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# ROBUST METHODS FOR LTE AND WIMAX DIMENSIONING

L.DECREUSEFOND, E. FERRAZ, P. MARTINS, AND T.T. VU

**ABSTRACT.** This paper proposes an analytic model for dimensioning OFDMA based networks like WiMAX and LTE systems. In such a system, users require a number of subchannels which depends on their SNR, hence of their position and the shadowing they experience. The system is overloaded when the number of required subchannels is greater than the number of available subchannels. We give an exact though not closed expression of the loss probability and then give an algorithmic method to derive the number of subchannels which guarantees a loss probability less than a given threshold. We show that Gaussian approximation lead to optimistic values and are thus unusable. We then introduce Edgeworth expansions with error bounds and show that by choosing the right order of the expansion, one can have an approximate dimensioning value easy to compute but with guaranteed performance. As the values obtained are highly dependent from the parameters of the system, which turned to be rather undetermined, we provide a procedure based on concentration inequality for Poisson functionals, which yields to conservative dimensioning. This paper relies on recent results on concentration inequalities and establish new results on Edgeworth expansions.

Concentration inequality, Edgeworth expansion, LTE, OFDMA

## 1. INTRODUCTION

Future wireless systems will widely rely on OFDMA (Orthogonal Frequency Division Multiple Access) multiple access technique. OFDMA can satisfy end user's demands in terms of throughput. It also fulfills operator's requirements in terms of capacity for high data rate services. Systems such as 802.16e and 3G-LTE (Third Generation Long Term Evolution) already use OFDMA on the downlink. Dimensioning of OFDMA systems is then of the utmost importance for wireless telecommunications industry.

OFDM (Orthogonal Frequency Division Multiplex) is a multi carrier technique especially designed for high data rate services. It divides the spectrum in a large number of frequency bands called (orthogonal) subcarriers that overlap partially in order to reduce spectrum occupation. Each subcarrier has a small bandwidth compared to the coherence bandwidth of the channel in order to mitigate frequency selective fading. User data is then transmitted in parallel on each sub carrier. In OFDM systems, all available subcarriers are affected to one user at a given time for transmission. OFDMA extends OFDM by making it possible to share dynamically the available subcarriers between different users. In that sense, it can then be seen as multiple access technique that both combines FDMA and TDMA features. OFDMA can also be possibly combined with multiple antenna (MIMO) technology to improve either quality or capacity of systems.

In practical systems, such as WiMAX or 3G-LTE, subcarriers are not allocated individually for implementation reasons mainly inherent to the scheduler design and physical layer signaling. Several subcarriers are then grouped in subchannels according to different strategies specific to each system. In OFDMA systems, the unit of resource allocation is mainly the subchannels. The number of subchannels

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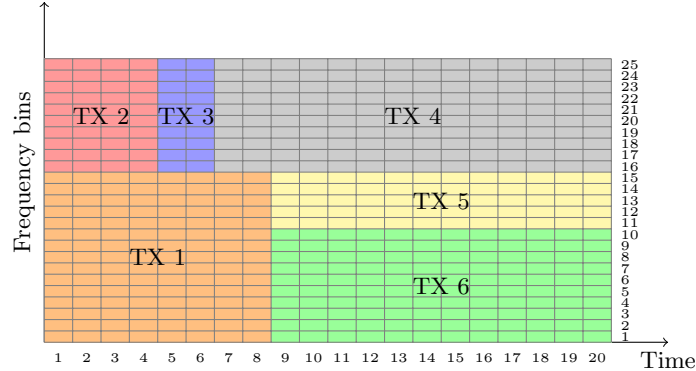


FIGURE 1. OFDMA principle : subcarriers are allocated according to the required transmission rate

required by a user depends on his channel's quality and the required bit rate. If the number of demanded subchannels by all users in the cell is greater than the available number of subchannel, the system is overloaded and suffer packet losses. The questions addressed here can then be stated as follows: how many subchannels must be assigned to a BS to ensure a small overloading probability ? Given the number of available subchannels, what is the maximum load, in terms of mean number of customers per unit of surface, that can be tolerated ? Both questions rely on accurate estimations of the loss probability.

The objectives of this paper are twofold: First, construct and analyze a general performance model for an isolated cell equipped with an OFDMA system as described above. We allows several classes of customers distinguished by their transmission rate and we take into account path-loss with shadowing. We then show that for a Poissonian configuration of users in the cell, the required number subchannels follows a compound Poisson distribution. The second objective is to compare different numerical methods to solve the dimensioning problem. In fact, there exists an algorithmic approach which gives the exact result potentially with huge memory consumption. On the other hand, we use and even extend some recent results on functional inequalities for Poisson processes to derive some approximations formulas which turn to be rather effective at a very low cost. When it comes to evaluate the performance of a network, the quality of such a work may be judged according to several criteria. First and foremost, the exactness is the most used criterion: it means that given the exact values of the parameters, the real system, the performances of which may be estimated by simulation, behaves as close as possible to the computed behavior. The sources of errors are of three kinds: The mathematical model may be too rough to take into account important phenomena which alter the performances of the system, this is known as the epistemic risk. Another source may be in the mathematical resolution of the model where we may be forced to use approximate algorithms to find some numerical values. The third source lies in the lack of precision in the determination of the parameters characterizing the system: They may be hard, if not impossible, to measure with the desired accuracy. It is thus our point of view that exactness of performance analysis is not all the matter of the problem, we must also be able to provide confidence intervals and robust analysis. That is why, we insist on error bounds in our approximations.

Resources allocation on OFDMA systems have been extensively studied over the last decade, often with joint power and subcarriers allocation, see for instance [1, 8, 14, 15]. The problem of OFDMA planning and dimensioning have been

more recently under investigation. In [7], the authors propose a dimensioning of OFDMA systems focusing on link outage but not on the other parameters of the systems. In [11], the authors give a general methodology for the dimensioning of OFDMA systems, which mixes a simulation based determination of the distribution of the signal-to-interference-plus-noise ratio (SINR) and a Markov chain analysis of the traffic. In [3, 9], the authors propose a dimensioning method for OFDMA systems using Erlang's loss model and Kaufman-Roberts recursion algorithm. In [4], the authors study the effect of Rayleigh fading on the performance of OFDMA networks.

