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# Dimensional reduction in QCD: Lessons from lower dimensions

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In this contribution we present the results of a series of investigations of dimensional reduction, applied to SU(3) gauge theory in 2 + 1 dimensions. We review earlier results, present a new reduced model with  $Z_3$  symmetry, and discuss the results of numerical simulations of this model.

## §1. Introduction

Dimensional reduction is a powerful method to study the long distance behaviour of field theories at high temperature. It was first introduced in [1, 2]. It was further developed for gauge theories in [3, 4, 5]. The first quantitative studies using lattice simulations of the reduced model were performed in [6, 7, 8, 9]. Quantitative results for QCD were further obtained in [10 - 15]. For a nice review see [16].

In order to find the region of validity of dimensional reduction we have investigated in detail the reduction of SU(3) gauge theory in 2+1 dimensions [17, 18, 19, 20, 21]. In section II we present the method, in Section III some of the results are described. In Section IV an alternative form of dimensional reduction is presented, which preserves the  $Z_3$  symmetry of the action. Section V, finally, gives our conclusions.

## §2. Perturbative dimensional reduction

The properties of gauge theories at finite temperature is described by the partition function

$$Z = \int \mathcal{D}\mathcal{A}_\mu^a \mathcal{D}\psi^{(f)c} \mathcal{D}\bar{\psi}^{(f)c} e^{-S(\mathcal{A}_\mu, \psi, \bar{\psi})} \quad (2.1)$$

where  $\mathcal{A}_\mu^a(x)$  are the gauge fields in the adjoint representation, and where we have included fermion fields of flavor  $f$  in the fundamental representation of SU(3). The action  $S$  at vanishing chemical potential is given by

$$S = \int_0^{1/T} dx_0 \int d^{d_s} x \left[ \frac{1}{4} \mathcal{F}_{\mu\nu}^a \mathcal{F}_{\mu\nu}^a + \sum_f \bar{\psi}^{(f)} (\not{D} + m_f) \psi^{(f)} \right] \quad (2.2)$$

The metric is Euclidean and  $d_s$  is the number of space dimensions,  $x = (x_0, \bar{x})$ . The physical case is  $d_s = 3$ . In this contribution we will mainly discuss a simpler theory,

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\*) Presented by B. Petersson

where the method of dimensional reduction can be tested in detail, namely pure SU(3) gauge theory in 2+1 dimensions ( $d_s = 2$ ) without fermion fields.

The  $x_0$ -integration goes from zero to  $1/T$ , where  $T$  is the temperature. For  $T \rightarrow \infty$ , and for distances  $R \gg 1/T$  it seems plausible that one can integrate out the non static modes of the field perturbatively. It is important to note that the perturbation theory of these modes does not have infrared divergences, in contrast to the perturbative expansion in the full theory. We thus obtain an effective model of the static modes in  $d_s$  dimensions. Note that the fermion fields, which obey Fermi statistics have antiperiodic boundary conditions in  $x_0$ , and thus have no static modes.

$$Z = \int \mathcal{D}\mathcal{A}_0^{st}(\bar{x}) \mathcal{D}\mathcal{A}_i^{st}(\bar{x}) e^{-S_{eff}(\mathcal{A}_0^{st}(\bar{x}), \mathcal{A}_i^{st}(\bar{x}))} \quad (2.3)$$

It can be shown, that for  $T$  large and  $R \gg 1/T$  ( $|\bar{p}| \ll T$ ),  $S_{eff}$  has a finite number of local terms at given order in  $g^2/T$ ,  $|\bar{p}|/T$  [5]. The coefficients are determined by the renormalized perturbative expansion, and there are thus no new free parameters. A pictorial description of dimensional reduction is given in Fig. 1, where the word Higgs refers to the field  $\mathcal{A}_0^{st}$ . Obviously the reduced theory demands much less computertime than the full theory.

