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HODGE METRICS AND THE CURVATURE OF HIGHER DIRECT IMAGES

CHRISTOPHE MOURougANE AND SHIGEHARU TAKAYAMA

1. INTRODUCTION

This is a continuation of our works [M] [MT] on the metric positivity of direct image sheaves of adjoint bundles. The goal of this paper is to prove the following

Theorem 1.1. *Let $f : X \rightarrow Y$ be a holomorphic map of complex manifolds, which is smooth, proper, Kähler, surjective, and with connected fibers. Let (E, h) be a holomorphic vector bundle on X with a Hermitian metric h of semi-positive curvature in the sense of Nakano. Then for any $q \geq 0$, the direct image sheaf $R^q f_* \Omega_{X/Y}^n(E)$ is locally free and Nakano semi-positive, where n is the dimension of fibers.*

A real $(1, 1)$ -form ω on X is said to be a *relative Kähler form* for $f : X \rightarrow Y$, if for every point $y \in Y$, there exists a local coordinate $(W; (t_1, \dots, t_m))$ around y such that $\omega + cf^*(\sqrt{-1} \sum_j dt_j \wedge \overline{dt_j})$ is a Kähler form on $f^{-1}(W)$ for a constant c . A morphism f is said to be *Kähler*, if there exists a relative Kähler form ω_f for f (see [Tk, 6.1]).

In case when E is a semi-positive line bundle and $q = 0$, Theorem 1.1 is a theorem of Berndtsson [B, 1.2]. In our previous paper [MT], we obtained independently from [B], a weaker semi-positivity: the Griffiths semi-positivity of $f_* \Omega_{X/Y}^n(E)$ for a semi-ample line bundle E . Right after two papers [B] [MT], especially [B] have appeared, the analogous statement for higher direct images has been considered as a next problem among others. Theorem 1.1 solves this problem for Nakano semi-positive vector bundles E .

For a Nakano semi-positive vector bundle E , the local freeness $R^q f_* \Omega_{X/Y}^n(E)$ is a consequence of Takegoshi's injectivity theorem [Tk]. Here is one point where we use the smoothness of f . We can only expect the torsion freeness in general, by Kollár [Ko1] ([Tk] in analytic setting). Another theorem in [Tk] shows that $R^q f_* \Omega_{X/Y}^n(E)$ can be embedded into $f_* \Omega_{X/Y}^{n-q}(E)$, and that $R^q f_* \Omega_{X/Y}^n(E) = 0$ for $q > n$. The sheaf $f_* \Omega_{X/Y}^{n-q}(E)$ has a natural Hermitian metric induced from a relative Kähler form ω_f and h (at least on which it is locally free) the so-called Hodge metric. For local sections $\sigma, \tau \in H^0(W, f_* \Omega_{X/Y}^{n-q}(E))$, the inner product at $y \in W \subset Y$ is given by $\int_{X_y} (\sigma|_{X_y}) \wedge \overline{*}_{h_y}(\tau|_{X_y})$, where $X_y = f^{-1}(y)$ is the fiber, and $\overline{*}_{h_y}$ is the "star"-operator with respect to $\omega_y = \omega_f|_{X_y}$ and $h_y = h|_{X_y}$. By pulling back this Hodge metric via the injection, say $S_f^q : R^q f_* \Omega_{X/Y}^n(E) \rightarrow f_* \Omega_{X/Y}^{n-q}(E)$,

we have a Hermitian metric on $R^q f_* \Omega_{X/Y}^n(E)$ in the usual sense. Our main theorem is that its curvature is Nakano semi-positive. The space $H^q(X_y, \Omega_{X_y}^n(E_y))$ has a natural inner product with respect to ω_y and h_y . For cohomology classes $u_y, v_y \in H^q(X_y, \Omega_{X_y}^n(E_y))$, it is given by $\int_{X_y} u'_y \wedge \bar{*}_{h_y} v'_y$, where u'_y and v'_y are the harmonic representatives of u_y and v_y respectively. These fiberwise inner products also define a Hermitian metric on $R^q f_* \Omega_{X/Y}^n(E)$. We first tried to compute its curvature, but we did not succeed it.

We follow [B], not [MT], for the method of computation of the curvature. Since one can directly see the original method in [B], let us explain how different from [B], i.e., the differences in cases $q = 0$ and $q > 0$. In case $q = 0$, the map $S_f^0 : f_* \Omega_{X/Y}^n(E) \longrightarrow f_* \Omega_{X/Y}^n(E)$ is an isomorphism, in fact the multiplication by a constant. Moreover $f_* \Omega_{X/Y}^n(E)$ is locally free thanks to Ohsawa-Takegoshi type L^2 -extension theorem [OT] [O] [Ma], and the $(1, 0)$ -derivative of $\sigma \in H^0(Y, f_* \Omega_{X/Y}^n(E)) = H^0(X, \Omega_{X/Y}^n(E))$ vanishes on each fiber X_y by simply a bidegree reason. However in case $q > 0$, we have no local freeness of $f_* \Omega_{X/Y}^{n-q}(E)$, nor the vanishing of the $(1, 0)$ -derivative of $\sigma \in H^0(Y, f_* \Omega_{X/Y}^{n-q}(E)) = H^0(X, \Omega_{X/Y}^{n-q}(E))$ on X_y . To overcome these difficulties, we need to restrict ourselves to consider the image of $S_f^q : R^q f_* \Omega_{X/Y}^n(E) \longrightarrow f_* \Omega_{X/Y}^{n-q}(E)$. Then we have a local freeness as we mentioned above, and the vanishing of the $(1, 0)$ -derivative thanks to an estimate by Takegoshi [Tk]. This is a key fact, and which is like the d -closedness of holomorphic p -forms on a compact Kähler manifold. After getting those key observations: the local freeness, the right Hodge metric to be considered, and the closedness of holomorphic sections, the computation of the curvature is a straight forward generalization of [B].

There are many positivity results of direct image sheaves of relative canonical bundles and of adjoint bundles, which are mostly about the positivity in algebraic geometry. We will recall only a few here. The origin is due to Griffiths in his theory on the variation of Hodge structures [Gr]. Griffiths' work has been generalized by Fujita [Ft], Kawamata [Ka1], Viehweg [Vi1], Kollár [Ko1], and so on, in more algebro-geometric setting. There are also positivity results on higher direct images by Moriwaki [Mw], Fujino [Fn] and so on. We refer to [Mr] [EV] [Vi2] [Ko2] for further remarks on related results. On the analytic side, it can be understood as the plurisubharmonic variation of related functions to the Robin constant [Y] [LY], or of the Bergman kernels [MY]. There is also a series of works by Yamaguchi. As he mentioned in [B], his method is inspired by those works.

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2. PRELIMINARIES

2.1. Hermitian vector bundles. Let X be a complex manifold of dimension n with a Hermitian metric ω , and let E be a holomorphic vector bundle of rank r on X with a Hermitian metric h . Let (E^*, h^*) be the dual vector bundle. Let $A^{p,q}(X, E)$ be the space of E -valued smooth (p, q) -forms, and $A_0^{p,q}(X, E)$ be the space of E -valued smooth (p, q) -forms with compact support. Let $*$: $A^{p,q}(X, E) \rightarrow A^{n-q, n-p}(X, E)$ be the Hodge star-operator with respect to ω . For any $u \in A^{p,q}(X, E)$ and $v \in A^{s,t}(X, E)$, we define $u \wedge h\bar{v} \in A^{p+s, q+t}(X, \mathbb{C})$ as follows. We take a local trivialization of E on an open subset $U \subset X$, and we regard $u = {}^t(u_1, \dots, u_r)$ as a row vector with (p, q) -forms u_j on U , and similarly for $v = {}^t(v_1, \dots, v_r)$. The Hermitian metric h is then a matrix valued function $h = (h_{jk})$ on U . We define $u \wedge h\bar{v}$ locally on U by

$$u \wedge h\bar{v} = \sum_{j,k} u_j \wedge h_{jk} \bar{v}_k \in A^{p+s, q+t}(U, \mathbb{C}).$$

We should write ${}^t u \wedge h\bar{v}$, but if there is no risk of confusions, we will write in this way. In this manner, we can define anti-linear isomorphisms $\sharp_h : A^{p,q}(X, E) \rightarrow A^{q,p}(X, E^*)$ by $\sharp_h u = h\bar{u}$, and $\bar{\sharp}_h = \sharp_h \circ * : A^{p,q}(X, E) \rightarrow A^{n-p, n-q}(X, E^*)$ by $\bar{\sharp}_h u = h\bar{*}u$. The inner product on $A_0^{p,q}(X, E)$ is defined by $(u, v)_h = \int_X u \wedge \bar{\sharp}_h v$. Denote by $D_h = \partial_h + \bar{\partial}$ the metric connection, and by $\Theta_h = D_h^2$ the curvature of (E, h) . The Hermitian vector bundle (E, h) is said to be *Nakano semi-positive* (resp. *Nakano positive*), if the $\text{End}(E)$ -valued real $(1, 1)$ -form $\sqrt{-1}\Theta_h$ is positive semi-definite (resp. positive definite) quadratic form on each fiber of the vector bundle $T_X \otimes E$.

We define $\vartheta_h : A^{p,q}(X, E) \rightarrow A^{p, q-1}(X, E)$ by $\vartheta_h = - * \partial_h * = -\bar{\sharp}_h * \bar{\partial} \bar{\sharp}_h$, which is the formal adjoint operator of $\bar{\partial} : A^{p,q}(X, E) \rightarrow A^{p, q+1}(X, E)$ with respect to the inner product $(\ , \)_h$. We also define $\bar{\vartheta} : A^{p,q}(X, E) \rightarrow A^{p-1, q}(X, E)$ by $\bar{\vartheta} = - * \bar{\partial} *$, which is the formal adjoint operator of $\partial_h : A^{p,q}(X, E) \rightarrow A^{p+1, q}(X, E)$ with respect to the inner product $(\ , \)_h$. We denote by $e(\theta)$ the left exterior product acting on $A^{p,q}(X, E)$ by a form $\theta \in A^{s,t}(X, \mathbb{C})$. Then the adjoint operator $e(\theta)^*$ with respect to the inner product $(\ , \)_h$ is defined by $e(\theta)^* = (-1)^{(p+q)(s+t+1)} * e(\bar{\theta}) *$. For instance we set $\Lambda_\omega = e(\omega)^*$. We recall the following very useful relation ([Huy, 1.2.31] [Vo, 6.29]):

Lemma 2.1. *For a primitive element $u \in A^{p,q}(X, E)$, i.e., $p+q = k \leq n$ and $\Lambda_\omega u = 0$, the Hodge $*$ -operator reads*

$$*(\omega^j \wedge u) = \sqrt{-1}^{p-q} (-1)^{\frac{k(k+1)}{2}} \frac{j!}{(n-k-j)!} \omega^{n-k-j} \wedge u$$

for every $0 \leq j \leq n-k$.