The article is organized as follows. In Section 2, we describe the system model and set up the problem. In Section 3, we examine four methods to derive an exact, approximate or robust value of the number of subchannels necessary to ensure a given loss probability. In Section 4, we apply these formulas to the particular situation of OFDMA systems. A new bound for the Edgeworth expansion is in Section B and Section C contains a new proof of the concentration inequality established for instance in [16].

## 2. SYSTEM MODEL

In practical systems, such as WiMAX or 3G-LTE, resource allocation algorithms work at subchannel level. The subcarriers are grouped into subchannels that the system allocates to different users according to their throughput demand and mobility pattern. For example, in WiMAX, there are three modes available for building subchannels: FUSC (Fully Partial Usage of Subchannels), PUSC (Partial Usage of SubChannels) and AMC (Adaptive modulation and coding). In FUSC, subchannels are made of subcarriers spread over all the frequency band. This mode is generally more adapted to mobile users. In AMC, the subcarriers of a subchannel are adjacent instead of being uniformly distributed over the spectrum. AMC is more adapted to nomadic or stationary users and generally provides higher capacity.

The grouping of subcarriers into subchannels raises the problem of the estimation of the quality of a subchannel. Theoretically channel quality should be evaluated on each subcarrier of the corresponding subchannel to compute the associated capacity. This work assumes that it is possible to consider a single channel gain for all the subcarriers making part of a subchannel (for example via channel gains evaluated on pilot subcarriers).

We consider a circular cell  $C$  of radius  $R$  with a base station (BS for short) at its center. The transmission power dedicated to each subchannel by the base station is denoted by  $P$ . Each subchannel has a total bandwidth  $W$  (in kHz). The received signal power for a mobile station at distance  $d$  from the BS can be expressed as

$$(1) \quad P(d) = \frac{PK_\gamma}{d^\gamma} GF := P_\gamma G d^{-\gamma},$$

where  $K_\gamma$  is a constant equal to the attenuation at a reference distance, denoted by  $d_{\text{ref}}$ , that separates far field from near field propagation. Namely,

$$K_\gamma = \left( \frac{c}{4\pi f d_{\text{ref}}} \right)^2 d_{\text{ref}}^\gamma,$$

where  $f$  is the radio-wave frequency. The variable  $\gamma$  is the path-loss exponent which indicates the power at which the path loss increases with distance. Its depends on the specific propagation environment, in urban area, it is in the range from 3 to 5. It must be noted that this propagation model is an approximate model, difficult to calibrate for real life situations. In particular, it might be reasonable to envision models where  $\gamma$  depends on the distance so that the attenuation would be proportional to  $d^{\gamma(d)}$ . Because of the complexity of such a model,  $\gamma$  is often

considered as constant but the path-loss is multiplied by two random variables  $G$  and  $F$  which represent respectively the shadowing, i.e. the attenuation due to obstacles, and the Rayleigh fading, i.e. the attenuation due to local movements of the mobile. Usually,  $G$  is taken as a log-normal distribution:  $G = 10^{S/10}$ , where  $S \sim \mathcal{N}(\kappa, v^2)$ . As to  $F$ , it is customary to choose an exponential distribution with parameter 1. Both, the shadowing and the fading experienced by each user are supposed to be independent from other users' shadowing and fading. For the sake of simplicity, we will here treat the situation where only shadowing is taken into account, the computations would be pretty much like the forthcoming ones and the results rather similar should we consider Rayleigh fading.

All active users in the cell compete to have access to some of the  $N_{\text{avail}}$  available subchannels. There are  $K$  classes of users distinguished by the transmission rate they require:  $C_k$  is the rate of class  $k$  customers and  $\tau_k$  denotes the probability that a customer belongs to class  $k$ . A user, at distance  $d$  from the BS, is able to receive the signal only if the signal-to-interference-plus-noise ratio  $\text{SNR} = \frac{P(d)}{I}$  is above some constant  $\beta_{\text{min}}$  where  $I$  is the noise plus interference power and  $P(d)$  is the received signal power at distance  $d$ , see (1). If the SNR is below the critical threshold, then the user is said to be in outage and cannot proceed with his communication.

To avoid excess demands, the operator may impose a maximum number  $N_{\text{max}}$  of allocated subchannels to each user at each time slot. According to the Shannon formula, for a user demanding a service of bit rate  $C_k$ , located at distance  $d$  from the BS and experiencing a shadowing  $g$ , the number of requires subchannels is thus the minimum of  $N_{\text{max}}$  and of

$$N_{\text{user}} = \begin{cases} \left\lceil \frac{C_k}{W \log_2(1 + P_\gamma g d^{-\gamma}/I)} \right\rceil & \text{if } P_\gamma g d^{-\gamma}/I \geq \beta_{\text{min}}, \\ 0 & \text{otherwise,} \end{cases}$$

where  $\lceil x \rceil$  means the minimum integer number not smaller than  $x$ .

We make the simplifying assumption that the allocation is made at every time slot and that there is no buffering neither in the access point nor in each mobile station. All the users have independently from others a probability  $p$  to have a packet to transmit at each slot. This means, that each user has a traffic pattern which follows a geometric process of intensity  $p$ . We also assume that users are dispatched in the cell according to a Poisson process of intensity  $\lambda_0$ . According to the thinning theorem for Poisson processes, this induces that active users form a Poisson process of intensity  $\lambda = \lambda_0 p$ . This intensity is kept fixed over the time. That may result from two hypothesis: Either we consider that for a small time scale, users do not move significantly and thus the configuration does not evolve. Alternatively, we may consider that statistically, the whole configuration of active users has reached its equilibrium so that the distribution of active users does not vary through time though each user may move.