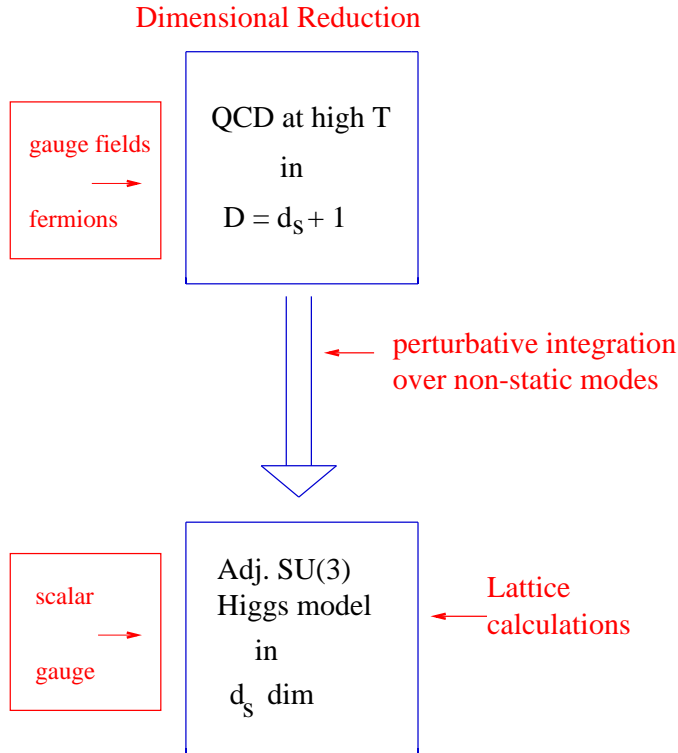


Fig. 1. Pictorial description of dimensional reduction

The static modes are defined by

$$\mathcal{A}_\mu(x) = \mathcal{A}_\mu^{st}(\bar{x}) + \mathcal{A}_\mu^{ns}(x); \quad \int_0^{1/T} dx_0 \mathcal{A}_\mu^{ns}(x) = 0 \quad (2.4)$$

Note that this splitting is not gauge invariant in general. The physical quantities, which we calculate are, however, gauge invariant. Define

$$\begin{aligned} \mathcal{A}_i^{st}(\bar{x}) &= \sqrt{T} A_i(\bar{x}) \\ \mathcal{A}_0^{st}(\bar{x}) &= \sqrt{T} \phi(\bar{x}) \end{aligned} \quad (2.5)$$

On the classical (tree) level we set

$$S_{eff} = \int d^{d_s} x \left[ \frac{1}{4} [F_{ij}^a(\bar{x})]^2 + \sum_{i=1}^{d_s} Tr[D_i \phi]^2 \right] \quad (2.6)$$

This we also call “naive reduction”. Because of the ultraviolet behaviour of the theory, we should, however, also take quantum effects into account. Restricting to the local terms, important for large distance physics, we find to one loop order the systematics described in Figure 2.

Quantum theory  
(one loop)

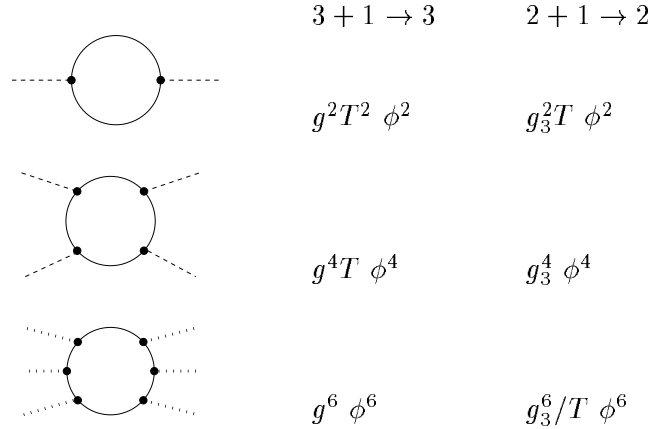


Fig. 2. Contributions to the  $\phi^2$ ,  $\phi^4$  and  $\phi^6$  terms in the reduced action. In the diagrams, dotted lines are static modes, full lines non-static modes. The numerical coefficients can be found in the original papers, for 3+1 dimensions in [6, 7, 8, 10, 11] and for 2+1 dimensions in [17].

Note that in  $d_s = 3$ ,  $g^2$  is dimensionless. For  $d_s = 2$ ,  $g_3^2$  has dimension energy, and  $\phi$  is dimensionless. In the latter case, obviously higher loop orders are suppressed at high  $T$  by powers of  $g_3^2/T$ .