As immediate consequences, we have

Corollary 2.2. Denote by $c_{n-q} = \sqrt{-1}^{(n-q)^2} = \sqrt{-1}^{n-q}(-1)^{(n-q)(n-q-1)/2}$.

(1) Let $a, b \in A^{n-q,0}(X, E)$. Then $*a = (\overline{c_{n-q}}/q!) \omega^q \wedge a$, $\overline{*}_h a = (c_{n-q}/q!) \omega^q \wedge h\overline{a}$, $a = (c_{n-q}/q!) * (\omega^q \wedge a)$, and $a \wedge \overline{*}_h b = (c_{n-q}/q!) \omega^q \wedge a \wedge h\overline{b}$.

(2) Let $u \in A^{n,q}(X, E)$. Then $u = (c_{n-q}/q!) \omega^q \wedge *u$.

Notation 2.3. We use the following conventions often. Denote by $c_d = \sqrt{-1}^{d^2} = \sqrt{-1}^d (-1)^{d(d-1)/2}$ for any non-negative integer d . Let $t = (t_1, \dots, t_m)$ be the coordinates of \mathbb{C}^m .

(1) $dt = dt_1 \wedge \dots \wedge dt_m$, $\overline{dt} = \overline{dt}$, and $dV_t := c_m dt \wedge \overline{dt} = \bigwedge_{j=1}^m \sqrt{-1} dt_j \wedge \overline{dt}_j > 0$.

(2) Let \widehat{dt}_j be a smooth $(m-1, 0)$ -form without dt_j such that $dt_j \wedge \widehat{dt}_j = dt$, and $\widehat{\overline{dt}_j} = \overline{\widehat{dt}_j}$.

(3) Let $\widehat{dt_j \wedge \overline{dt}_k}$ be a smooth $(m-1, m-1)$ -form without dt_j and \overline{dt}_k such that $\sqrt{-1} dt_j \wedge \overline{dt}_k \wedge \widehat{dt_j \wedge \overline{dt}_k} = c_m dt \wedge \overline{dt}$.

2.2. Set up. In the rest of this paper, we will use the following set up.

Let X and Y be complex manifolds of $\dim X = n+m$ and $\dim Y = m$. Let $f : X \rightarrow Y$ be a holomorphic map, which is smooth, proper, Kähler, surjective, and with connected fibers. Let (E, h) be a holomorphic vector bundle on X of rank r , with a Hermitian metric h whose curvature Θ_h is semi-positive in the sense of Nakano.

(I) a general setting: $f : (X, \omega_f) \rightarrow Y$. We take a relative Kähler form ω_f for f , and let $\kappa_f = \{\omega_f\}$ be the de Rham cohomology class. On each fiber X_y , we have a Kähler form $\omega_y = \omega_f|_{X_y}$, and a Nakano semi-positive vector bundle $(E_y, h_y) = (E, h)|_{X_y}$.

(II) a localized setting of (I): $f : (X, \omega) \rightarrow Y \subset \mathbb{C}^m$. We further assume that the base Y is a unit ball in \mathbb{C}^m with coordinates $t = (t_1, \dots, t_m)$ and with admissible charts over Y (see below). We take a global Kähler form $\omega = \omega_f + cf^*(\sqrt{-1} \sum dt_j \wedge \overline{dt}_j)$ on X for large $c > 0$, without changing the class κ_f nor the fiberwise Kähler forms ω_y .

Since $f : X \rightarrow Y$ is smooth, for every point $y \in Y$, we can take a local coordinate $(W; t = (t_1, \dots, t_m))$ centered at y , so that $(W; t)$ is a unit ball in \mathbb{C}^m , and a system of local coordinates $\mathcal{U} = \{(U_\alpha; z_\alpha, t); \alpha = 1, 2, 3, \dots\}$ of $f^{-1}(W)$ which is locally finite, and every U_α is biholomorphic to a product $D_\alpha \times W$ for a domain D_α in \mathbb{C}^n ; $x \mapsto (z_\alpha^1(x), \dots, z_\alpha^n(x), t_1, \dots, t_m)$, namely the projection from $D_\alpha \times W$ to W is compatible with the map $f|_{U_\alpha}$. We can write $z_\alpha^j = f_{\alpha\beta}^j(z_\beta^1, \dots, z_\beta^n, t_1, \dots, t_m)$ for $1 \leq j \leq n$ on $U_\alpha \cap U_\beta$. All $f_{\alpha\beta}^j(z_\beta, t)$ are holomorphic in z_β and t . We call it *admissible charts* \mathcal{U} over W (cf. [Kd, §2.3]).

Since our assertions are basically local on Y , we will mostly use the set up (II). The set up (I) will be used in subsections 3.1, 3.3 and 4.3.

3. GENERALITIES OF RELATIVE DIFFERENTIAL FORMS

Let $f : (X, \omega_f) \longrightarrow Y$ and (E, h) be as in §2.2.I. We recall the complex analytic properties of the relative cotangent bundle $\Omega_{X/Y} = \Omega_X/f^*\Omega_Y$ and the bundle of relative holomorphic p -forms $\Omega_{X/Y}^p = \bigwedge^p \Omega_{X/Y}$. We will not distinguish a vector bundle and the corresponding locally free sheaf. For a subset $S \subset Y$, we denote by $X_S = f^{-1}(S)$ and $E_S = E|_{X_S}$.

3.1. Definition of relative differential forms. Let $U \subset X$ be an open subset.

(1) For a form $u \in A^{p,q}(U, E)$, we have the restriction $u|_{X_y \cap U} \in A^{p,q}(X_y \cap U, E)$ on each fiber over $y \in Y$, which is the pull-back as a form via the inclusion $X_y \longrightarrow X$. Two forms $u, v \in A^{p,q}(U, E)$ are said to be *f-equivalent* “ $u \sim v$ ”, if $u|_{X_y \cap U} = v|_{X_y \cap U}$ for any $t \in Y$. We denote the set of equivalence classes by

$$A^{p,q}(U/Y, E) = A^{p,q}(U, E) / \sim .$$

The set $A^{p,q}(U/Y, E)$ will be called the space of relative differential forms on U . We denote by $[u] \in A^{p,q}(U/Y, E)$ the equivalence class of $u \in A^{p,q}(U, E)$.

(2) A form $u \in A^{p,0}(U, E)$ is said to be *holomorphic on each fiber*, if the restriction $u|_{X_y}$ is holomorphic, i.e., $u|_{X_y} \in H^0(X_y, \Omega_{X_y}^p(E_y))$ for every $y \in Y$. A form $u \in A^{p,0}(U, E)$ is said to be *relatively holomorphic*, if for any local chart $(W; t = (t_1, \dots, t_m))$ of Y , the form $u \wedge f^*dt$ is holomorphic on $X_W \cap U$, i.e., $u \wedge f^*dt \in H^0(X_W \cap U, \Omega_X^{p+m}(E))$.

(3) For a function $\alpha \in A^0(Y, \mathbb{C})$ and $[u] \in A^{p,0}(U/Y, E)$, we can define $\alpha[u] := [(f^*\alpha)u] \in A^{p,0}(U/Y, E)$. For each open subset $W \subset Y$, we set

$$\begin{aligned} A^0(W, f_*\Omega_{X/Y}^p(E)) &:= \{[u] \in A^{p,0}(X_W/W, E); u \text{ is holomorphic on each fiber}\}, \\ H^0(W, f_*\Omega_{X/Y}^p(E)) &:= \{[u] \in A^{p,0}(X_W/W, E); u \text{ is relatively holomorphic}\}. \end{aligned}$$

We can see that $A^0(W, f_*\Omega_{X/Y}^p(E))$ becomes an $A^0(W, \mathbb{C})$ -module, and $H^0(W, f_*\Omega_{X/Y}^p(E))$ becomes an $H^0(W, \mathcal{O}_W)$ -module.

(4) It is some times convenient to use local coordinates to look at those properties above. Let $u \in A^{p,q}(X, E)$. On an admissible chart $(U_\alpha; z_\alpha, t)$ as above, we can write

$$u = \sum_{I \in I_p, J \in I_q} u_{IJ\alpha} dz_\alpha^I \wedge \overline{dz}_\alpha^J + R,$$

where $u_{IJ\alpha} = u_{IJ\alpha}(z_\alpha, t) \in A^0(U_\alpha, \mathbb{C}^r)$, and $R \in \sum_j A^{p-1,q}(U_\alpha, \mathbb{C}^r) \wedge dt_j + \sum_j A^{p,q-1}(U_\alpha, \mathbb{C}^r) \wedge \overline{dt}_j$. Here we use a standard convention. We set $I_p = \{\{i_1, i_2, \dots, i_p\}; 1 \leq i_1 < i_2 < \dots < i_p \leq n\}$, and I_0 is empty. For $I = \{i_1, i_2, \dots, i_p\} \in I_p$, we denote by $dz_\alpha^I = dz_\alpha^{i_1} \wedge \dots \wedge dz_\alpha^{i_p}$. Similar for $J \in I_q$ and \overline{dz}_α^J . The restriction on a fiber is locally given by

$$u|_{X_y} = \sum_{I \in I_p, J \in I_q} u_{IJ\alpha}|_{X_y} dz_\alpha^I \wedge \overline{dz}_\alpha^J.$$

In particular, for two forms $u, v \in A^{p,q}(X, E)$, they are f -equivalent $u \sim v$ if and only if $u_{IJ\alpha} = v_{IJ\alpha}$ for any $(I, J) \in I_p \times I_q$ on any admissible chart $(U_\alpha; z_\alpha, t)$.

(5) Let $u \in A^{p,0}(X, E)$. On an admissible chart $(U_\alpha; z_\alpha, t)$, we have

$$u = \sum_{I \in I_p} u_{I\alpha} dz_\alpha^I + R$$

as above. Therefore u is holomorphic on each fiber (resp. relatively holomorphic) if and only if every $u_{I\alpha}$ is holomorphic in z_α (resp. holomorphic in z_α and t) for any $I \in I_p$ on any admissible chart $(U_\alpha; z_\alpha, t)$.