From the previous considerations, a user is characterized by three independent parameters: his position, his class and the intensity of the shadowing he is experiencing. We model this as a Poisson process on  $E = B(0, R) \times \{1, \dots, K\} \times \mathbf{R}^+$  of intensity measure

$$\lambda \, d\nu(x) := \lambda (dx \otimes d\tau(k) \otimes d\rho(g))$$

where  $B(0, R) = \{x \in \mathbf{R}^2, \|x\| \leq R\}$ ,  $\tau$  is the probability distribution of classes given by  $\tau(\{k\}) = \tau_k$  and  $\rho$  is the distribution of the random variable  $G$  defined

above. We set

$$f(x, k, g) = \min(N_{\max}, \mathbf{1}_{\{P_\gamma g \|x\|^{-\gamma} \geq I\beta_{\min}\}} \left[ \frac{C_k}{W \log_2(1 + P_\gamma g \|x\|^{-\gamma}/I)} \right]).$$

With the notations of Section B,

$$N_{\text{tot}} = \int_{\text{cell}} f(x, k, g) \, d\omega(x, k, g).$$

We are interested in the loss probability which is given by

$$\mathbf{P}(N_{\text{tot}} \geq N_{\text{avail}}).$$

We first need to compute the different moment of  $f$  with respect to  $\nu$  in order to apply Theorem 2 and Theorem 3. For, we set

$$l_k = N_{\max} \wedge \left\lceil \frac{C_k}{W \log_2(1 + \beta_{\min})} \right\rceil,$$

where  $a \wedge b = \min(a, b)$ . Furthermore, we introduce  $\beta_{k,0} = \infty$ ,

$$\beta_{k,l} = \frac{I}{P} \left( 2^{C_k/Wl} - 1 \right), \quad 1 \leq k \leq K, \quad 1 \leq l \leq l_k - 1,$$

and  $\beta_{k,l_k} = I\beta_{\min}/P$ .

By the very definition of the ceiling function, we have

$$\begin{aligned} \int_E f^p \, d\nu &= \sum_{k=1}^K \tau_k \sum_{l=1}^{l_k} l^p \int_{\text{cell}} \int_{\mathbf{R}} \mathbf{1}_{[\beta_{k,l}; \beta_{k,l-1})}(g \|x\|^{-\gamma}) \, d\rho(g) \, dx. \end{aligned}$$

According to the change of variable formula, we have

$$\begin{aligned} \int_{\text{cell}} \mathbf{1}_{[\beta_{k,l}; \beta_{k,l-1})}(g \|x\|^{-\gamma}) \, dx &= \pi(\beta_{k,l}^{-2/\gamma} \wedge R^2 - \beta_{k,l-1}^{-2/\gamma} \wedge R^2) g^{2/\gamma}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \int_{\text{cell}} \int_{\mathbf{R}} \mathbf{1}_{[\beta_{k,l}; \beta_{k,l-1})}(g \|x\|^{-\gamma}) \, d\rho(g) \, dx &= \pi(\beta_{k,l}^{-2/\gamma} \wedge R^2 - \beta_{k,l-1}^{-2/\gamma} \wedge R^2) \mathbf{E} \left[ 10^{S/5\gamma} \right] \\ &= \pi(\beta_{k,l}^{-2/\gamma} \wedge R^2 - \beta_{k,l-1}^{-2/\gamma} \wedge R^2) 10^{(\kappa + \frac{\nu^2}{10\gamma} \ln 10)/5\gamma} := \zeta_{k,l}. \end{aligned}$$

We thus have proved the following theorem.

**Theorem 1.** *For any  $p \geq 0$ , with the same notations as above, we have:*

$$(2) \quad \int f^p \, d\nu = \sum_{k=1}^K \tau_k \sum_{l=1}^{l_k} l^p \zeta_{k,l}.$$

### 3. LOSS PROBABILITY

**3.1. Exact method.** Since  $f$  is deterministic,  $N_{\text{tot}}$  follows a compound Poisson distribution: it is distributed as

$$\sum_{k=1}^K \sum_{l=1}^{l_k} l N_{k,l}$$

where  $(N_{k,l}, 1 \leq k \leq K, 1 \leq l \leq l_k)$  are independent Poisson random variables, the parameter of  $N_{k,l}$  is  $\lambda \tau_k \zeta_{k,l}$ . Using the properties of Poisson random variables, we can reduce the complexity of this expression. Let  $L = \max(l_k, 1 \leq k \leq K)$  and for  $l \in \{1, \dots, L\}$ , let  $K_l = \{k, l_k \geq l\}$ . Then,  $N_{\text{tot}}$  is distributed as

$$\sum_{l=1}^L l M_l$$

where  $(M_l, 1 \leq l \leq L)$  are independent Poisson random variables, the parameter of  $M_l$  being  $m_l := \sum_{k \in K_l} \lambda \tau_k \zeta_{k,l}$ . For each  $l$ , it is easy to construct an array which represents the distribution of  $lM_l$  by the following rule:

$$p_l(w) = \begin{cases} 0 & \text{if } w \bmod l \neq 0, \\ \exp(-m_l) m_l^q / q! & \text{if } w = ql. \end{cases}$$

By discrete convolution, the distribution of  $N_{\text{tot}}$  and then its cumulative distribution function, are easily calculable. The value of  $N_{\text{avail}}$  which ensures a loss probability below the desired threshold is found by inspection. The only difficulty with this approach is to determine where to truncate the Poisson distribution functions for machine representation. According to large deviation theory [6],

$$\mathbf{P}(\text{Poisson}(\theta) \geq a\theta) \leq \exp(-\theta(a \ln a + 1 - a)).$$

When  $\theta$  is known, it is straightforward to choose  $a(\theta)$  so that the right-hand-side of the previous equation is smaller than the desired threshold. The total memory size is thus proportional to  $\max(m_l a(m_l) l, 1 \leq l \leq L)$ . This may be memory (and time) consuming if the parameters of some Poisson random variables or the threshold are small. This method is well suited to estimate loss probability since it gives exact results within a reasonable amount of time but it is less useful for dimensioning purpose. Given  $N_{\text{avail}}$ , if we seek for the value of  $\lambda$  which guarantees a loss probability less than the desired threshold, there is no better way than trial and error. At least, the subsequent methods even imprecise may help to evaluate the order of magnitude of  $\lambda$  for the first trial.