The form of  $S_{eff}$  calculated through perturbation theory of the non static modes in the lattice regularization can be found in [5, 6, 7, 8], and in dimensional regularization in [10] [11]. Here I present the result for  $d_s = 2$ , pure gauge theory, which we

derived in [16]. We have chosen the static Landau gauge [STALG]

$$\begin{aligned}\partial_0 \mathcal{A}_0(x) &= 0 \\ \int_0^{1/T} \partial_i \mathcal{A}_i(x) dx_0 &= 0\end{aligned}\tag{2.7}$$

We find to one-loop order in lattice regularization with spacing  $a$

$$\begin{aligned}S_2 \equiv S_{eff} &= \int d^2x \left[ \frac{1}{4} F_{ij}^a F_{ij}^a + \sum_i^{d_s} Tr[D_i \phi]^2 \right. \\ &\quad \left. - \frac{3g_3^2 T}{2\pi} \left[ \frac{5}{2} \log 2 - 1 - \log aT \right] Tr \phi^2 + \frac{g_3^4}{32\pi} Tr \phi^4 \right]\end{aligned}\tag{2.8}$$

The effective model is a Higgs gauge model in two dimensions, with the Higgs field  $\phi$  in the adjoint representation. The action has a particular symmetry,  $R$ -symmetry, under  $\phi \rightarrow -\phi$ , which comes from the  $T$ -reflection symmetry in the 2+1 dimensional theory. Including fermions in the original model will only make a change of the value of the coefficients. At finite density, however,  $R$ -symmetry and hermiticity is broken, and there are also terms with odd powers of  $\phi$  and imaginary coefficients [15].

There is a logarithmic divergence in the coefficient of the quadratic term in  $\phi$ . This will be cancelled by the ultraviolet divergence of the two dimensional model. The perturbation theory of the model is infrared divergent. And although the pure gauge theory in two dimensions has been solved on the lattice [22], and in the continuum in the large  $N$  limit [23], there is no known solution for 2d gauge theory coupled to Higgs fields in the adjoint representation. Instead we solve the model numerically, using lattice Monte Carlo simulations. As a lattice version of this action we choose

$$\begin{aligned}S_{2L} &= \beta_3 L_0 \sum_{\bar{x}} \left( 1 - \frac{1}{3} Re Tr U_p \right) \\ &\quad + \sum_{\bar{x}, i} Tr \left( U(\bar{x}; i) \phi(\bar{x} + a\hat{i}) U(\bar{x}, i)^{-1} - \phi(\bar{x}) \right)^2 \\ &\quad - \frac{9}{\pi L_0 \beta_3} \left[ \frac{5}{2} \log 2 - 1 + \log L_0 \right] Tr \phi^2(\bar{x}) + \frac{9}{8\pi \beta_3^2} Tr \phi^4\end{aligned}\tag{2.9}$$

The parameters  $\beta_3$  and  $L_0$  which appear in this action are related to the coupling  $g_3^2$  and the temperature  $T$  of  $QCD_{2+1}$  by

$$\beta_3 = \frac{6}{ag_3^2}; \quad L_0 = 1/aT\tag{2.10}$$

where  $a$  is the lattice spacing.

The results for physical quantities using this effective action are compared with the full 2+1 dimensional  $SU(3)$  gauge theory, calculated with the lattice action

$$S_{3L} = \beta_3 \sum_x \sum_{\mu < \nu} \left( 1 - \frac{1}{3} Re Tr \mathcal{U}_p(x, \mu, \nu) \right),\tag{2.11}$$

with the same spacing.

The matrices  $\mathcal{U}_p$  and  $U_p$  are the products of  $\mathcal{U}(x, i)$  resp  $U(\bar{x}, i)$  around a plaquette in 3 resp 2 dimensions,  $\mathcal{U}(x, i)$  and  $U(\bar{x}, i) \in \text{SU}(3)$ . The matrix  $\phi(\bar{x})$  is, however, in the algebra of  $\text{SU}(3)$ ; thus the  $Z_3$ -symmetry of  $S_{3L}$  is broken in  $S_{2L}$ .

We also define a dimensionless reduced temperature

$$\tau \equiv \frac{T}{g_3^2} = \frac{\beta_3}{6L_0}. \quad (2.12)$$

The 2+1 dimensional model has a deconfining second order phase transition at [24]

$$\tau = \tau_c = 0.614(3). \quad (2.13)$$

### §3. Some results:

Our main observable will be the correlation function of Polyakov loops  $L(\bar{x})$  and the corresponding screening mass i.e. inverse correlation length.