3.2. Holomorphic structure of $f_*\Omega_{X/Y}^p(E)$. We also give the holomorphic structure on $f_*\Omega_{X/Y}^p(E)$ by an action of the $\bar{\partial}$ -operator. Let $(W; (t_1, \dots, t_m)) \subset Y$ be a local chart, over which we have admissible charts. Let $[\sigma] \in A^0(W, f_*\Omega_{X/Y}^p(E))$, which can be seen as a differentiable family of holomorphic forms. Since $(\bar{\partial}\sigma)|_{X_y} = \bar{\partial}(\sigma|_{X_y}) = 0$, we can write as

$$\bar{\partial}\sigma = \sum_j \eta^j \wedge dt_j + \sum_j \nu^j \wedge d\bar{t}_j$$

with some $\eta^j \in A^{p-1,1}(X_W, E)$ and some $\nu^j \in A^{p,0}(X_W, E)$. In particular $\bar{\partial}(\sigma \wedge dt) = \sum_j \nu^j \wedge d\bar{t}_j \wedge dt$. On an admissible chart $(U; z, t) = (U_\alpha; z_\alpha, t)$, we write $\sigma = \sum_{I \in I_p} \sigma_I dz^I + R$ with $R \in \sum_j A^{p-1,0}(U_\alpha, \mathbb{C}^r) \wedge dt_j$. Then we have $\nu^j|_{X_y \cap U} = (-1)^p \sum_{I \in I_p} (\partial\sigma_I / \partial\bar{t}_j)|_{X_y \cap U} dz^I \in H^0(X_y \cap U, \Omega_{X_y}^p(E_y))$ for every j . In particular, the class $[\nu^j] \in A^0(W, f_*\Omega_{X/Y}^p(E))$ is well-defined for $[\sigma]$. For $[\sigma] \in A^0(W, f_*\Omega_{X/Y}^p(E))$, we define

$$\bar{\partial}[\sigma] = \sum_j [\nu^j] d\bar{t}_j \in A^{0,1}(W, f_*\Omega_{X/Y}^p(E)).$$

Here $A^{0,1}(W, f_*\Omega_{X/Y}^p(E)) = A^0(W, f_*\Omega_{X/Y}^p(E)) \otimes A^{0,1}(W, \mathbb{C})$ as $A^0(W, \mathbb{C})$ -module, but it has only a formal meaning. Then, $[\sigma] \in H^0(W, f_*\Omega_{X/Y}^p(E))$ if and only if $\bar{\partial}[\sigma] \equiv 0$. In fact both of them are characterized by the holomorphicity of all σ_I in z and t locally.

Lemma 3.1. *Let $(W; (t_1, \dots, t_m)) \subset Y$ be a local chart as above in §2.2. Let $\sigma \in A^{p,0}(X_W, E)$ such that $[\sigma] \in H^0(W, f_*\Omega_{X/Y}^p(E))$. Then (1)*

$$\bar{\partial}\sigma = \sum_j \eta^j \wedge dt_j$$

with some $\eta^j \in A^{p-1,1}(X_W, E)$,

- (2) these η^j are not unique, but $[\eta^j] \in A^{p-1,1}(X_W/W, E)$ are well-defined for σ ,
- (3) all $\eta^j|_{X_y}$ are $\bar{\partial}$ -closed on any X_y , and
- (4) [B, Lemma 4.3] all $\eta^j|_{X_y} \wedge \omega_y^{q+1}$ are $\bar{\partial}$ -exact on any X_y .

Proof. (1) is now clear. We show (2) and (3). For each j , $\sigma \wedge \widehat{dt}_j \in A^{p+m-1,0}(X_W, E)$ is well-defined for σ , and so is $\bar{\partial}(\sigma \wedge \widehat{dt}_j) = \eta^j \wedge dt$. Hence $[\eta^j]$ are well-defined for

σ , by Remark 3.2 below. Moreover $(\bar{\partial}\eta^j) \wedge dt = \bar{\partial}\bar{\partial}(\sigma \wedge \widehat{dt}_j) = 0$. Hence we obtain $\bar{\partial}(\eta^j|_{X_y}) = (\bar{\partial}\eta^j)|_{X_y} = 0$ by Remark 3.2 again.

(4) We fix j . By a bidegree reason, we can write as $\sigma \wedge \widehat{dt}_j \wedge \omega^{q+1} = a^j \wedge dt$ with some $a^j \in A^{n,q+1}(X, E)$. We note that the class $[a^j] \in A^{n,q+1}(X/Y, E)$ is well-defined by Remark 3.2. By taking $\bar{\partial}$, we have $\eta^j \wedge dt \wedge \omega^{q+1} = (\bar{\partial}a^j) \wedge dt$. Then $[\eta^j \wedge \omega^{q+1}] = [\bar{\partial}a^j]$ in $A^{n,q+2}(X/Y, E)$ by Remark 3.2, and hence $\eta^j|_{X_y} \wedge \omega_y^{q+1} = \bar{\partial}(a^j|_{X_y})$ on any X_y . \square

Remark 3.2. For $u, v \in A^{p,q}(X_W, E)$, a relation $u \wedge dt = v \wedge dt$ implies $[u] = [v]$ in $A^{p,q}(X_W/W, E)$, and the converse holds true in case $q = 0$.

Remark 3.3. Each cohomology class $\{\eta^j|_{X_y}\} \in H^{p-1,1}(X_y, E_y)$ is well-defined for $[\sigma] \in H^0(W, f_*\Omega_{X/Y}^p(E))$. As it is well-known, $\{\eta^j|_{X_y}\}$ is obtained by the cup-product (up to a sign) with the Kodaira-Spencer class of $f : X \rightarrow Y$ at $y \in Y$ for t_j -direction. However we will not use these remarks.

3.3. Canonical pairing. There is a canonical pairing on each stalk $f_*\Omega_{X/Y}^p(E)_y$ with respect to ω_y and h_y , via the natural inclusion $f_*\Omega_{X/Y}^p(E)_y \subset H^0(X_y, \Omega_{X_y}^p(E_y))$. At each point $y \in Y$, we have the fiberwise inner product

$$g_y(\sigma_y, \tau_y) := (\sigma_y, \tau_y)_{h_y} = \int_{X_y} (c_p/(n-p)!) \omega_y^{n-p} \wedge \sigma_y \wedge h\bar{\tau}_y$$

for $\sigma_y, \tau_y \in A^{p,0}(X_y, E_y)$. As germs $\sigma_y, \tau_y \in f_*\Omega_{X/Y}^p(E)_y$, we will denote by $g_y(\sigma_y, \tau_y)$. On the other hand, as forms $\sigma_y, \tau_y \in H^0(X_y, \Omega_{X_y}^p(E_y))$, we will denote by $(\sigma_y, \tau_y)_{h_y}$. These two are the same, but our standing points are different, i.e., at a point $y \in Y$, or on the fiber X_y .

For relative forms $[\sigma], [\tau] \in A^{p,0}(X_W/W, E)$ (or $H^0(W, f_*\Omega_{X/Y}^p(E))$) over an open subset $W \subset Y$, the above fiberwise inner product gives

$$g([\sigma], [\tau]) := f_*((c_p/(n-p)!) \omega_f^{n-p} \wedge \sigma \wedge h\bar{\tau}),$$

where the right hand side is a push-forward as a current. For a test (m, m) -form φ on W (i.e., a smooth form with compact support), we have $f_*((c_p/(n-p)!) \omega_f^{n-p} \wedge \sigma \wedge h\bar{\tau})(\varphi) := \int_X (c_p/(n-p)!) \omega_f^{n-p} \wedge \sigma \wedge h\bar{\tau} \wedge f^*\varphi$. Hence the right hand side does not depend on representatives σ nor τ (see Remark 3.2). Since the map f is smooth, $g([\sigma], [\tau])$ is in fact a smooth function on W .

We remark that the definition of the pairing g depends only on the fiberwise Kähler forms $\{\omega_y\}_{y \in Y}$. For example, over a local chart $(W; t) \subset Y$, we can replace ω_f by $\omega_f + c f^*(\sqrt{-1} \sum dt_j \wedge \bar{dt}_j)$ for any $c \in \mathbb{R}$ in the definition of $g([\sigma], [\tau])$. The pairing g defines a Hermitian metric on every locally free subsheaf of $f_*\Omega_{X/Y}^p(E)$ in the usual sense, which we call the Hodge metric. As a matter of fact, g itself is called the Hodge metric on $f_*\Omega_{X/Y}^p(E)$ commonly, although it may not be locally free.

4. HARMONIC THEORY FOR NAKANO SEMI-POSITIVE VECTOR BUNDLES

We collect some fundamental results of Takegoshi [Tk], and immediate consequences from them.

4.1. Absolute setting. Let (X_0, ω_0) be an n -dimensional compact Kähler manifold and let (E_0, h_0) be a holomorphic Hermitian vector bundle on X_0 . We have an inner product $(\ , \)_{h_0}$ and the associated norm $\| \ \|_{h_0}$ on each $A^{p,q}(X_0, E_0)$. Let $\mathcal{H}^{p,q}(X_0, E_0)$ be the space of harmonic (p, q) -forms. From Bochner-Kodaira-Nakano formula, it then follows that if (E_0, h_0) is furthermore Nakano semi-positive and Nakano positive at one point, then $H^q(X_0, \Omega_{X_0}^n(E_0))$ vanish for all $q > 0$.

Enoki [E] and Takegoshi [Tk] (a special case of Theorem 4.1 below) show that if (E_0, h_0) is Nakano semi-positive, then the Hodge $*$ -operator yields injective homomorphism $*_0 : \mathcal{H}^{n,q}(X_0, E_0) \longrightarrow H^0(X_0, \Omega_{X_0}^{n-q}(E_0))$. Recalling that $(c_{n-q}/q!) \omega_0^q \wedge *_0 u = u$ for $u \in A^{n,q}(X_0, E_0)$, it then follows that the Lefschetz operator $L_0^q : H^0(X_0, \Omega_{X_0}^{n-q}(E_0)) \longrightarrow H^q(X_0, \Omega_{X_0}^n(E_0))$ is surjective. Hence we have $H^0(X_0, \Omega_{X_0}^{n-q}(E_0)) = \text{Ker } L_0^q \oplus *_0 \mathcal{H}^{n,q}(X_0, E_0)$.