**3.2. Approximations.** We begin by the classical Gaussian approximation. It is clear that

$$\begin{aligned} \mathbf{P}\left(\int_E f \, d\omega \geq N_{\text{avail}}\right) &= \mathbf{P}\left(\int_E f_\sigma(\, d\omega - \lambda \, d\nu) \geq N_\sigma\right) \\ &= \mathbf{E}_{\lambda\nu} \left[ \mathbf{1}_{[N_\sigma, +\infty)}\left(\int_E f_\sigma(\, d\omega - \lambda \, d\nu)\right) \right] \end{aligned}$$

where  $N_\sigma = (N_{\text{avail}} - \int f \lambda \, d\nu) / \sigma$ . Since the indicator function  $\mathbf{1}_{[N_\sigma, +\infty)}$  is not Lipschitz, in principle, we can not apply the bound given by Theorem 2. In practice,

it is still reasonable to do it. Hence,

$$(3) \quad 1 - Q(N_\sigma) - \frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{m(3, 1)}{\sqrt{\lambda}} \\ \leq \mathbf{P}\left(\int_E f \, d\omega \geq N_{\text{avail}}\right) \leq \\ 1 - Q(N_\sigma) + \frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{m(3, 1)}{\sqrt{\lambda}},$$

where  $Q$  is the cumulative distribution function of a standard Gaussian random variable.

From (10), it follows that

$$(4) \quad 1 - Q(N_\sigma) - \frac{m(3, 1)}{6\sqrt{\lambda}} Q^{(3)}(N_\sigma) - E_\lambda \\ \leq \mathbf{P}\left(\int_E f \, d\omega \geq N_{\text{avail}}\right) \leq \\ 1 - Q(N_\sigma) + \frac{m(3, 1)}{6\sqrt{\lambda}} Q^{(3)}(N_\sigma) + E_\lambda$$

where  $E_\lambda$  is the right-hand-side of (13) with  $\|F^{(3)}\|_\infty = 1$ .

Going again one step further, following the same lines, according to (15), one can show that

$$(5) \quad \mathbf{P}\left(\int_E f \, d\omega \geq N_{\text{avail}}\right) \leq 1 - Q(N_\sigma) \\ + \frac{m(3, 1)}{6\sqrt{\lambda}} Q^{(3)}(N_\sigma) + \frac{m(3, 1)^2}{72\lambda} Q^{(5)}(N_\sigma) \\ + \frac{m(4, 1)}{24\lambda} Q^{(3)}(N_\sigma) + F_\lambda$$

where  $F_\lambda$  is bounded above in (16).

For all the approximations given above, for a fixed value of  $N_{\text{avail}}$ , an approximate value of  $\lambda$  can be obtained by solving numerically an equation in  $\sqrt{\lambda}$ .

**3.3. Robust upper-bound.** If we seek for robustness and not precision, it may be interesting to consider the so-called concentration inequality. We remark that in the present context,  $f$  is non-negative and bounded by  $L = \max_k l_k$  so that we are in position to apply Theorem 4. We obtain that

$$(6) \quad \mathbf{P}\left(\int_E f \, d\omega \geq \int_E f \, d\nu + a\right) \\ \leq \exp\left(-\frac{\int_E f^2 \lambda \, d\nu}{L^2} g\left(\frac{aL}{\int_E f^2 \lambda \, d\nu}\right)\right),$$

where  $g$  is defined in Section C.

#### 4. APPLICATIONS TO OFDMA AND LTE

In such systems, there is a huge number of physical parameters with a wide range of variations, it is thus rather hard to explore the while variety of sensible scenarios. For illustration purposes, we chose a circular cell of radius  $R = 300$  meters equipped with an isotropic antenna such that the transmitted power is 1 W and the reference distance is 10 meters. The mean number of active customers per unit of surface, denoted by  $\lambda$ , was chosen to vary between 0,001 and 0.000 1, this corresponds to an average number of active customers varying from 3 to 30, a realistic value for



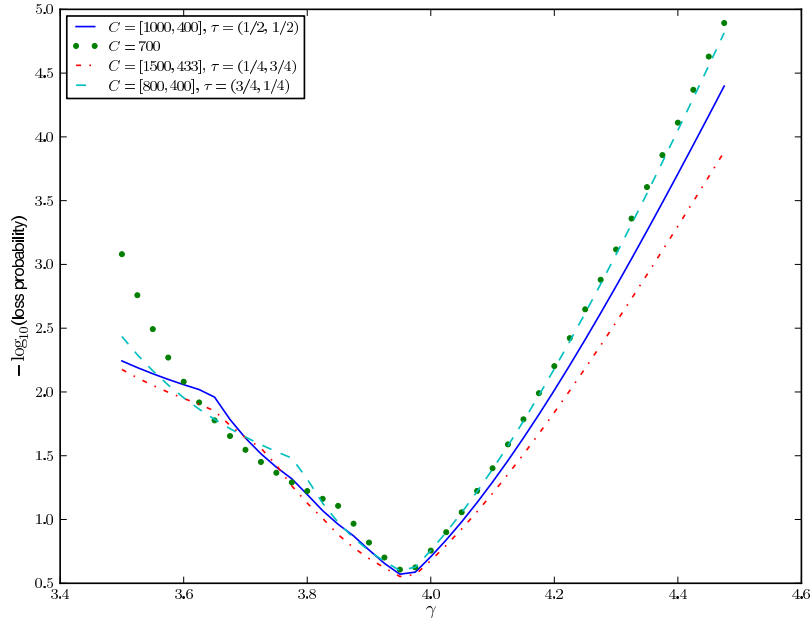


FIGURE 2. Impact of  $\gamma$  and  $\tau$  on the loss probability ( $N_{\text{avail}} = 92$ ,  $\lambda = 0.0001$ )

the systems under consideration. The minimum SINR is 0.3 dB and the random variable  $S$  defined above is a centered Gaussian with variance equal to 10. There are two classes of customers,  $C_1 = 1,000$  kb/s and  $C_2 = 400$  kb/s. It must be noted that our set of parameters is not universal but for the different scenarios we tested, the numerical facts we want to point out were always apparent. Since the time scale is of the order of a packet transmission time, the traffic is defined as the mean number of required subchannels at each slot provided that the time unit is the slot duration, that is to say that the load is defined as  $\rho = \lambda \int_{\text{cell}} f \, d\nu$ .

Figure 2 shows, the loss probability may vary up to two orders of magnitude when the rate and the probability of each class change even if the mean rate  $\sum_k \tau_k C_k$  remains constant. Thus mean rate is not a sufficient parameter to predict the performances of such a system. The load  $\rho$  is neither a pertinent indicator as the computations show that the loads of the various scenarios differs from less than 3%.