In STALG we have

$$L(\bar{x}) = \frac{1}{3} \text{Tr} V(\bar{x}) \quad (3.1)$$

$$V(\bar{x}) = e^{i\phi(\bar{x})/\sqrt{\tau}}, \quad (3.2)$$

so that  $L(\bar{x})$  is a static operator and can be taken out of the integration over non-static variables. Therefore the comparison with the full model is straightforward.

In Figure 3 we plot the screening mass as defined above. For details, see [17]. The screening masses in the naive tree level reduction do not agree with those of the full model. The agreement is, however, very good in the one-loop approximation form down to  $T \approx 1.5T_c$ . At  $T_c$  the screening mass goes to zero in the full model.

The reduced model does not have this second order phase transition, and thus there will be necessarily a discrepancy in the values of the screening masses around  $T = T_c$ .

We have also measured the spacelike string tension. Also here there is good agreement between the full and the reduced model. In this case at high temperature there is also good agreement with the analytic result of pure gauge theory in 2d.

In the 2d model one may in fact identify two screening masses, the lowest states  $m_s$  of  $\text{Tr}\phi^2$  ( $R = 1$ ) and  $m_{ps}$  of  $\text{Tr}\phi^3$  ( $R = -1$ ). The corresponding correlation functions are nearly straight exponentials, corresponding to well isolated poles. The ratio  $m_{ps}/m_s$  between the two masses are between 1.5 and 2 depending on the temperature. Only a more detailed analysis could tell if there is agreement with the naive counting rule  $m_{ps}/m_s = 1.5$  [25] For further details see [18].

The reduced model has two phases, a symmetric phase and a phase where the  $R$ -symmetry is spontaneously broken (Higgs phase). The transition in between is first order. The parameters in the one-loop approximation are in the unphysical broken phase, but very near the transition. In fact, we have performed the measurements at those values, but in the metastable symmetric phase region [17].

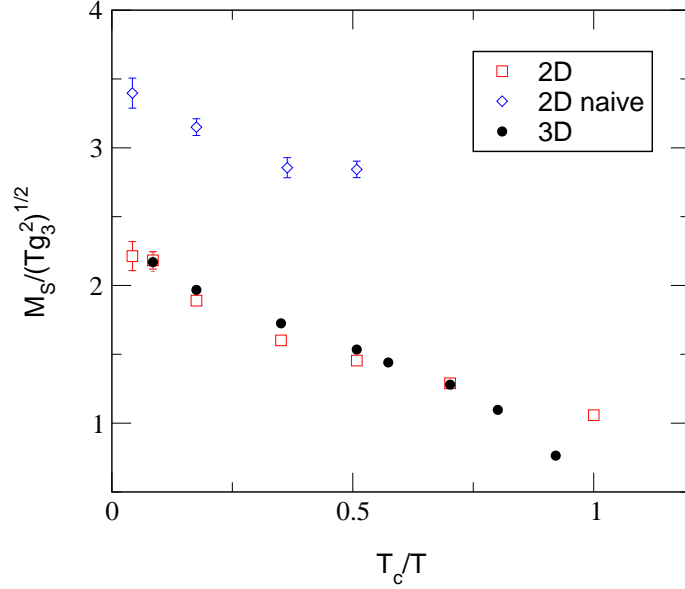


Fig. 3. Physical screening masses  $M_S$  in units of  $g_3\sqrt{T}$  versus  $T_c/T$ , in (2+1)  $D$  (black points) and  $2D$  (squares). Also shown are the masses obtained with the tree level reduced action (diamonds).

#### §4. $Z_3$ -symmetric dimensional reduction

To obtain a reduced  $Z_3$  symmetric model we define instead of  $U, \phi$  effective variables  $U(\bar{x}, i)$  and  $V(\bar{x})$  which are  $SU(3)$  matrices, where  $\frac{1}{3}\text{Tr}V(\bar{x})$  is the Polyakov loop.