4.2. Localized relative setting. Let $f : (X, \omega) \longrightarrow Y \subset \mathbb{C}^m$ and (E, h) be as in §2.2.II. We take a C^∞ plurisubharmonic exhaustion function $\Phi = f^* \sum_{j=1}^m |t_j|^2$ on X . We take any $0 \leq q \leq n$. Following [Tk, 4.3], we set the following subspace of E -valued harmonic $(n+m, q)$ -forms with respect to ω and h :

$$\mathcal{H}^{n+m,q}(X, E, \Phi) = \{u \in A^{n+m,q}(X, E); \bar{\partial}u = \vartheta_h u = 0 \text{ and } e(\bar{\partial}\Phi)^*u = 0 \text{ on } X\}.$$

By [Tk, 4.3.i], $u \in \mathcal{H}^{n+m,q}(X, E, \Phi)$ if and only if $\bar{\vartheta}u = 0$, $(\sqrt{-1}e(\Theta_h + \partial\bar{\partial}\Phi)\Lambda_\omega u) \wedge h\bar{u} = 0$ and $e(\bar{\partial}\Phi)^*u = 0$ on X . One can check easily that $(f^*\alpha)u$ satisfies those latter three conditions, if $\alpha \in H^0(Y, \mathcal{O}_Y)$ and if $u \in \mathcal{H}^{n+m,q}(X, E, \Phi)$.

Theorem 4.1. [Tk, 4.3]. (1) *The space $\mathcal{H}^{n+m,q}(X, E, \Phi)$ does not depend on C^∞ plurisubharmonic exhaustion functions Φ , and has a natural structure of $H^0(Y, \mathcal{O}_Y)$ -module.*

(2) *For $u \in \mathcal{H}^{n+m,q}(X, E, \Phi)$, one has $\bar{\partial} * u = 0$ and $\partial_h * u = 0$. In particular, the Hodge $*$ -operator yields an injective homomorphism $* : \mathcal{H}^{n+m,q}(X, E, \Phi) \longrightarrow H^0(X, \Omega_X^{n+m-q}(E))$, and $\mathcal{H}^{n+m,q}(X, E, \Phi)$ becomes a torsion free $H^0(Y, \mathcal{O}_Y)$ -module.*

Let $\iota' : Z_{\bar{\partial}}^{n+m,q}(X, E) \longrightarrow H^q(X, \Omega_X^{n+m}(E))$ be the quotient map which induces the Dolbeault's isomorphism.

Theorem 4.2. [Tk, 5.2]. (1) *The space $\mathcal{H}^{n+m,q}(X, E, \Phi)$ represents $H^q(X, \Omega_X^{n+m}(E))$ as a torsion free $H^0(Y, \mathcal{O}_Y)$ -module, in particular the quotient map ι' induces an isomorphism $\iota : \mathcal{H}^{n+m,q}(X, E, \Phi) \longrightarrow H^q(X, \Omega_X^{n+m}(E))$.*

(2) The injective homomorphism $*$: $\mathcal{H}^{n+m,q}(X, E, \Phi) \longrightarrow H^0(X, \Omega_X^{n+m-q}(E))$ induces a splitting homomorphism (up to a constant)

$$* \circ \iota^{-1} : H^q(X, \Omega_X^{n+m}(E)) \longrightarrow H^0(X, \Omega_X^{n+m-q}(E))$$

for the Lefschetz homomorphism

$$L^q : H^0(X, \Omega_X^{n+m-q}(E)) \longrightarrow H^q(X, \Omega_X^{n+m}(E)).$$

such that $(c_{n+m-q}/q!)L^q \circ * \circ \iota^{-1} = id$.

(3) Let $u \in \mathcal{H}^{n+m,q}(X, E, \Phi)$. Then the form $*u \in H^0(X, \Omega_X^{n+m-q}(E))$ is saturated in base variables, i.e., $*u = \sigma_u \wedge dt$ for some $[\sigma_u] \in H^0(X, \Omega_{X/Y}^{n-q}(E))$ (see the proof of [Tk, 5.2.ii]). In particular, $u = (c_{n+m-q}/q!)\omega^q \wedge \sigma_u \wedge dt$ and the map $u \mapsto [\sigma_u]$ is well-defined. Thus the Hodge $*$ -operator induces an injective homomorphism

$$S^q : \mathcal{H}^{n+m,q}(X, E, \Phi) \longrightarrow H^0(X, \Omega_{X/Y}^{n-q}(E)).$$

We take a trivialization $\mathcal{O}_Y \xrightarrow{\sim} \Omega_Y^m$ given by $1 \mapsto dt$, which induces isomorphisms of sheaves $\Omega_{X/Y}^n \cong \Omega_{X/Y}^n \otimes f^*\Omega_Y^m \cong \Omega_X^{n+m}$ by $[u] \mapsto u \wedge dt$, and hence of cohomology groups $\alpha^q : H^q(X, \Omega_{X/Y}^n(E)) \xrightarrow{\sim} H^q(X, \Omega_X^{n+m}(E))$. We also have an injection $\Omega_{X/Y}^{n-q} \longrightarrow \Omega_X^{n+m-q}$ by $[\sigma] \mapsto \sigma \wedge dt$, and hence an injection $\beta^0 : H^0(X, \Omega_{X/Y}^{n-q}(E)) \longrightarrow H^0(X, \Omega_X^{n+m-q}(E))$. Combining with Theorem 4.2, we have

$$\begin{aligned} \iota^{-1} \circ \alpha^q &: H^q(X, \Omega_{X/Y}^n(E)) \xrightarrow{\sim} H^q(X, \Omega_X^{n+m}(E)) \xrightarrow{\sim} \mathcal{H}^{n+m,q}(X, E, \Phi), \\ * &= \beta^0 \circ S^q : \mathcal{H}^{n+m,q}(X, E, \Phi) \longrightarrow H^0(X, \Omega_{X/Y}^{n-q}(E)) \longrightarrow H^0(X, \Omega_X^{n+m-q}(E)), \\ (\alpha^q)^{-1} \circ L^q &: H^0(X, \Omega_X^{n+m-q}(E)) \longrightarrow H^q(X, \Omega_X^{n+m}(E)) \xrightarrow{\sim} H^q(X, \Omega_{X/Y}^n(E)). \end{aligned}$$

Then Theorem 4.2 (2) reads the following relative version:

Corollary 4.3. *Let*

$$\begin{aligned} S_f^q &= S^q \circ \iota^{-1} \circ \alpha^q : H^q(X, \Omega_{X/Y}^n(E)) \longrightarrow H^0(X, \Omega_{X/Y}^{n-q}(E)), \\ L_f^q &= (\alpha^q)^{-1} \circ L^q \circ \beta^0 : H^0(X, \Omega_{X/Y}^{n-q}(E)) \longrightarrow H^q(X, \Omega_{X/Y}^n(E)). \end{aligned}$$

Then $(c_{n+m-q}/q!)L_f^q \circ S_f^q = id$ on $H^q(X, \Omega_{X/Y}^n(E))$.

We can also see, thanks to [Tk, 5.2.iv] (see also [Tk, 6.5.i]) that those constructions can be localized on Y , and induce homomorphisms of direct image sheaves.

Corollary 4.4. *There exist homomorphisms induced from the Hodge $*$ -operator and the Lefschetz homomorphism:*

$$S_f^q : R^q f_* \Omega_{X/Y}^n(E) \longrightarrow f_* \Omega_{X/Y}^{n-q}(E), \quad L_f^q : f_* \Omega_{X/Y}^{n-q}(E) \longrightarrow R^q f_* \Omega_{X/Y}^n(E)$$

so that $(c_{n+m-q}/q!)L_f^q \circ S_f^q = id$ on $R^q f_* \Omega_{X/Y}^n(E)$. In particular

$$f_* \Omega_{X/Y}^{n-q}(E) = F^{n-q} \oplus \mathcal{K}^{n-q}, \quad \text{with } F^{n-q} = \text{Im } S_f^q \text{ and } \mathcal{K}^{n-q} = \text{Ker } L_f^q.$$

We translate some results above into explicit forms.

Lemma 4.5. *Let $\sigma \in A^{n-q,0}(X, E)$ such that $[\sigma] \in H^0(Y, F^{n-q})$. Then (1)*

$$\partial_h \sigma = \sum_j \mu^j \wedge dt_j$$

for some $\mu^j \in A^{n-q,0}(X, E)$,

- (2) these μ^j are not unique, but $[\mu^j] \in A^{n-q,0}(X/Y, E)$ are well-defined for σ , and
- (3) $\partial_{h_y}(\mu^j|_{X_y}) = 0$ on any X_y and all j .

Proof. There exists $u \in \mathcal{H}^{n+m,q}(X, E, \Phi)$ such that $*u = \sigma \wedge dt \in H^0(X, \Omega_X^{n+m-q}(E))$. By Takegoshi: Theorem 4.1, we have $\partial_h *u = 0$. Hence $(\partial_h \sigma) \wedge dt = \partial_h *u = 0$, and we have (1). We can show (2) and (3) by the same method in Lemma 3.1. \square

Remark 4.6. Unlike in the case $q = 0$ that is treated by degree considerations [B, §4], we used the semi-positivity here. In general, for $[\sigma] \in H^0(Y, f_*\Omega_{X/Y}^{n-q}(E))$ with $q > 0$, we can not have $\partial_h \sigma = \sum \mu^j \wedge dt_j$ for some $\mu^j \in A^{n-q,0}(X, E)$. This is in fact a key property, and it makes various computations possible. We also note that η^j in Lemma 3.1 and μ^j are not well-defined for a class $[\sigma] \in H^0(Y, F^{n-q})$, and that means, we have some freedom of choices in a class.

Stalks or fibers at a point $y \in Y$ will be denoted by $f_*\Omega_{X/Y}^{n-q}(E)_y, F_y^{n-q}, \mathcal{K}_y^{n-q}$ respectively. Those stalks can be seen as subspaces of $H^0(X_y, \Omega_{X_y}^{n-q}(E_y))$, i.e., $F_y^{n-q} \oplus \mathcal{K}_y^{n-q} = f_*\Omega_{X/Y}^{n-q}(E)_y \subset H^0(X_y, \Omega_{X_y}^{n-q}(E_y))$.