Comparatively, Figure 2 shows that variations of  $\gamma$  have tremendous effects on the loss probability: a change of a few percents of the value of  $\gamma$  induces a variation of several order of magnitude for the loss probability. It is not surprising that the loss probability increases as a function of  $\gamma$ : as  $\gamma$  increases, the radio propagation conditions worsen and for a given transmission rate, the number of necessary subchannels increases, generating overloading. Beyond a certain value of  $\gamma$  (apparently around 3.95 on Figure 2), the radio conditions are so harsh that a major part of the customers are in outage since they do not satisfy the SNR criterion any longer. We remark here that the critical value of  $\gamma$  is almost the same for all configurations of classes. Indeed, the critical value  $\gamma_c$  of  $\gamma$  can be found by a simple reasoning: When  $\gamma < \gamma_c$ , a class  $k$  customer uses less than the allowed  $l_k$  subchannels because the radio conditions are good enough for  $\beta_{k,j}^{1/\gamma} \geq R$  for some  $j < l_k$  so that the load

increases with  $\gamma$ . For  $\gamma > \gamma_c$ , all the  $\beta_{k,l}^{-1/\gamma}$  are lower than  $R$  and the larger  $\gamma$ , the wider the gap. Hence the number of customers in outage increases as  $\gamma$  increases and the load decreases. Thus,

$$\gamma_c \simeq \inf\{\gamma, \beta_{s,l_s-1}^{-1/\gamma} \leq R\} \text{ for } s = \arg \max_k l_k.$$

If we proceed this way for the data of Figure 2, we retrieve  $\gamma_c = 3.95$ . This means that for a conservative dimensioning, in the absence of estimate of  $\gamma$ , computations may be done with this value of  $\gamma$ .

For a threshold given by  $\epsilon = 10^{-4}$ , we want to find  $N_{\text{avail}}$  such that  $\mathbf{P}(N_{\text{tot}} \geq N_{\text{avail}}) \leq \epsilon$ . As said earlier, the exact method gives the result at the price of a sometimes lengthy process. In view of 3, one could also search for  $\alpha$  such that

$$(7) \quad 1 - Q(\alpha) + \frac{1}{2} \sqrt{\frac{2}{\pi}} m(3, \lambda) = \epsilon$$

and then consider  $[\int_E f d\nu + \alpha\sigma]$  as an approximate value of  $N_{\text{avail}}$ . Unfortunately and as was expected since the Gaussian approximation is likely to be valid for *large* values of  $\lambda$ , the corrective term in (7) is far too large (between 30 and 500 depending on  $\gamma$ ) for (7) to have a meaning. Hence, we must proceed as usual and find  $\alpha$  such that  $1 - Q(\alpha) = \epsilon$ , i.e.  $\alpha \simeq 3.71$ . The approximate value of  $N_{\text{avail}}$  is thus given by  $[\int_E f d\nu + 3.71\sigma]$ . The consequence is that we do not have any longer any guarantee on the quality of this approximation, how close it is to the true value and even more basic, whether it is greater or lower than the correct value. In fact, it is absolutely impossible to choose a dimensioning value lower than the true value since there is no longer a guarantee that the loss probability is lower than  $\epsilon$ . As shows Figure 3, it turns out that the values returned by the Gaussian method are always under the true value. Thus this annihilates any possibility to use the Gaussian approximation for dimensioning purposes.

Going one step further, according to (4), one may find  $\alpha$  such that

$$1 - Q(\alpha) - \frac{m(3, \lambda)}{6} Q^{(3)}(\alpha) + E_\lambda = \epsilon$$

and then use

$$[3.5 + \int_E f d\nu + \alpha\sigma]$$

as an approximate *guaranteed* value of  $N_{\text{avail}}$ . By *guaranteed*, we mean that according to (4), it holds for sure that the loss probability with this value of  $N_{\text{avail}}$  is smaller than  $\epsilon$  even if there is an approximation process during its computation. Since the error in the Edgeworth approximation is of the order of  $1/\lambda$ , instead of  $1/\sqrt{\lambda}$  for the Gaussian approximation, one may hope that this method will be efficient for smaller values of  $\lambda$ . It turns out that for the data sets we examined,  $E_\lambda$  is of the order of  $10^{-7}/\lambda$ , thus this method can be used as long as  $10^{-7}/\lambda \ll \epsilon$ . Otherwise, as for the Gaussian case, we are reduced to find  $\alpha$  such that

$$1 - Q(\alpha) - \frac{m(3, \lambda)}{6} Q^{(3)}(\alpha) = \epsilon$$

and consider  $[\int_E f d\nu + \alpha\sigma]$  but we no longer have any guarantee on the validity of the value. As Figure 3 shows, for the considered data set, Edgeworth methods leads to an optimistic value which is once again absolutely not acceptable. One can pursue the development as in (15) and use (5), thus we have to solve

$$1 - Q(\alpha) - \frac{m(3, \lambda)}{6} Q^{(3)}(\alpha) - \frac{m(3, 1)^2}{72\lambda} Q^{(6)}(\alpha) + \frac{m(4, 1)}{24\lambda} Q^{(4)}(\alpha) - F_\lambda = \epsilon.$$

For the values we have, it turns out that  $F_\lambda$  is of the order of  $10^{-9}\lambda^{-3/2}$  which is negligible compared to  $\epsilon = 10^{-4}$ , so that we can effectively use this method for  $\lambda \geq 10^{-4}$ . As it is shown in Figure 3, the values obtained with this development are very close to the true values but always greater as it is necessary for the guarantee. The procedure should thus be the following: compute the error bounds given by (3), (13) and (5) and find the one which gives a value negligible with respect to the threshold  $\epsilon$ , then use the corresponding dimensioning formula. If none is suitable, use a finer Edgeworth expansion or resort to the concentration inequality approach.