Define

$$\begin{aligned}
 Z_{eff} &= \int d[U]d[V]e^{-S_{eff}} \\
 S_U &= \beta_3 L_0 \sum_{\bar{x}} \left(1 - \frac{1}{3}\text{ReTr}U_p\right) \\
 S_{U,V} &= \frac{\beta_3}{L_0} \sum_{\bar{x}, i} \left(1 - \frac{1}{3}\text{ReTr}U(\bar{x}, i)V(\bar{x} + a\hat{i})U(\bar{x}, i)^+V(\bar{x})^+\right) \\
 S_V &= \lambda_2 \sum_x |\text{Tr}V(x)/3|^2 \\
 S_{eff} &= S_U + S_{U,V} + S_V
 \end{aligned} \tag{4.1}$$

The term  $S_{U,V}$  is constructed such that developing  $V(\bar{x})$  in  $\phi$  we get in lowest order the kinetic term of  $S_{2L}$  above. Similarly, developing  $S_V$  we obtain terms proportional to  $\text{Tr}\phi^2$  and higher orders.

We investigated this model numerically, keeping  $\lambda_2$  a free parameter [20, 21]. Similar models have been proposed in Refs [26, 27] and for imaginary chemical potential in Refs [28, 29].

At sufficiently high temperature in the deconfined phase this model should be

similar to the perturbative reduction. It has, however, an explicit  $Z_3$ -symmetry in the action, which may be broken at high temperature, but becomes restored at lower  $T$ . We have performed numerical simulations of the model, keeping  $L_0 = 4$ , and scanning the  $(\beta_3, \lambda_2)$  plane. With  $L_0$  fixed, the temperature is proportional to  $\beta_3$ . To characterize the phases we use the distribution of the Polyakov loop.

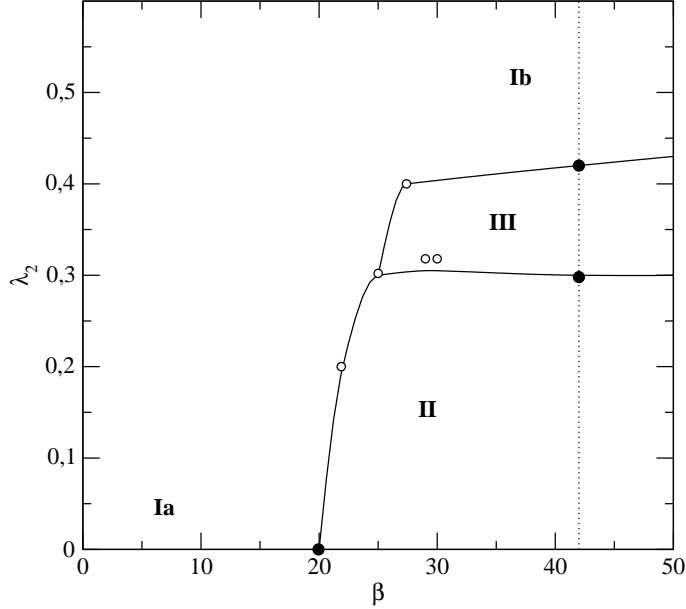


Fig. 4. A tentative phase diagram. Open points denote the location of peaks in susceptibility observed on  $16 \times 16$  lattices. Black points denote the position of phase transitions established on  $32 \times 32$  lattices with big statistics. The lines are hand drawn sketches of where transitions take place.

We find a quite non-trivial phase structure, as shown in Figure 4. For  $\lambda_2 = 0$  there is in fact a transition to a confining phase at  $\beta_3 = 20$ , corresponding to  $\tau_c = 0.83$ , about 25% higher than in the full model. We checked that the screening mass actually vanishes there, as it should. Then fixing  $\beta_3$  at 42 ( $\tau \approx 2\tau_c$ ), we made a more detailed investigation varying  $\lambda_2$ . The results for the susceptibility are plotted in Fig. 5.

We find in fact two transitions, corresponding to three different phases. In the phase II of Fig. 4 with small  $\lambda_2$ , large  $\beta_3$ , the values of  $L$  are complex and concentrated near  $(1, 0)$  up to  $Z_3$  rotations, i.e. their phases are close to  $2i\pi n/3$ . This is the normal plasma phase. In the middle phase (phase III)  $L$  has also non vanishing values near the border of phase space, but with phases close to  $2i\pi(n + 1/2)/3$ . This is a new phase. Finally for large  $\lambda_2$ ,  $L \approx 0$ , as it is in the deconfined phase, and we propose that this phase (Ib) is connected to the low temperature phase (Ia).

It would be interesting to investigate, if the new phase plays a role in finite density QCD.