Lemma 4.7. *Let $\sigma_y \in F_y^{n-q}$ and $\tau_y \in \mathcal{K}_y^{n-q}$, and regard them as elements of $H^0(X_y, \Omega_{X_y}^{n-q}(E_y))$.*

Then, (1) $\partial_{h_y} \sigma_y = 0$ in $A^{n-q+1,0}(X_y, E_y)$,

(2) $\omega_y^q \wedge \tau_y \in A^{n,q}(X_y, E_y)$ is $\bar{\partial}$ -exact, and

(3) $(\sigma_y, \tau_y)_{h_y} = \int_{X_y} (c_{n-q}/q!) \omega_y^q \wedge \sigma_y \wedge h_y \bar{\tau}_y = 0$.

Proof. We will argue at $y = 0$.

(1) Since Y is a unit ball in \mathbb{C}^m , there exists $[\sigma] \in H^0(Y, F^{n-q})$ such that $\sigma|_{X_0} = \sigma_0$. By Lemma 4.5, we have $\partial_{h_0}(\sigma|_{X_0}) = 0$.

(2) We take $[\tau] \in H^0(Y, \mathcal{K}^{n-q})$ such that $\tau|_{X_0} = \tau_0$. We have $L_f^q([\tau]) = 0$. Recall the definition of $L_f^q = (\alpha^q)^{-1} \circ L^q \circ \beta^0$, where $\beta^0([\tau]) = \tau \wedge dt$, and $(\alpha^q)^{-1}$ is an isomorphism. Then we have $L^q \circ \beta^0([\tau]) = 0$ in $H^q(X, \Omega_X^{n+m}(E))$, namely $\omega^q \wedge \tau \wedge dt = \bar{\partial}a$ for some $a \in A^{n+m,q-1}(X, E)$. By a bidegree reason, a can be written as $a = b \wedge dt$ for some $b \in A^{n,q-1}(X, E)$. Then $(\omega^q \wedge \tau - \bar{\partial}b) \wedge dt = 0$. By restricting on X_0 , we have $\omega_0^q \wedge \tau_0 - \bar{\partial}b_0 = 0$, where $b_0 = b|_{X_0}$.

(3) By (2), we have $\int_{X_0} \omega_0^q \wedge \sigma_0 \wedge h_0 \overline{\tau_0} = \int_{X_0} \sigma_0 \wedge h_0 \overline{\partial b_0}$. Since $\partial(\sigma_0 \wedge h_0 \overline{b_0}) = (\partial_{h_0} \sigma_0) \wedge h_0 \overline{b_0} + (-1)^{n-q} \sigma_0 \wedge h_0 \overline{\partial b_0}$, and since $\partial_{h_0} \sigma_0 = 0$ by (1), we have $\int_{X_0} \sigma_0 \wedge h_0 \overline{\partial b_0} = (-1)^{n-q} \int_{X_0} \partial(\sigma_0 \wedge h_0 \overline{b_0}) = 0$. \square

4.3. Local freeness. We shall show that the direct image sheaves $R^q f_* \Omega_{X/Y}^n(E)$ are locally free. This is an immediate consequence of a result of Takegoshi [Tk]. We start with recalling a general remark.

Lemma 4.8. *Let X and Y be varieties (reduced and irreducible), $f : X \rightarrow Y$ be a proper surjective morphism, and let \mathcal{E} be a coherent sheaf on X which is flat over Y . Assume that the natural map $\varphi^q(y) : R^q f_* \mathcal{E} \otimes \mathbb{C}(y) \rightarrow H^q(X_y, \mathcal{E}_y)$ is surjective for any $y \in Y$ and any $q \geq 0$, where X_y is the fiber over y , and \mathcal{E}_y is the induced sheaf ([Ha, III.9.4]). Then $R^q f_* \mathcal{E}$ is locally free for any $q \geq 0$, and $\varphi^q(y) : R^q f_* \mathcal{E} \otimes \mathbb{C}(y) \rightarrow H^q(X_y, \mathcal{E}_y)$ is an isomorphism for any $y \in Y$ and any $q \geq 0$.*

Proof. By [Ha, III.12.11(a)] (cohomology and base change), the surjectivity of $\varphi^q(y)$ implies that it is an isomorphism. By [Ha, III.12.11(b)], the local freeness of $R^q f_* \mathcal{E}$ in a neighborhood of $y \in Y$ follows from the surjectivities of $\varphi^q(y)$ and of $\varphi^{q-1}(y)$.

We can find the corresponding results in the category of complex spaces, for example [BS, III.3.4, III.3.7]. \square

Lemma 4.9. *(cf. [Tk, 6.8]) Let $f : (X, \omega_f) \rightarrow Y$ and (E, h) be as in §2.2.I. Then*

(1) *the natural restriction map $R^q f_* \Omega_{X/Y}^n(E) \rightarrow H^q(X_y, \Omega_{X_y}^n(E_y))$ is surjective for any $y \in Y$ and any $q \geq 0$, and*

(2) *$R^q f_* \Omega_{X/Y}^n(E)$ is locally free for any $q \geq 0$, and $\varphi^q(y) : R^q f_* \Omega_{X/Y}^n(E) \otimes \mathbb{C}(y) \rightarrow H^q(X_y, \Omega_{X_y}^n(E_y))$ is an isomorphism for any $y \in Y$ and any $q \geq 0$.*

Proof. (1) Fix $y \in Y$. Since our assertion is local on Y , we may assume that Y is a unit ball in \mathbb{C}^m with coordinates $t = (t_1, \dots, t_m)$ centered at $y = \{t = 0\}$. We take a trivialization $\mathcal{O}_Y \cong \Omega_Y^m$ given by $1 \mapsto dt = dt_1 \wedge \dots \wedge dt_m$. For every i with $1 \leq i \leq m$, we let $Y_i = \{t_1 = \dots = t_i = 0\}$ be a complex sub-manifold of Y , $X_i = f^{-1}(Y_i)$, and let $f_i : X_i \rightarrow Y_i$ be the induced morphism. We denote by $X_0 = X, Y_0 = Y$ and $f_0 = f$. By the injectivity theorem of Takegoshi with $F = \mathcal{O}_X$ in [Tk, 6.8.i], the sheaf homomorphism $R^q f_{0*}(f_0^* t_1) : R^q f_* \Omega_{X/Y}^n(E) \otimes \Omega_Y^m \rightarrow R^q f_* \Omega_{X/Y}^n(E) \otimes \Omega_Y^m$ induced by the product with the holomorphic function $f^* t_1$ is injective for any $q \geq 0$. Hence the restriction map $R^q f_{0*} \Omega_{X/Y}^n(E) \rightarrow R^q f_{1*}(\Omega_{X/Y}^n(E) \otimes \mathcal{O}_{X_1})$ is surjective for any $q \geq 0$. By the adjunction formula, we have $\Omega_{X/Y}^n \otimes \mathcal{O}_{X_1} = \Omega_{X_1/Y_1}^n$. Hence inductively, we obtain a surjection $R^q f_* \Omega_{X/Y}^n(E) \rightarrow H^q(X_y, \Omega_{X_y}^n(E_y))$.

(2) This follows from (1) and Lemma 4.8 \square

We now realize $R^q f_* \Omega_{X/Y}^n(E)$ as a sub-bundle of $f_* \Omega_{X/Y}^{n-q}(E)$ globally.

Corollary 4.10. (cf. [Tk, 6.5.i]) *Let $f : (X, \omega_f) \rightarrow Y$ and (E, h) be as in §2.2.I. Then for any $0 \leq q \leq n$, the homomorphism $L_f^q : f_* \Omega_{X/Y}^{n-q}(E) \rightarrow R^q f_* \Omega_{X/Y}^n(E)$ induced by the q -times left exterior product by the cohomology class $\kappa_f = \{\omega_f\}$ admits a splitting homomorphism $S_f^q : R^q f_* \Omega_{X/Y}^n(E) \rightarrow f_* \Omega_{X/Y}^{n-q}(E)$ with $(c_{n+m-q}/q!) L_f^q \circ S_f^q = \text{id}$.*

Proof. Since the problem is local on Y , we may assume that Y is a unit ball in \mathbb{C}^m , and moreover that X admits a global Kähler form ω which is cohomologous to ω_f . We can now apply Corollary 4.4. \square

5. THE HODGE METRIC

Let $f : (X, \omega) \rightarrow Y \subset \mathbb{C}^m$ and (E, h) be as in §2.2.II, and fix $0 \leq q \leq n$. We denote by $F = F^{n-q}$ the image of $S_f^q : R^q f_* \Omega_{X/Y}^n(E) \rightarrow f_* \Omega_{X/Y}^{n-q}(E)$, and we say a sub-bundle $F \subset f_* \Omega_{X/Y}^{n-q}(E)$.

5.1. Definition of the Hodge metric.

Definition 5.1. The Hodge metric g of F is the Hermitian metric induced from the canonical pairing g on $f_* \Omega_{X/Y}^{n-q}(E)$ in §3.3.

We have the so-called the metric connection (or the Chern connection) D_g of (F, g) , and the curvature $\Theta_g = D_g^2$. As we remarked in §3.3, the Nakano semi-positivity of (F, g) , namely of Θ_g in the local setting §2.2.II implies Theorem 1.1.

Although we do not need the explicit description of the induced metric on $R^q f_* \Omega_{X/Y}^n(E)$ via the injective homomorphism $S_f^q : R^q f_* \Omega_{X/Y}^n(E) \rightarrow f_* \Omega_{X/Y}^{n-q}(E)$, we write down the metric \tilde{g} on $R^q f_* \Omega_{X/Y}^n(E)$ for the sake of completeness. Let $u \in H^q(X, \Omega_{X/Y}^n(E))$ be a cohomology class. Then the cohomology class $u \wedge dt \in H^q(X, \Omega_X^{n+m}(E))$ is represented by a unique harmonic form $u' \wedge dt \in \mathcal{H}^{n+m, q}(X, E, \Phi)$, and $*(u' \wedge dt) = \sigma_u \wedge dt$ for some $[\sigma_u] \in H^0(X, \Omega_{X/Y}^{n-q}(E))$, i.e., $S_f^q(u) = [\sigma_u]$. For cohomology classes $u, v \in H^q(X, \Omega_{X/Y}^n(E))$, the induced metric \tilde{g} is

$$\tilde{g}(u, v) = g([\sigma_u], [\sigma_v]).$$

A short computation shows that, for a test (m, m) -form $\beta c_m dt \wedge \bar{d}\bar{t}$ on Y , the right hand side takes the value $f_*((c_{n-q}/q!) \omega^q \wedge \sigma_u \wedge h \bar{\sigma}_v)(\beta c_m dt \wedge \bar{d}\bar{t}) = \int_X (f^* \beta)(u' \wedge dt) \wedge \bar{\kappa}_h(v' \wedge \bar{d}\bar{t})$.