Note that the Edgeworth method requires the computations of the first three (or five) moments, whose lengthiest part is to compute the  $\zeta_{k,l}$  which is also a step required by the exact method. Thus Edgeworth methods are dramatically simpler than the exact method and may be as precise. However, both the exact and Edgeworth methods suffer from the same flaw: There are precise as long as the parameters, mainly  $\lambda$  and  $\gamma$ , are perfectly well estimated. The value of  $\gamma$  is often set empirically (to say the least) so that it seems important to have dimensioning values robust to some estimate errors. This is the goal of the last method we propose.

According to (6), if we find  $\alpha$  such that

$$g\left(\frac{\alpha L}{\int_E f^2 \lambda \, d\nu}\right) = -\frac{\log(\epsilon)L^2}{\int_E f^2 \lambda \, d\nu}$$

and

$$(8) \quad N_{\text{avail}} = \int_E f \, d\nu + \frac{\alpha}{L^2} \int_E f^2 \lambda \, d\nu,$$

we are sure that the loss probability will fall under  $\epsilon$ . However, we do not know a priori how larger this value of  $N_{\text{avail}}$  than the true value. It turns out that the relative oversizing increases with  $\gamma$  from a few percents to 40% for the large value of  $\gamma$  and hence small values of  $N_{\text{avail}}$ . For instance, for  $\gamma = 4.2$ , the value of  $N_{\text{avail}}$  given by (8) is 40 whereas the exact value is 32 hence an oversizing of 25%. However, for  $\gamma = 4.12$ , which is 2% away from 4.2, the required number of subchannels is also 40. The oversizing is thus not as bad as it may seem since it may be viewed as a protection against traffic increase, epistemic risk (model error) and estimate error.

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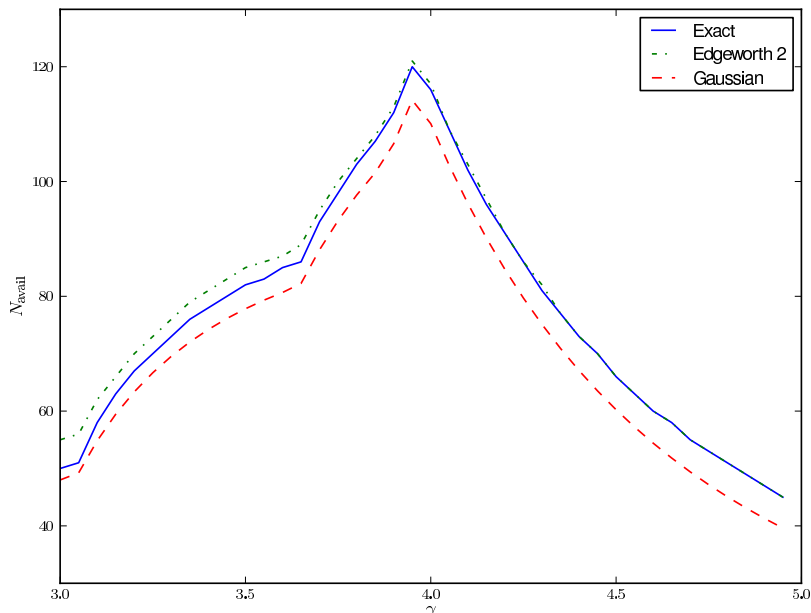


FIGURE 3. Estimates of  $N_{\text{avail}}$  as a function of  $\gamma$  by the different methods

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#### APPENDIX A. HERMITE POLYNOMIALS

Let  $\Phi$  be the Gaussian probability density function:  $\Phi(x) = \exp(-x^2/2)/\sqrt{2\pi}$  and  $\mu$  the Gaussian measure on  $\mathbf{R}$ . Hermite polynomials ( $H_k$ ,  $k \geq 0$ ) are defined by the recursion formula:

$$H_k(x)\Phi(x) = (-1)^k \frac{d^k}{dx^k} \Phi(x).$$

For the sake of completeness, we recall that

$$\begin{aligned} H_0(x) &= 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x \\ H_4(x) &= x^4 - 6x^2 + 3, \quad H_5(x) = x^5 - 10x^3 + 15x. \end{aligned}$$

Thus, for  $F \in \mathcal{C}_b^k$ , using integration by parts, we have

$$(9) \quad \int_{\mathbf{R}} F^{(k)}(x) \, d\mu(x) = \int_{\mathbf{R}} F(x) H_k(x) \, d\mu(x).$$

Let  $Q(x) = \int_{-\infty}^x \Phi(u) \, du = \int_{\mathbf{R}} \mathbf{1}_{(-\infty; x]}(u) \Phi(u) \, du$ . Then,  $Q' = \Phi$  and

$$(10) \quad \begin{aligned} & \int_{\mathbf{R}} \mathbf{1}_{(-\infty; x]}(u) H_k(u) \, d\mu(u) \\ &= (-1)^k \int_{-\infty}^x \Phi^{(k)}(u) \, du = (-1)^k Q^{(k)}(x) = -H_{k-1}(x) \Phi(x). \end{aligned}$$

#### APPENDIX B. EDGEWORTH EXPANSION

For details on Poisson processes, we refer to [2, 5]. For  $E$  a Polish space equipped with a Radon measure  $\nu$ ,  $\Gamma_E$  denotes the set of locally finite discrete measures on  $E$ . The generic element  $\omega$  of  $\Gamma_E$  may be identified with a set  $\omega = \{x_n, n \geq 1\}$  such that  $\omega \cap K$  has finite cardinal for any  $K$  compact in  $E$ . We denote by  $\int_E f \, d\omega$  the sum  $\sum_{x \in \omega} f(x)$  provided that it exists as an element of  $\mathbf{R} \cup \{+\infty\}$ . A Poisson process of intensity  $\nu$  is a probability  $\mathbf{P}_\nu$  on  $\Gamma_E$ , such that for any  $f \in \mathcal{C}_K(E, \mathbf{R})$ ,

$$\mathbf{E}_\nu \left[ \exp\left(-\int_E f \, d\omega\right) \right] = \exp\left(-\int_E 1 - e^{-f(x)} \, d\nu(x)\right).$$