We have investigated the behaviour of the screening mass. It is shown in Figure 6. Since here  $\lambda_2$  is a free parameter, we cannot predict the screening mass from the



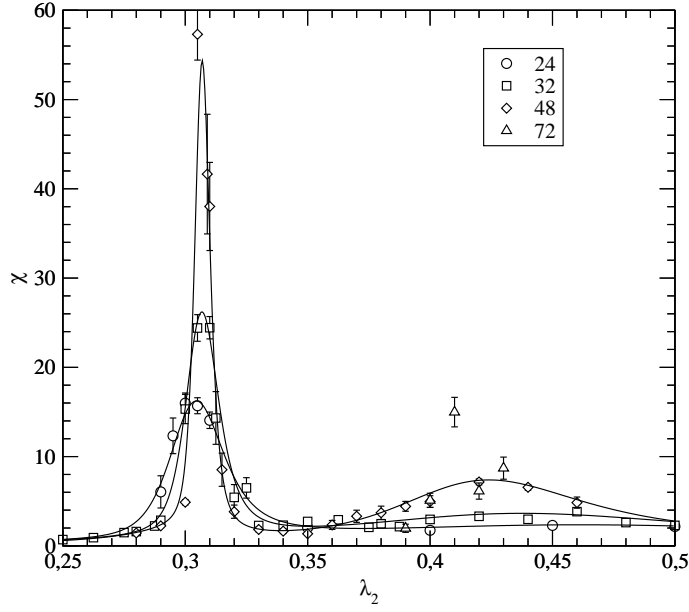


Fig. 5. Polyakov loop susceptibility as a function of  $\lambda_2$  for  $\beta_3 = 42$  on lattices of various sizes. The continuous lines were obtained through Swendsen–Ferrenberg reweightings.

reduced model. If we demand on the other hand that the screening mass in the full and reduced models should be the same, we find this to happen at  $\lambda_2 \approx 0.18$ , well inside the physical phase II, in contrast to the earlier method.

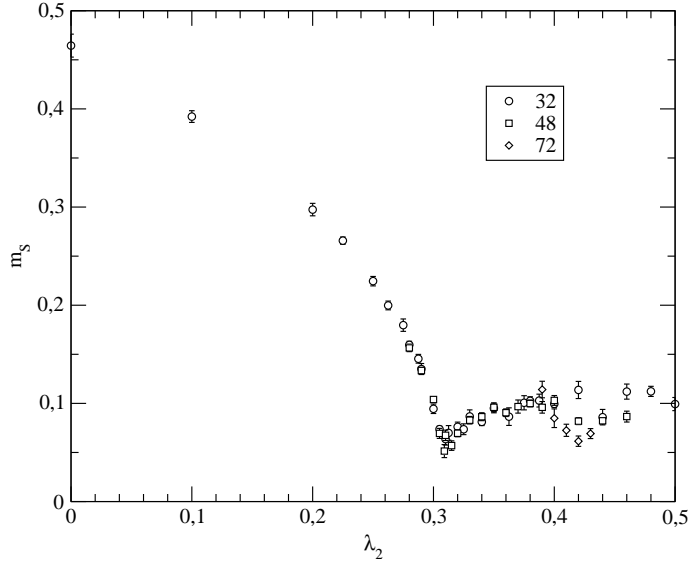


Fig. 6. The screening masses  $m_S$  for  $\beta_3 = 42$  and various lattice sizes.

We have also supplemented the numerical investigation with an analytic mean field calculation, which gives qualitatively the same results, in particular the phase

structure. This calculation can easily be generalized to the 3+1 dimensional case. Again one observes the same phase structure. For further details on the  $Z_3$  symmetric reduction see [20, 21].

## §5. Conclusions

The usual method may be called perturbative reduction, although the reduced theory is calculated non perturbatively. In this method there is a systematic expansion in  $g^2/T$  and  $\frac{1}{RT}$ , where  $R$  is the typical distance of interest. No free parameters are introduced. As we have shown, for finite temperature SU(3) gauge theory in 2+1 dimension this reduction works very well for screening masses and spacelike string tension down to  $T \approx 1.5T_c$ . A difficulty of principle is that the parameters determined from the reduction are not in the stable physical phase of the two dimensional model but in the metastable physical phase slightly beyond the transition. Furthermore in this method, as  $Z_3$  symmetry is explicitly broken, there is no confining transition.