5.2. The metric connection. We shall construct the metric connection D_g of (F, g) . Recall Lemma 4.5 that $\partial_h \sigma = \sum_j \mu^j \wedge dt_j$ with some $\mu^j \in A^{n-q, 0}(X, E)$ for $[\sigma] \in H^0(Y, F)$. Since $A^0(Y, F) = A^0(Y, \mathbb{C}) \otimes H^0(Y, F)$ as $A^0(Y, \mathbb{C})$ -module, this formula holds for $[\sigma] \in A^0(Y, F)$, too. We consider the fiberwise orthogonal projection $P_y : A^{n-q, 0}(X_y, E_y) \rightarrow F_y$

given by $u_y \mapsto \sum_{j=1}^{\ell} g_y(u_y, \sigma_{jy}) \sigma_{jy}$, where $\sigma_{1y}, \dots, \sigma_{\ell y} \in F_y$ is a basis of F_y . Since F is locally free, the family $\{P_y\}_{y \in Y}$ induces a map

$$P : A^{n-q,0}(X, E) \longrightarrow \{u \in A^{n-q,0}(X, E); u|_{X_y} \in F_y \text{ for any } y \in Y\}$$

Then for $[\sigma] \in A^0(Y, F)$ with $\partial_h \sigma = \sum_j \mu^j \wedge dt_j$, we define

$$\partial_g[\sigma] = \sum [P(\mu^j)] dt_j \in A^{1,0}(Y, F).$$

Lemma 5.2. *The class $[P(\mu^j)]$ is well-defined for $[\sigma] \in A^0(Y, F)$.*

Proof. (1) We shall show that $\mu^j|_{X_y}$ are perpendicular to $H^0(X_y, \Omega_{X_y}^{n-q}(E_y))$ under the condition $[\sigma] = [0]$, namely $\sigma|_{X_y} = 0$ for any $y \in Y$. We write as $\sigma = \sum \sigma_j \wedge dt_j$ with some $\sigma_j \in A^{n-q-1,0}(X, E)$. We note that we can take $\mu^j = \partial_h \sigma_j$. We take any $s \in H^0(X_y, \Omega_{X_y}^{n-q}(E_y))$. Then $\partial(\omega_y^q \wedge \sigma_j|_{X_y} \wedge h_y \bar{s}) = \omega_y^q \wedge \partial_{h_y}(\sigma_j|_{X_y}) \wedge h_y \bar{s} + (-1)^{n-q} \omega_y^q \wedge \sigma_j|_{X_y} \wedge h_y \bar{\partial} s$. Because of $\bar{\partial} s = 0$, we have $g_y((\partial_h \sigma_j)|_{X_y}, s) = (c_{n-q}/q!) \int_{X_y} \omega_y^q \wedge \partial_{h_y}(\sigma_j|_{X_y}) \wedge h_y \bar{s} = (c_{n-q}/q!) \int_{X_y} \partial(\omega_y^q \wedge \sigma_j|_{X_y} \wedge h_y \bar{s}) = 0$.

(2) The above (1) is enough to show that $[P(\mu^j)]$ is well-defined. But in fact, (1) said slightly more. \square

Lemma 5.3. *The sum $D_g := \partial_g + \bar{\partial}$ is the metric connection of the Hermitian vector bundle (F, g) .*

Proof. It is not difficult to see that it is a connection. Let us check the compatibility with the metric g . Let $[\sigma], [\tau] \in H^0(Y, F)$, and write $\partial_h \tau = \sum_j \mu^j(\tau) \wedge dt_j$. Then $\bar{\partial} g([\sigma], [\tau]) = (-1)^{n-q} f_*((c_{n-q}/q!) \omega^q \wedge \sigma \wedge h \bar{\partial}_h \tau) = \sum_j f_*((c_{n-q}/q!) \omega^q \wedge \sigma \wedge h \overline{\mu^j(\tau)}) d\bar{t}_j$. Since $\sigma|_{X_y} \in F_y$, the last term becomes $\sum_j f_*((c_{n-q}/q!) \omega^q \wedge \sigma \wedge h \overline{P(\mu^j(\tau))}) d\bar{t}_j$, and it is $\sum_j g([\sigma], [P(\mu^j(\tau))]) d\bar{t}_j$. In the notation of §2.1, we can write as $g([\sigma], [\tau]) = [\sigma] \wedge g[\bar{\tau}]$ and $\bar{\partial} g([\sigma], [\tau]) = [\sigma] \wedge g \overline{\partial_g[\tau]}$. \square

5.3. Curvature formula. We describe the Nakano semi-positivity of a Hermitian holomorphic vector bundle. Since it is a local property, we will discuss on a local chart. Let $Y \subset \mathbb{C}^m$ be a unit ball centered at 0 with coordinates $t = (t_1, \dots, t_m)$, and let $F = Y \times \mathbb{C}^\ell$ be a trivial vector bundle with a non-trivial Hermitian metric g . (This (F, g) may not necessarily be our original bundle.) We write $\Theta_g = \sum \Theta_{jk} dt_j \wedge d\bar{t}_k$ with $\Theta_{jk} \in \text{End}(Y, F)$. Then (F, g) is Nakano semi-positive at $t = 0$, if and only if for any tensor $s = \sum_{j=1}^m \partial/\partial t_j \otimes \sigma_j^0 \in (T_Y \otimes F)_0$, we have $\Theta_g(s) = \sum_{j,k} g_0(\Theta_{jk} \sigma_j^0, \sigma_k^0) \geq 0$. Moreover the last quantity can be obtained another way from local sections. If $\sigma, \tau \in H^0(Y, F)$, we have $\frac{\partial^2}{\partial t_j \partial \bar{t}_k} g(\sigma, \tau) = g((\partial_g \sigma)^j, (\partial_g \tau)^k) - g(\Theta_{jk} \sigma, \tau)$, where $\partial_g \sigma = \sum_j (\partial_g \sigma)^j dt_j \in A^{1,0}(Y, F)$ and so on. Hence if σ and τ are *normal* at 0 with respect to g (i.e., $\partial_g \sigma = \partial_g \tau = 0$ at 0), we have $(\frac{\partial^2}{\partial t_j \partial \bar{t}_k} g(\sigma, \tau))|_{t=0} = -g_0(\Theta_{jk} \sigma|_{t=0}, \tau|_{t=0})$.

Notation 5.4. (1) Let V be a continuous (m, m) -form on $Y \subset \mathbb{C}^m$. Then we can write $V = v(t)dV_t$ with a unique continuous function v on Y , and we define $V_{t=0} := v(0)$.

(2) Associated to m -ple $\sigma_1, \dots, \sigma_m \in H^0(Y, F)$, we let

$$T(\sigma) = \sum_{j,k} g(\sigma_j, \sigma_k) \widehat{dt_j} \wedge \widehat{dt_k} \in A^{m-1, m-1}(Y, \mathbb{C}).$$

In case all σ_j are normal at $t = 0$, we have $\sqrt{-1}\partial\bar{\partial}T(\sigma)_{t=0} = -\sum_{j,k} g_0(\Theta_{jk}\sigma_j|_{x_0}, \sigma_k|_{x_0})$. Hence we have

Lemma 5.5. [B, §2] *A Hermitian vector bundle (F, g) on an open subset $Y \subset \mathbb{C}^m$ is Nakano semi-positive at $t = 0$, if for any m -ple vectors $\sigma_1^0, \dots, \sigma_m^0 \in F_0$, there exist extensions $\sigma_j \in H^0(Y, F)$ of σ_j^0 , all of which are normal at $t = 0$ and satisfy $\sqrt{-1}\partial\bar{\partial}T(\sigma)_{t=0} \leq 0$.*

We go back to our original situation. We prepare the following notations.

Notation 5.6. Let $f : (X, \omega) \rightarrow Y \subset \mathbb{C}^m$ and (E, h) be as in §2.2.II. Let $\sigma_1, \dots, \sigma_m \in A^{n-q, 0}(X, E)$ such that $[\sigma_j] \in H^0(Y, F)$ for all j .

(1) We set

$$\widehat{\sigma} = \sum \sigma_j \wedge \widehat{dt_j} \in A^{n-q+m-1, 0}(X, E).$$

Then

$$T([\sigma]) = \sum_{j,k} g([\sigma_j], [\sigma_k]) \widehat{dt_j} \wedge \widehat{dt_k} = f_*((c_N/q!) \omega^q \wedge \widehat{\sigma} \wedge h\widehat{\sigma}).$$

Here $N = n - q + m - 1$.

(2) We write $\partial_h \sigma_j = \sum_k \mu_j^k \wedge dt_k$. Then

$$\partial_h \widehat{\sigma} = \sum_j \mu_j^j \wedge dt =: \mu \wedge dt$$

with $\mu \in A^{n-q, 0}(X, E)$, or rather $[\mu] \in A^{n-q, 0}(X/Y, E)$.

(3) We write $\bar{\partial} \sigma_j = \sum_k \eta_j^k \wedge dt_k$. Then

$$\bar{\partial} \widehat{\sigma} = \sum_j \eta_j^j \wedge dt =: \eta \wedge dt$$

with $\eta \in A^{n-q-1, 1}(X, E)$, or rather $[\eta] \in A^{n-q-1, 1}(X/Y, E)$. □

Lemma 5.7. (cf. [B, (4.4)]) *In Notation 5.6, one has*

$$\begin{aligned} -\sqrt{-1}\partial\bar{\partial}T([\sigma])_{t=0} &= f_*((c_N/q!) \omega^q \wedge \sqrt{-1}\Theta_h \wedge \widehat{\sigma} \wedge h\widehat{\sigma})_{t=0} \\ &\quad - \int_{X_0} (c_{n-q}/q!) (\omega^q \wedge \mu \wedge h\bar{\mu})|_{X_0} - \int_{X_0} (c_{n-q}/q!) (\omega^q \wedge \eta \wedge h\bar{\eta})|_{X_0}. \end{aligned}$$

Remark 5.8. The first term comes from the curvature of E , and contributes positively.