For  $f \in L^1(\nu)$ , the Campbell formula states that

$$\mathbf{E}_\nu \left[ \int f \, d\omega \right] = \int f \, d\nu.$$

We introduce the discrete gradient  $D$  defined by

$$D_x F(\omega) = F(\omega \cup \{x\}) - F(\omega), \text{ for all } x \in E.$$

In particular, for  $f \in L^1(\nu)$ , we have

$$D_x \int_E f \, d\omega = f(x).$$

The domain of  $D$ , denoted by  $\text{Dom } D$  is the set of functionals  $F : \Gamma_E \rightarrow \mathbf{R}$  such that

$$\mathbf{E}_\nu \left[ \int_E |D_x F(\omega)|^2 \, d\nu(x) \right] < \infty.$$

The integration by parts then says that, for any  $F \in \text{Dom } D$ , any  $u \in L^2(\nu)$ ,

$$(11) \quad \begin{aligned} \mathbf{E}_\nu \left[ F \int_E u(x) (d\omega(x) - d\nu(x)) \right] \\ = \mathbf{E}_\nu \left[ \int_E D_x F u(x) \, d\nu(x) \right]. \end{aligned}$$

We denote by  $\sigma = \|f\|_{L^2(\nu)} \sqrt{\lambda}$  and  $f_\sigma = f/\sigma$ . Note that  $\|f_\sigma\|_{L^2(\nu)} = 1/\lambda$  and that

$$m(p, \lambda) := \int_E |f_\sigma(x)|^p \lambda \, d\nu(x) = \|f\|_{L^2(\nu)}^{-p} \|f\|_{L^p(\nu)}^p \lambda^{1-p/2}.$$

The proof of the following theorem may be found in [5, 12, 13].

**Theorem 2.** *Let  $f \in L^2(\nu)$ . For  $\lambda > 0$ , let*

$$N^\lambda = \int_E f_\sigma(x) (d\omega(x) - \lambda \, d\nu(x)).$$

Then, for any Lipschitz function  $F$  from  $\mathbf{R}$  to  $\mathbf{R}$ , we have

$$\left| \mathbf{E}_{\lambda\nu} [F(N^\lambda)] - \int_{\mathbf{R}} F \, d\mu \right| \leq \frac{1}{2} \sqrt{\frac{\pi}{2}} m(3, \lambda) \|F\|_{Lip}.$$

To prove the Edgeworth expansion and its error bound, we introduce some notions of Gaussian calculus. For  $F \in \mathcal{C}_b^2(\mathbf{R}; \mathbf{R})$ , we consider

$$AF(x) = xF'(x) - F''(x), \text{ for any } x \in \mathbf{R}.$$

The Ornstein-Uhlenbeck semi-group is defined by

$$P_t F(x) = \int_{\mathbf{R}} F(e^{-t}x + \sqrt{1 - e^{-2t}}y) \, d\mu(y) \text{ for any } t \geq 0.$$

The infinitesimal generator  $A$  and  $P_t$  are linked by the following identity

$$(12) \quad F(x) - \int_{\mathbf{R}} F(y) \, d\mu(y) = - \int_0^\infty AP_t F(x) \, dt.$$

**Theorem 3.** For  $F \in \mathcal{C}_b^3(\mathbf{R}, \mathbf{R})$ ,

$$(13) \quad \left| \mathbf{E}_{\lambda\nu} [F(N^\lambda)] - \int_{\mathbf{R}} F(y) \, d\mu(y) - \frac{1}{6} m(3, \lambda) \int_{\mathbf{R}} F(y) H_3(y) \, d\mu(y) \right| \leq \left( \frac{m(3, 1)^2}{6} + \frac{m(4, 1)}{9} \sqrt{\frac{2}{\pi}} \right) \frac{\|F^{(3)}\|_\infty}{\lambda}.$$

*Proof.* According to the Taylor formula,

$$(14) \quad \begin{aligned} D_x G(N^\lambda) &= G(N^\lambda + f_\sigma(x)) - G(N^\lambda) \\ &= G'(N^\lambda) f_\sigma(x) + \frac{1}{2} f_\sigma^2(x) G''(N^\lambda) \\ &\quad + \frac{1}{2} f_\sigma(x)^3 \int_0^1 r^2 G^{(3)}(rN^\lambda + (1-r)f_\sigma(x)) \, dr. \end{aligned}$$

Hence, according to (11) and (14),

$$\begin{aligned} \mathbf{E}_{\lambda\nu} [N^\lambda (P_t F)'(N^\lambda)] &= \mathbf{E}_{\lambda\nu} \left[ \int_E f_\sigma(x) D_x (P_t F)'(N^\lambda) \lambda \, d\nu(x) \right] \\ &= \mathbf{E}_{\lambda\nu} [(P_t F)''(N^\lambda)] \\ &\quad + \frac{1}{2} \int_E f_\sigma^3(x) \lambda \, d\nu(x) \mathbf{E}_{\lambda\nu} [(P_t F)^{(3)}(N^\lambda)] \\ &\quad + \frac{1}{2} \int_E f_\sigma^4(x) \lambda \, d\nu(x) \\ &\quad \times \mathbf{E}_{\lambda\nu} \left[ \int_0^1 (P_t F)^{(4)}(rN^\lambda + (1-r)f_\sigma(x)) r^2 \, dr \right] \\ &= A_1 + A_2 + A_3. \end{aligned}$$

It is well known that for  $F \in \mathcal{C}^k$ ,  $(x \mapsto P_t F(x))$  is  $k + 1$ -times differentiable and that we have two expressions of the derivatives (see [10]):

$$\begin{aligned} (P_t F)^{(k+1)}(x) &= \frac{e^{-(k+1)t}}{\sqrt{1 - e^{-2t}}} \int_{\mathbf{R}} F^{(k)}(e^{-t}x + \sqrt{1 - e^{-2t}}y) y \, d\mu(y). \end{aligned}$$

and  $(P_t F)^{(k+1)}(x) = e^{-(k+1)t} P_t F^{(k)}(x)$ . The former equation induces that

$$\begin{aligned} \|(P_t F)^{(k+1)}\|_\infty &\leq \frac{e^{-(k+1)t}}{\sqrt{1-e^{-2t}}} \|F^{(k)}\|_\infty \int_{\mathbf{R}} |y| \, d\mu(y) \\ &= \frac{e^{-(k+1)t}}{\sqrt{1-e^{-2t}}} \sqrt{\frac{2}{\pi}} \|F^{(k)}\|_\infty. \end{aligned}$$