The  $Z_3$  symmetric reduction, which we propose, works all the way down to the transition, and the parameters can be chosen in the physical phase.

However, in this case we have not fixed the parameters of the reduced model from the full model. In order to keep the quantitative predictive power, this has to be done.

Qualitatively the reduced model is in good agreement with the full model. We have also found a  $Z_3$  symmetric new phase, which may be of importance at finite density.

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## References

- 1) P. Ginsparg, Nucl. Phys. B170 (1980) 388.
- 2) T. Appelquist and R. Pisarski, Phys. Rev. D23 (1981) 2305.
- 3) S. Nadkarni, Phys. Rev. D27 (1983) 917; Phys. Rev. D38 (1988) 3287.
- 4) N.P. Landsman, Nucl. Phys. B322 (1989) 498.
- 5) T. Reisz, Z.f.Phys. C53 (192) 169.
- 6) P. Lacock, D.E. Miller and T. Reisz, Nucl. Phys. B369 (1992) 501.
- 7) L. Kärkäinen, P. Lacock, D.E. Miller, B. Petersson and T. Reisz, Phys. Lett. B282 (1992) 121.
- 8) L. Kärkäinen, P. Lacock, D.E. Miller, B. Petersson and T. Reisz, Nucl. Phys. B395 (1993) 733.
- 9) L. Kärkäinen, P. Lacock, D.E. Miller, B. Petersson and T. Reisz, Nucl. Phys. B418 (1994) 3.
- 10) K. Kajantie, M. Laine, K. Rummukainen and M. Shaposhnikov, Nucl. Phys. B503 (1997) 357.

- 11) K. Kajantie, M. Laine, A. Rajantie, K. Rummukainen and M. Tsypin, JHEP 9811 (1998) 011.
- 12) S. Datta and S. Gupta, Phys. Lett. B471 (2000) 382; Nucl. Phys. B534 (1998) 392.
- 13) M. Laine and O. Philipsen, Nucl. Phys. B523 (1998) 291; Phys. Lett. B459 (1999) 259.
- 14) F. Karsch, M. Oevers and P. Petreczky, Phys. Lett. B442 (1998) 291; A. Cucchieri, F. Karsch and P. Petreczky, Phys. Rev. D64 (2001) 036001.
- 15) A. Hart, M. Laine and O. Philipsen, Nucl. Phys. B586 (2000) 443.
- 16) O. Philipsen, Nucl. Phys. Suppl. 94 (2001) 49.
- 17) P. Bialas, A. Morel, B. Petersson, K. Petrov and T. Reisz, Nucl. Phys. B581 (2000) 477.
- 18) P. Bialas, A. Morel, B. Petersson, K. Petrov and T. Reisz, Nucl. Phys. B603 (2001) 369.
- 19) P. Bialas, A. Morel, B. Petersson and K. Petrov, Nucl. Phys. Proc. Suppl. 106 (2002) 882.
- 20) P. Bialas, A. Morel, B. Petersson and K. Petrov, “Z(3) Symmetric Dimensional Reduction of (2+1)D QCD”, Proceedings of the International Symposium on Statistical QCD, [arXiv: hep-lat/0112008].
- 21) P. Bialas, A. Morel and B. Petersson, “Dimensional reduction and a Z(3) symmetric model” to appear in the Proceedings of the XXI International Symposium on Lattice Field Theory, (Tsukuba, Japan, 2003) [arXiv: hep-lat/0309190] and paper in preparation.
- 22) D.J. Gross and E. Witten, Phys. Rev. D21 (1980) 446.
- 23) G. 't Hooft, Nucl. Phys. B75 (1974) 461.
- 24) C. Legeland, Ph.D. Thesis “Aspects of (2+1) Dimensional Lattice Gauge Theory” (Univ. of Bielefeld, 1998)
- 25) W. Buchmüller and O. Philipsen, Phys. Lett. B397 (1997) 112.
- 26) T. Banks and A. Ukawa, Nucl. Phys. B225 (1983) 145.
- 27) R.D. Pisarski, Phys. Rev. D62 (2000) 111501.
- 28) A. Roberge and N. Weiss, Nucl. Phys. B275 (1986) 734.
- 29) Ph. de Forcrand and O. Philipsen, Nucl. Phys. B642 (2002) 290.