The second term is $-\|\mu|_{X_0}\|_{h_0}^2$, and it can be seen as the “second fundamental form” of $F \subset \bigcup_{t \in Y} A^{n-q,0}(X_y, E_y)$ at $t = 0$. This negative contribution will be eliminated by a careful choice of forms σ_j , in §6.2.

The third term is not a definite form. In general one can write $\eta|_{X_0}$ as a sum $\eta|_{X_0} = \eta'_0 + \omega_0 \wedge \eta''_0$ for primitive forms η'_0 and η''_0 on X_0 , and then $-\int_{X_0} (c_{n-q}/q!) (\omega^q \wedge \eta \wedge h\bar{\eta})|_{X_0} = \|\eta'_0\|_{h_0}^2 - \|\eta''_0\|_{h_0}^2$. In §6.2, we will show that we can take σ_j so that all $\eta_j^k|_{X_0}$ and hence $\eta|_{X_0}$ are primitive on X_0 . In that case, the third term is $-\int_{X_0} (c_{n-q}/q!) (\omega^q \wedge \eta \wedge h\bar{\eta})|_{X_0} = \|\eta|_{X_0}\|_{h_0}^2 \geq 0$. We should read the Kodaira-Spencer class contributes positively.

Proof of Lemma 5.7. The proof will be done by direct computations. We first note that $f_*(\omega^q \wedge \bar{\partial}\hat{\sigma} \wedge h\bar{\hat{\sigma}}) = f_*(\omega^q \wedge \eta \wedge dt \wedge h\bar{\hat{\sigma}}) = 0$ as an $(m-1, m)$ -current on Y , because it contains dt . By the same reason, we have $f_*(\omega^q \wedge \hat{\sigma} \wedge h\bar{\partial}\bar{\hat{\sigma}}) = 0$, and hence, by taking $\bar{\partial}$, we have $f_*(\omega^q \wedge \hat{\sigma} \wedge h\bar{\partial}_h\bar{\partial}\bar{\hat{\sigma}}) = -(-1)^N f_*(\omega^q \wedge \bar{\partial}\hat{\sigma} \wedge h\bar{\partial}\bar{\hat{\sigma}})$. Then we have $\bar{\partial}f_*(\omega^q \wedge \hat{\sigma} \wedge h\bar{\hat{\sigma}}) = (-1)^N f_*(\omega^q \wedge \hat{\sigma} \wedge h\bar{\partial}_h\bar{\hat{\sigma}})$, and then

$$\bar{\partial}f_*(\omega^q \wedge \hat{\sigma} \wedge h\bar{\hat{\sigma}}) = (-1)^N f_*(\omega^q \wedge \partial_h\hat{\sigma} \wedge h\bar{\partial}_h\bar{\hat{\sigma}}) + f_*(\omega^q \wedge \hat{\sigma} \wedge h\bar{\partial}\bar{\partial}_h\bar{\hat{\sigma}}).$$

Since $\partial_h\bar{\partial} + \bar{\partial}\partial_h = e(\Theta_h)$, we have $f_*(\omega^q \wedge \hat{\sigma} \wedge h\bar{\partial}\bar{\partial}_h\bar{\hat{\sigma}}) = f_*(\omega^q \wedge \hat{\sigma} \wedge h\bar{\Theta}_h \wedge \bar{\hat{\sigma}}) - f_*(\omega^q \wedge \hat{\sigma} \wedge h\bar{\partial}_h\bar{\partial}\bar{\hat{\sigma}})$. Using $f_*(\omega^q \wedge \hat{\sigma} \wedge h\bar{\partial}_h\bar{\partial}\bar{\hat{\sigma}}) = -(-1)^N f_*(\omega^q \wedge \bar{\partial}\hat{\sigma} \wedge h\bar{\partial}\bar{\hat{\sigma}})$, we can write

$$\begin{aligned} \bar{\partial}f_*(\omega^q \wedge \hat{\sigma} \wedge h\bar{\hat{\sigma}}) &= -f_*(\omega^q \wedge \Theta_h \wedge \hat{\sigma} \wedge h\bar{\hat{\sigma}}) + (-1)^{N+(n-q)m} f_*(\omega^q \wedge \mu \wedge h\bar{\mu} \wedge dt \wedge d\bar{t}) \\ &\quad + (-1)^{N+(n-q)m} f_*(\omega^q \wedge \eta \wedge h\bar{\eta} \wedge dt \wedge d\bar{t}). \end{aligned}$$

Here we mind that $\sqrt{-1}\Theta_h$ is real. Hence $-\sqrt{-1}\bar{\partial}f_*((c_N/q!)\omega^q \wedge \hat{\sigma} \wedge h\bar{\hat{\sigma}})$ is

$$\begin{aligned} f_*((c_N/q!)\omega^q \wedge \sqrt{-1}\Theta_h \wedge \hat{\sigma} \wedge h\bar{\hat{\sigma}}) &- f_*((c_{n-q}/q!)\omega^q \wedge \mu \wedge h\bar{\mu} \wedge c_m dt \wedge d\bar{t}) \\ &- f_*((c_{n-q}/q!)\omega^q \wedge \eta \wedge h\bar{\eta} \wedge c_m dt \wedge d\bar{t}). \end{aligned}$$

By taking their values at $t = 0$, we have our assertion. \square

6. NORMAL AND “PRIMITIVE” SECTIONS, AND THE PROOF OF THEOREM 1.1

Let $f : (X, \omega) \longrightarrow Y \subset \mathbb{C}^m$ and (E, h) as in §2.2.II, and keep the notations in §5.

6.1. Effect of normality. We control $\partial_h\sigma$ at one point for $[\sigma] \in H^0(Y, F)$, when it is normal at $t = 0$. Recall $\partial_h\sigma = \sum \mu^j \wedge dt_j$ with some $\mu^j \in A^{n-q,0}(X, E)$. To go further, we need a genericity condition over the base Y . We will assume that the function $y \mapsto \dim H^0(X_y, \Omega_{X_y}^{n-q}(E_y))$ is constant around $t = 0$. This assumption implies that $f_*\Omega_{X/Y}^{n-q}(E)$ is locally free around $t = 0$, and that the fiber $f_*\Omega_{X/Y}^{n-q}(E)_y$ coincides with $H^0(X_y, \Omega_{X_y}^{n-q}(E_y))$ around $t = 0$ ([GR, 10.5.5], [Ha, III §12]). In case $q = 0$, i.e., the case in [B, §4], this assumption holds true thanks to Ohsawa-Takegoshi type L^2 -extension theorem [OT] [O] [Ma]. Recall Corollary 4.4 that $f_*\Omega_{X/Y}^{n-q}(E) = F \oplus \mathcal{K}$, where $\mathcal{K} = \mathcal{K}^{n-q} = \text{Ker } L_f^q$.

Lemma 6.1. *Assume that the function $y \mapsto \dim H^0(X_y, \Omega_{X_y}^{n-q}(E_y))$ is constant around $t = 0$. Let $[\sigma] \in H^0(Y, F)$ with $\partial_h \sigma = \sum \mu^j \wedge dt_j$, and suppose $\partial_g[\sigma] = 0$ at $t = 0$. Then all $\mu^j|_{X_0}$ are perpendicular to $H^0(X_0, \Omega_{X_0}^{n-q}(E_0))$.*

Proof. We will use notations in §4.1 for (X_0, ω_0) and (E_0, h_0) . Let $(\cdot, \cdot)_{h_0}$ be the inner product of $A^{n-q,0}(X_0, E_0)$ in terms of the metrics ω_0 and h_0 on X_0 . We have $H^0(X_0, \Omega_{X_0}^{n-q}(E_0)) = F_0 \oplus \mathcal{K}_0$, which is an orthogonal direct sum by our assumption and by Lemma 4.7(3). We fix j . Let $\mu^j|_{X_0} = \tau_0 + a_0 \in A^{n-q,0}(X_0, E_0)$ be the Hodge decomposition of forms so that $\tau_0 \in H^0(X_0, \Omega_{X_0}^{n-q}(E_0))$ and $a_0 \in \vartheta_{h_0} A^{n-q,1}(X_0, E_0)$. We would like to show that $\tau_0 = 0$.

Since $\partial_g[\sigma] = 0$ at $t = 0$, namely all $\mu^k|_{X_0}$ are perpendicular to F_0 , it follows that $\tau_0 \in \mathcal{K}_0$ by Lemma 4.7(3). Then $\omega_0^q \wedge \tau_0 = \bar{\partial} b_0$ for some $b_0 \in A^{n,q-1}(X_0, E_0)$ by Lemma 4.7(2). Combining with Lemma 4.5 that $\partial_{h_0}(\mu^j|_{X_0}) = 0$, we have $\int_{X_0} \omega_0^q \wedge \mu^j|_{X_0} \wedge h_0 \bar{\tau}_0 = 0$ by integration by parts as in Lemma 4.7(3). Then $\|\tau_0\|_{h_0}^2 = (\tau_0 + a_0, \tau_0)_{h_0} = \int_{X_0} (c_{n-q}/q!) \omega_0^q \wedge \mu^j|_{X_0} \wedge h_0 \bar{\tau}_0 = 0$, and hence $\tau_0 = 0$. \square

Lemma 6.2. *Let $[\sigma] \in H^0(Y, F)$ with $\partial_h \sigma = \sum \mu^j \wedge dt_j$, and suppose that $\mu^j|_{X_0}$ is perpendicular to $H^0(X_0, \Omega_{X_0}^{n-q}(E_0))$ for some j . Then there exists $\xi_0^j \in A^{n-q-1,0}(X_0, E_0)$ such that $\partial_{h_0} \xi_0^j = \mu^j|_{X_0}$ and that $\bar{\partial} \xi_0^j$ is primitive.*

Proof. We will use notations in §4.1 for (X_0, ω_0) and (E_0, h_0) . Recall Theorem 4.1 with $\dim Y = 0$ that the Hodge $*$ -operator yields an injective homomorphism $*_0 : \mathcal{H}^{n,q}(X_0, E_0) \longrightarrow H^0(X_0, \Omega_{X_0}^{n-q}(E_0))$.

