\| \right] \leq \left(E^Q \left[\left(\frac{dP}{dQ} \right)^2 \right] \right)^{1/2} \times \left\| \sum_{i=1}^n \int_0^T \theta_i(t) d(l^i(t, T) - p^i(t, T)) \right\|_{2, Q}.$$

Lemma (3.3) allows to conclude. The terms $E^Q [[B_i]_T]$ are related to the riskiness associated with credit spreads. The smaller the "volatility" associated with the credit spreads, the better the approximation hedge will be. Provided that the $E^Q [[B_i]_T]$ are uniformly bounded, that the risk premium term $E^Q \left[\left(\frac{dP}{dQ} \right)^2 \right]$ also remains bounded and that the credit spread market is complete, the previous proposition states that the $\mathcal{L}^1(P)$ norm of the hedging error tends to zero at the speed $n^{-1/2}$ as n tends to infinity.

Let us remark that we apply the hedging strategy $\theta_1(\cdot), \dots, \theta_n(\cdot)$ to the actual defaultable claims with associated price processes $l^1(\cdot, T), \dots, l^n(\cdot, T)$. When $\theta_1(t) = \dots = \theta_n(t) = \theta(t)$, the underlying aggregate portfolio becomes the single hedging instrument and $\frac{1}{n} \sum_{i=1}^n \int_0^T \theta_i(t) dl^i(t, T) = \int_0^T \theta(t) dl_n(t, T)$. When $\theta(t) = 1$, we simply hold the aggregate portfolio. However, even when τ_1, \dots, τ_n are exchangeable, there is no reason why the $\theta_i(t)$ should not depend upon the name i for $t > 0$. The name per name (or individual) model involves a larger number of credit deltas but can account for dispersion in the credit spreads which is problematic in an aggregate (or collective) loss model. When the aggregate portfolio is actively traded (say in the case of iTraxx or CDX indices), one may use an exposure of $\theta(t) = \frac{1}{n} \sum_{i=1}^n \theta_i(t)$ to the index and of $\frac{\theta_i(t) - \theta(t)}{n}$ to the individual names in order to minimize transaction costs.

3.2 projection of the option payoff on \mathcal{F}_T

In the previous subsection, we considered the risk management of $(l_n(T) - K)^+$ through the hedging of $(E^Q [l_n(T) | \mathcal{F}_T] - K)^+$. We might also have considered the hedging of $E^Q [(l_n(T) - K)^+ | \mathcal{F}_T]$. We show here that for large portfolios, these two approaches are equivalent.

Lemma 3.4

$$E^Q [(p_n(T) - K)^+ | \mathcal{F}_t] \leq E^Q [(l_n(T) - K)^+ | \mathcal{F}_t], \quad (3.6)$$

for all $t \in [0, T]$ and $K \in [0, 1]$.

PROOF: let us remark that $p_n(T) = E^Q [l_n(T) | \mathcal{F}_T]$. From conditional Jensen inequality, we have: $(p_n(T) - K)^+ \leq E^Q [(l_n(T) - K)^+ | \mathcal{F}_T]$ which yields the stated result.

Thus, $E^Q [(p_n(T) - K)^+] \leq E^Q [(l_n(T) - K)^+]$, which is consistent with the smaller "volatility" of $p_n(T)$ compared with $l_n(T)$.

Lemma 3.5

$$\left\| E^Q [(p_n(T) - K)^+ | \mathcal{F}_t] - E^Q [(l_n(T) - K)^+ | \mathcal{F}_t] \right\|_{2, Q}^2 \leq \frac{1}{2n},$$

for $0 \leq K \leq 1$ and $0 \leq t \leq T$.

PROOF: let us denote by $u = \left| E^Q [(p_n(T) - K)^+ - (l_n(T) - K)^+ | \mathcal{F}_t] \right|$. Thanks to conditional Jensen inequality, we can bound u by $E^Q \left[\left| (p_n(T) - K)^+ - (l_n(T) - K)^+ \right| | \mathcal{F}_t \right]$ and thus by $E^Q [|p_n(T) - l_n(T)| | \mathcal{F}_t]$, which is itself bounded by $\left(E^Q [(p_n(T) - l_n(T))^2 | \mathcal{F}_t] \right)^{1/2}$.