Hence,

$$|A_3| \leq \frac{e^{-4t}}{6\sqrt{1-e^{-2t}}} \sqrt{\frac{2}{\pi}} m(4, \lambda) \|F^{(3)}\|_\infty.$$

Moreover, according to Theorem 2,

$$\begin{aligned} \left| \mathbf{E}_{\lambda\nu} \left[ (P_t F)^{(3)}(N^\lambda) \right] - \int_{\mathbf{R}} (P_t F)^{(3)}(x) \, d\mu(x) \right| \\ \leq \frac{1}{2} \sqrt{\frac{\pi}{2}} m(3, \lambda) \|(P_t F)^{(4)}\|_\infty \\ \leq \frac{1}{2} m(3, \lambda) \frac{e^{-4t}}{\sqrt{1-e^{-2t}}} \|F^{(3)}\|_\infty. \end{aligned}$$

Then, we have,

$$\begin{aligned} |A_2 - \frac{1}{2} m(3, \lambda) \int_{\mathbf{R}} (P_t F)^{(3)}(x) \, d\mu(x)| \\ \leq \frac{1}{4} m(3, \lambda)^2 \frac{e^{-4t}}{\sqrt{1-e^{-2t}}} \|F^{(3)}\|_\infty. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbf{E}_{\lambda\nu} \left[ N^\lambda (P_t F)'(N^\lambda) - (P_t F)''(N^\lambda) \right] \\ = \frac{1}{2} m(3, \lambda) \int_{\mathbf{R}} (P_t F)^{(3)}(x) \, d\mu(x) + R(t), \end{aligned}$$

where

$$R(t) \leq \left( \frac{m(3, \lambda)^2}{4} + \frac{m(4, \lambda)}{6} \sqrt{\frac{2}{\pi}} \right) \|F^{(3)}\|_\infty \frac{e^{-4t}}{\sqrt{1-e^{-2t}}}.$$

Now then,

$$\begin{aligned} \int_{\mathbf{R}} (P_t F)^{(3)}(x) \, d\mu(x) \\ = e^{-3t} \int_{\mathbf{R}} \int_{\mathbf{R}} F^{(3)}(e^{-t}x + \sqrt{1-e^{-2t}}y) \, d\mu(y) \\ = e^{-3t} \int_{\mathbf{R}} F^{(3)}(y) \, d\mu(y) \\ = e^{-3t} \int_{\mathbf{R}} F(y) H_3(y) \, d\mu(y), \end{aligned}$$

since the Gaussian measure on  $\mathbf{R}^2$  is rotation invariant and according to (9). Remark that

$$\int_0^\infty e^{-4t} (1-e^{-2t})^{-1/2} \, dt = 2/3$$

and applying (12) to  $x = N^\lambda$ , the result follows.  $\square$

This development is not new in itself but to the best of our knowledge, it is the first time that there is an estimate of the error bound. Following the same lines, we can pursue the expansion up to any order provided that  $F$  be sufficiently differentiable. Namely, for  $F \in \mathcal{C}_b^5$ , we have

$$(15) \quad \mathbf{E}_{\lambda\nu} [F(N^\lambda)] = \int_{\mathbf{R}} F(y) \, d\mu(y) \\ + \frac{m(3,1)}{6\sqrt{\lambda}} \int_{\mathbf{R}} F^{(3)}(y) \, d\mu(y) + \frac{m(3,1)^2}{72\lambda} \int_{\mathbf{R}} F^{(6)}(y) \, d\mu(y) \\ + \frac{m(4,1)}{24\lambda} \int_{\mathbf{R}} F^{(4)}(y) \, d\mu(y) + F_\lambda \|F^{(5)}\|_\infty.$$

where

$$(16) \quad F_\lambda \leq \frac{m(3,1)}{\lambda^{3/2}} \left( \frac{2}{45} m(3,1)^2 \right. \\ \left. + \left( \frac{4}{135} + \frac{\pi^2}{128} \right) \sqrt{\frac{2}{\pi}} m(4,1) \right).$$

#### APPENDIX C. CONCENTRATION INEQUALITY

We are now interested in an upper bound, which is called concentration inequality.

**Theorem 4.** *Let  $M, a > 0$ . Assume that  $|f(z)| \leq M$   $\nu$ -a.s and  $f \in L^2(E, \nu)$ , then*

$$(17) \quad \mathbf{P}(F > \mathbf{E}[F] + a) \leq \exp \left\{ -\frac{M^2}{\mathbf{V}[F]} g \left( \frac{aM}{\mathbf{V}[F]} \right) \right\}$$

where  $g(u) = (1+u) \ln(1+u) - u$ .

The above theorem can be directly derived from [16]. However let us take this opportunity to prove this theorem in a very nice, simple and elementary fashion, exactly the same way as Bennett built his concentration inequality for the sum of  $n$  i.i.d random variables.

*Proof.* Using Chernoff's bound we have:

$$\mathbf{P}(F > \mathbf{E}[F] + a) \leq \mathbf{E} [e^{\theta F}] / e^{\theta(\mathbf{E}[F] + a)} \\ = e^{\int_E (e^{\theta f(z)} - 1 - \theta f(z)) \, d\nu(z) - \theta a}$$

Now assume that  $|f(z)| \leq M$   $\nu$ -a.s. Observe that the function  $(e^x - 1 - x)/x^2$  is increasing on  $\mathbf{R}$  (the value at 0 is 1/2), we have that

$$e^{\theta f(z)} - \theta f(z) - 1 \leq \frac{e^{\theta M} - 1 - \theta M}{M^2} f^2(z) \quad \nu \text{ a.s.}$$

Thus,

$$\mathbf{P}(F > \mathbf{E}[F] + a) \\ \leq \exp \left\{ \int_E \left( \frac{e^{\theta M} - \theta M - 1}{M^2} f^2(z) \right) \, d\nu(z) - \theta a \right\} \\ = \exp \left\{ \frac{e^{\theta M} - 1 - \theta M}{M^2} \mathbf{V}[F] - \theta a \right\}.$$

We find that  $\theta = \ln(1 + aM/\mathbf{V}[F])/M$  minimizes the right-hand-side and thus we obtain (17).  $\square$



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