We consider $u := \omega_0^q \wedge \mu^j|_{X_0} \in A^{n,q}(X_0, E_0)$, and recall $(c_{n-q}/q!) *_0 u = \mu^j|_{X_0}$. Let $u = a + \bar{\partial} b + \vartheta_{h_0} c$ be the Hodge decomposition of forms so that $a \in \mathcal{H}^{n,q}(X_0, E_0)$, b is ϑ_{h_0} -exact, and that c is $\bar{\partial}$ -exact.

We first show that $\bar{\partial} b = 0$. Using integration by parts and by Lemma 4.5(3), we have $\int_{X_0} \bar{\partial} b \wedge h_0 \bar{\mu}^j|_{X_0} = 0$. Since $\|\bar{\partial} b\|_{h_0}^2 = (\bar{\partial} b, u)_{h_0} = \int_{X_0} \bar{\partial} b \wedge \bar{*}_{h_0} u$, and since the last term is $c_{n-q} q! \int_{X_0} \bar{\partial} b \wedge h_0 \bar{\mu}^j|_{X_0} = 0$, we have $\bar{\partial} b = 0$.

We next show that $a = 0$. Recall in general, $(v, w)_{h_0} = (*_0 v, *_0 w)_{h_0}$ holds for $v, w \in A^{p,q}(X_0, E_0)$ ([Huy, 1.2.20]). Since $\vartheta_{h_0} c \in (\mathcal{H}^{n,q}(X_0, E_0))^\perp$ the orthogonal complement in $A^{n,q}(X_0, E_0)$, we have $*_0(\vartheta_{h_0} c) \in (*_0 \mathcal{H}^{n,q}(X_0, E_0))^\perp$. We also have $*_0 u = c_{n-q}^{-1} q! \mu^j|_{X_0} \in H^0(X_0, \Omega_{X_0}^{n-q}(E_0))^\perp \subset (*_0 \mathcal{H}^{n,q}(X_0, E_0))^\perp$. On the other hand $*_0 a \in *_0 \mathcal{H}^{n,q}(X_0, E_0)$, hence the both sides of $*_0 a = *_0 u - *_0(\vartheta_{h_0} c)$ have to be 0.

Now we had $u = \vartheta_{h_0} c$ for a $\bar{\partial}$ -exact form $c \in A^{n,q+1}(X_0, E_0)$. By the Lefschetz isomorphism on forms ([Huy, 1.2.30]), there exists $\xi \in A^{n-q-1,0}(X_0, E_0)$ such that $\omega_0^{q+1} \wedge \xi = c$. We have $\omega_0^{q+1} \wedge \bar{\partial} \xi = \bar{\partial} c = 0$, namely $\bar{\partial} \xi$ is primitive. We also have $*_0 c = *_0(\omega_0^{q+1} \wedge \xi) = c_{n-q-1}^{-1} (q+1)! \xi$ by Lemma 2.1. Then $\mu^j|_{X_0} = (c_{n-q}/q!) *_0 u = (c_{n-q}/q!) (-*_0 \circ *_0$

$\partial_{h_0}(*_0 c) = -(-1)^{n-q}(c_{n-q}/q!)\partial_{h_0}(c_{n-q-1}^{-1}(q+1)!\xi) = -\sqrt{-1}(q+1)\partial_{h_0}\xi$. We finally take $\xi_0 = -\sqrt{-1}(q+1)\xi$. \square

6.2. Existence of strongly normal and “primitive” sections. Here we state a key result for the curvature estimate of our Hodge metric, as a consequence of Lemma 3.1 and Lemma 6.2. The assertion (i) below in fact holds not only for F , but also for any locally free subsheaf of $f_*\Omega_{X/Y}^{n-q}(E)$. The property (3) (respectively, (4)) below will be referred as “primitive” (respectively, strongly normal) at $t = 0$.

Proposition 6.3. (cf. [B, Proposition 4.2]) *Let $\sigma^0 \in F_0$ be a vector at $t = 0$.*

(i) *Then, there exists $\sigma \in A^{n-q,0}(X, E)$ such that $[\sigma] \in H^0(Y, F)$ with the following properties:*

(1) $\sigma|_{X_0} = \sigma^0$,

(2) $\partial_g[\sigma] = 0$ at $t = 0$,

(3) $\eta^j|_{X_0} \wedge \omega_0^{q+1} = 0$ for any j , where $\bar{\partial}\sigma = \sum \eta^j \wedge dt_j$, and

(ii) *If the function $y \mapsto \dim H^0(X_y, \Omega_{X_y}^{n-q}(E_y))$ is constant around $t = 0$, one can take σ in (i) with the following additional property:*

(4) $\mu^j|_{X_0} = 0$ for any j , where $\partial_h\sigma = \sum \mu^j \wedge dt_j$.

Proof. (i) A local extension as in (1) and (2) is possible for any Hermitian vector bundle. Hence we start with a local extension $[\sigma] \in H^0(Y, F)$ satisfying (1) and (2). We write $\bar{\partial}\sigma = \sum \eta^j \wedge dt_j$. By Lemma 3.1 (4), we have $(\eta^j \wedge \omega_0^{q+1})|_{X_0} = \bar{\partial}a_0^j$ for some $a_0^j \in A^{n,q+1}(X_0, E_0)$. By the Lefschetz isomorphism on forms ([Huy, 1.2.30]), we can write $a_0^j = b_0^j \wedge \omega_0^{q+1}$ for some $b_0^j \in A^{n-q-1,0}(X_0, E_0)$. We take smooth extensions $b^j \in A^{n-q-1,0}(X, E)$ so that $b^j|_{X_0} = b_0^j$, and we let $\tilde{\sigma} = \sigma - \sum b^j \wedge dt_j \in A^{n-q,0}(X, E)$. We check $[\tilde{\sigma}]$ is what we are looking for. Since $[\tilde{\sigma}] = [\sigma]$ in $A^{n-q,0}(X/Y, E)$, we see $[\tilde{\sigma}] \in H^0(Y, F)$, and (1) and (2) for $\tilde{\sigma}$. Moreover $\bar{\partial}\tilde{\sigma} = \sum(\eta^j - \bar{\partial}b^j) \wedge dt_j$, and $(\eta^j - \bar{\partial}b^j)|_{X_0} \wedge \omega_0^{q+1} = \bar{\partial}a_0^j - \bar{\partial}(b_0^j \wedge \omega_0^{q+1}) = 0$. Hence we have $\eta^j(\tilde{\sigma})|_{X_0} \wedge \omega_0^{q+1} = 0$, i.e., (3) for $\tilde{\sigma}$.

(ii) We assume that the function $y \mapsto \dim H^0(X_y, \Omega_{X_y}^{n-q}(E_y))$ is constant around $t = 0$. We take $\sigma \in A^{n-q,0}(X, E)$ which satisfies all the properties in (i). We write $\partial_h\sigma = \sum \mu^j \wedge dt_j$ and $\bar{\partial}\sigma = \sum \eta^j \wedge dt_j$. By Lemma 6.1 and 6.2, for every j , there exists $\xi_0^j \in A^{n-q-1,0}(X_0, E_0)$ such that $\partial_{h_0}\xi_0^j = \mu^j|_{X_0}$ and that $\bar{\partial}\xi_0^j$ is primitive. We take $\xi^j \in A^{n-q-1,0}(X, E)$ such that $\xi^j|_{X_0} = \xi_0^j$ for every j . We consider $\tilde{\sigma} = \sigma - \sum_j \xi^j \wedge dt_j$. Since $[\tilde{\sigma}] = [\sigma]$ in $A^{n-q,0}(X, E)$, we see $[\tilde{\sigma}] \in H^0(Y, F)$, and (1) and (2) for $\tilde{\sigma}$. We have $\partial_h\tilde{\sigma} = \sum(\mu^j - \partial_h\xi^j) \wedge dt_j$ and $\bar{\partial}\tilde{\sigma} = \sum(\eta^j - \bar{\partial}\xi^j) \wedge dt_j$. The property (3) for $\tilde{\sigma}$ follows from the primitivity of $\eta^j|_{X_0}$ for σ and of $(\bar{\partial}\xi^j)|_{X_0} = \bar{\partial}\xi_0^j$. The property (4) for $\tilde{\sigma}$ follows from $\mu^j|_{X_0} - (\partial_h\xi^j)|_{X_0} = 0$. \square

6.3. Nakano semi-positivity.

Proposition 6.4. *Let $\sigma_1, \dots, \sigma_m \in A^{n-q,0}(X, E)$ with $[\sigma_1], \dots, [\sigma_m] \in H^0(Y, F)$ and satisfying the properties (3) and (4) in Proposition 6.3. Then $\sqrt{-1}\partial\bar{\partial}T([\sigma])_{t=0} \leq 0$ in Notation 5.6 for these $\sigma_1, \dots, \sigma_m$.*

Proof. We will use the notations in 5.6. By the property (4) in Proposition 6.3, we have $\mu|_{X_0} = 0$. The property (3) in Proposition 6.3 implies that $\eta|_{X_0}$ is primitive. In particular, by using Lemma 2.1, $-\int_{X_0}(c_{n-q}/q!)(\omega^q \wedge \eta \wedge h\bar{\eta})|_{X_0} = \|\eta|_{X_0}\|_{h_0}^2$ the square norm with respect to ω_0 and h_0 . Then the formula in Lemma 5.7 is

$$-\sqrt{-1}\partial\bar{\partial}T([\sigma])_{t=0} = f_*((c_N/q!)\omega^q \wedge \sqrt{-1}\Theta_h \wedge \hat{\sigma} \wedge h\bar{\sigma})_{t=0} + \|\eta|_{X_0}\|_{h_0}^2.$$

The right hand side is non-negative, since the curvature Θ_h is Nakano semi-positive. \square

Corollary 6.5. *(F, g) is Nakano semi-positive, and hence so is $R^q f_*\Omega_{X/Y}^n(E)$.*

Proof. Since g is a smooth Hermitian metric of F , to show the Nakano semi-positivity, it is enough to show it on the complement of an analytic subset of Y . By Grauert ([GR, 10.5.4] [Ha, III.12.8, 12.9]), there exists an analytic subset $Z \subset Y$ such that the function $y \mapsto \dim H^0(X_y, \Omega_{X_y}^{n-q}(E_y))$ is constant on $Y \setminus Z$. We apply the criterion in Lemma 5.5 at each point on $Y \setminus Z$. Then thanks to Proposition 6.3, Proposition 6.4 in fact shows that g is Nakano semi-positive on $Y \setminus Z$. \square

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