Using lemma (3.1), we have $E^Q \left[(p_n(T) - l_n(T))^2 \mid \mathcal{F}_T \right] = \frac{1}{n^2} \sum_{i=1}^n \text{Var}^Q [N_i(T) \mid \mathcal{F}_T] \leq \frac{1}{2n}$. Using the monotonicity of conditional expectations and the law of total expectation, $E^Q \left[(p_n(T) - l_n(T))^2 \mid \mathcal{F}_t \right] \leq \frac{1}{2n}$ and thus $u \leq \frac{1}{\sqrt{2n}}$. Thus $E^Q[u^2] \leq \frac{1}{2n}$ which yields the stated result. Let us remark that:

$$E^Q \left[(p_n(T) - K)^+ \mid \mathcal{F}_t \right] \leq E^Q \left[(l_n(T) - K)^+ \mid \mathcal{F}_t \right] \leq E^Q \left[(p_n(T) - K)^+ \mid \mathcal{F}_t \right] + \frac{1}{\sqrt{2n}}, \quad (3.7)$$

for all $t \in [0, T]$ and $K \in [0, 1]$.

When the credit spread market is complete, $E^Q \left[(p_n(T) - K)^+ \mid \mathcal{F}_t \right]$ is the time t price of $(p_n(T) - K)^+$ while $E^Q \left[(l_n(T) - K)^+ \mid \mathcal{F}_t \right]$ is the time t price of $E^Q \left[(l_n(T) - K)^+ \mid \mathcal{F}_T \right]$. In such a complete credit spread market, the above price processes are unambiguously defined and are both \mathcal{F} and \mathcal{G} martingales thanks to the martingale invariance property. For large n , i.e. when the *granularity* of the portfolio is small, these two price processes are close (the inequalities hold almost surely).

Assumption 3 *There exists some bounded \mathcal{F} -predictable processes $\tilde{\theta}_1(\cdot), \dots, \tilde{\theta}_n(\cdot)$ such that:*

$$E^Q \left[(l_n(T) - K)^+ \mid \mathcal{F}_T \right] = E^Q \left[(l_n(T) - K)^+ \right] + \frac{1}{n} \sum_{i=1}^n \int_0^T \tilde{\theta}_i(t) dp^i(t, T) + \tilde{z}_n, \quad (3.8)$$

where \tilde{z}_n is \mathcal{F}_T -measurable, of Q -mean zero and Q -strongly orthogonal to $p^1(\cdot, T), \dots, p^n(\cdot, T)$.

The previous equation is the Galtchouk - Kunita - Watanabe decomposition of $E^Q \left[(l_n(T) - K)^+ \mid \mathcal{F}_T \right]$ for (Q, \mathcal{F}) . $\tilde{\theta}_1(\cdot), \dots, \tilde{\theta}_n(\cdot)$ correspond to the optimal (Q, \mathcal{F}) mean-variance hedging strategy based upon the abstract forward price processes $p^1(\cdot, T), \dots, p^n(\cdot, T)$. When the credit spread market is complete, $\tilde{z}_n = 0$. As above, we will thereafter assume that $0 \leq \tilde{\theta}_i(\cdot) \leq 1$ for $i = 1, \dots, n$.

Proposition 2 *Under assumptions (1) and (3), the hedging error $\tilde{\varepsilon}_n$ defined as:*

$$\tilde{\varepsilon}_n = (l_n(T) - K)^+ - E^Q \left[(l_n(T) - K)^+ \right] - \frac{1}{n} \sum_{i=1}^n \int_0^T \tilde{\theta}_i(t) dl^i(t, T), \quad (3.9)$$

is such that $E^P[\|\tilde{\varepsilon}_n\|]$ is bounded by:

$$\left(E^Q \left[\left(\frac{dP}{dQ} \right)^2 \right] \right)^{1/2} \left(\sqrt{\frac{2}{n}} + \frac{1}{n} \left(\sum_{i=1}^n (Q(\tau_i \leq T) + E^Q[B_i|T]) \right)^{1/2} \right) + E^P[\|\tilde{z}_n\|] \quad (3.10)$$

PROOF: $\left| (l_n(T) - K)^+ - E^Q \left[(l_n(T) - K)^+ \mid \mathcal{F}_T \right] \right|$ can be bounded by $\left| (l_n(T) - K)^+ - (p_n(T) - K)^+ \right| + \left| (p_n(T) - K)^+ - E^Q \left[(l_n(T) - K)^+ \mid \mathcal{F}_T \right] \right|$. Using lemma (3.5) and the proof of proposition (1), we have:

$$E^P \left[\left| (l_n(T) - K)^+ - E^Q \left[(l_n(T) - K)^+ \mid \mathcal{F}_T \right] \right| \right] \leq \left(E^Q \left[\left(\frac{dP}{dQ} \right)^2 \right] \right)^{1/2} \sqrt{\frac{2}{n}}.$$

The stochastic integral terms are treated as in lemma (3.3). This shows that when considering the risk-management of the CDO payoff $(l_n(T) - K)^+$ we may as well choose the strategy $\theta_1(\cdot), \dots, \theta_n(\cdot)$ or the strategy $\tilde{\theta}_1(\cdot), \dots, \tilde{\theta}_n(\cdot)$.

3.3 study of $E^Q [(l_n(T) - K)^+ | \mathcal{G}_t]$

$E^Q [(l_n(T) - K)^+ | \mathcal{G}_t]$ is the expectation of the payoff under "the" risk-neutral probability Q^9 . Though there is no theoretical background based on dynamical replication at this stage, it is "tempting" to consider $E^Q [(l_n(T) - K)^+ | \mathcal{G}_t]$ as the time t "price" of the CDO tranche. We can actually relate $E^Q [(l_n(T) - K)^+ | \mathcal{G}_t]$ and $E^Q [(l_n(T) - K)^+ | \mathcal{F}_t]$. We cannot expect the same a.s. inequalities as above since on $\{l_n(t) = 1\}$, $E^Q [(l_n(T) - K)^+ | \mathcal{G}_t]$ reaches the upper bound $1 - K$. We can however state that the processes are close with respect to the $\mathcal{L}^2(Q)$ -norm.

Lemma 3.6

$$\| E^Q [(l_n(T) - K)^+ | \mathcal{G}_t] - E^Q [(l_n(T) - K)^+ | \mathcal{F}_t] \|_{2,Q}^2 \leq \frac{2}{n} \quad (3.11)$$

PROOF: let us denote by $u = E^Q [(l_n(T) - K)^+ | \mathcal{G}_t] - E^Q [(l_n(T) - K)^+ | \mathcal{F}_t]$. From the martingale invariance property, $E^Q [(p_n(T) - K)^+ | \mathcal{G}_t] = E^Q [(p_n(T) - K)^+ | \mathcal{F}_t]$. Thus,

$$|u| \leq |E^Q [(l_n(T) - K)^+ - (p_n(T) - K)^+ | \mathcal{G}_t]| + |E^Q [(p_n(T) - K)^+ - (l_n(T) - K)^+ | \mathcal{F}_t]|.$$

From the proof of lemma (3.5), we already have $|E^Q [(p_n(T) - K)^+ - (l_n(T) - K)^+ | \mathcal{F}_t]| \leq \frac{1}{\sqrt{2n}}$. Using conditional Jensen inequality yields:

$$|E^Q [(l_n(T) - K)^+ - (p_n(T) - K)^+ | \mathcal{G}_t]| \leq E^Q [|(l_n(T) - K)^+ - (p_n(T) - K)^+| | \mathcal{G}_t].$$

The right-hand side of the inequality is bounded by $E^Q [|l_n(T) - p_n(T)| | \mathcal{G}_t]$ which is itself bounded by $(E^Q [(l_n(T) - p_n(T))^2 | \mathcal{G}_t])^{1/2}$. Thus $u^2 \leq 2E^Q [(l_n(T) - p_n(T))^2 | \mathcal{G}_t] + \frac{1}{n}$ and $E^Q [u^2] \leq \frac{2}{n}$ thanks to the law of total expectation and lemma (3.1).

Conclusion

This note shows some simplification in the risk management of CDOs when a large portfolio is involved. According to market practice, a greater consideration is given to the dynamic hedging of credit spread risks, while default risks are mitigated. The Cox modelling assumption is crucial for disentangling default and credit spread risks. In our framework, defaults do not occur simultaneously and are not informative. There are no jumps in credit spreads or related contagion effects after default of one name. In contagion models, we could not think of default and credit spread risks independently.

Though theoretical results in the note suggest that, for infinitely granular portfolios, we could only care of credit spread risks, the practical application to CDX or iTraxx CDO tranches remains to be studied. As far as the number of names is concerned, we are an intermediate stance. It is likely that the credit spread risks should be managed by taking into account observed defaults. It is also likely that the hedging of a tranche should also take into account both default and credit spread risks, for instance using CDS of different maturities.

⁹such that the *traded* defaultable bond price processes are (Q, \mathcal{G}) -martingales.

We did not specialize the credit spread dynamics nor discussed in detail the actual computation of the credit deltas. Using a Markovian framework would certainly help understanding the various effects involved in the hedging of a tranche. Since we are likely to be in a high dimensional framework, efficient numerical approaches must be considered. This is left for future work.

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