Upper bound of loss probability in an OFDMA system with randomly located users

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Abstract

For OFDMA systems, we find a rough but easily computed upper bound for the probability of loosing communications by insufficient number of sub-channels on downlink. We consider as random the positions of receiving users in the system as well as the number of sub-channels dedicated to each one. We use recent results of the theory of point processes which reduce our calculations to that of the first and second moments of the total required number of sub-carriers.

1 Introduction

The demand for high data rate wireless applications with restrictions in the RF signal bandwidth requires bandwidth efficient air interface schemes. It is known that OFDM yields a relatively simple solution to these problems [7]. Based on the OFDM system, OFDMA can achieve a larger capacity. Furthermore, this latter system is more flexible, since it can be easily scaled to fit in a certain piece of spectrum simply by changing the number of used subcarriers [9]. However, as any wireless systems, OFDM and OFDMA have physical limitations which cause loss of communications. This loss can be caused by insufficient power or by low signal-to-interference ratio, for instance. In this paper we are interested in the calculation of an upper bound of the probability of loosing a communication due to an insufficient number of sub-channels in the downlink.

We say that the system is overloaded when all non-used sub-channels are not enough to warrant a minimum data rate for an incoming demand. We consider a system with N_0 sub-carriers and N_i is the number of sub-carriers used by the i-th user in the cell. As it is usually done, we substitute the finite number of subcarriers by infinity and substitute the loss probability by $P_{loss} = P(\sum_i N_i > N_0)$. It is well known that this consideration gives us an upper bound for the actual loss probability.

A user i requires a capacity C_i depending on the service he uses. Considering a system with just one kind of service, all users require the same capacity C_0.

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Even so, the number of subcarriers for each user varies according to the channel conditions. These conditions can be summarized into two kinds of gains, one depending only on the position of the user i, the path loss G_{pl_i}, and a gain G_i, which classically may include the shadowing and the Rayleigh fading. We choose the simplest model to represent the path loss:

$$G_{pl_i} = \frac{K}{D_i^\gamma}$$

where K and γ are constants and D_i is the distance between the user i and the antenna.

Shannon's maximum achievable capacity implies that:

$$N_i = \left\lceil \frac{C_0}{W \log_2 \left(1 + \frac{P_t G_{pl_i} G_i}{I}\right)} \right\rceil$$

where W is the bandwidth of each sub-carrier, P_t is the mean transmitted power by sub-channel and I is the power of the additive Gaussian white noise by sub-channel.

We consider that the number as well as the position of users in the cells are random. After some natural assumptions done in the following section, we conclude that the configuration of users in the cell is a Poisson point process (see section 2).

After a summary on Poisson point process, we consider three different cases to calculate an upper bound for the loss probability. First we consider the simplest case with deterministic gain. In Section 4, we consider a non-selective frequency gain, the shadowing. In section 5, we consider a general case from which all other cases could be derived but for which no closed form formula exists.

2 Poisson point processes

For details on point processes, we refer to [1, 4, 5, 6]. A configuration η in $\mathbf{R}^k$ is a set $\{x_n,\, n \geq 1\}$ where for each $n \geq 1$, $x_n \in \mathbf{R}^k$, $x_n \neq x_m$ for $n \neq m$ and each compact subset of $\mathbf{R}^k$ contains only a finite subset of η. We denote by $\Gamma_{\mathbf{R}^k}$ the set of configurations in $\mathbf{R}^k$. Equipped with the vague topology of discrete measures, $\Gamma_{\mathbf{R}^k}$ is a complete, separable metric space. A point process Φ is a random variable with values in $\Gamma_{\mathbf{R}^k}$, i.e., $\Phi(\omega) = \{X_n(\omega),\, n \geq 1\} \in \Gamma_{\mathbf{R}^k}$. For $A \subset \mathbf{R}^k$, we denote by Φ_A the random variable which counts the number of atoms of $\Phi(\omega)$ in A:

$$\Phi_A(\omega) = \sum_{n \geq 1} \mathbf{1}_{X_n(\omega) \in A} \ \in \mathbf{N} \cup \{+\infty\}.$$

Poisson point processes are particular instances of point processes such that:

Definition 1. *Let Λ be a σ finite measure on $\mathbf{R}^k$. A point process Φ is a Poisson process of intensity Λ whenever the two following properties hold.*

1 - For any compact subset $A \in \mathbf{R}^k$, Φ_A is a Poisson random variable of parameter $\Lambda(A)$, i.e.,

$$\mathbf{P}(\Phi_A = k) = e^{-\Lambda(A)} \frac{\Lambda(A)^k}{k!}.$$

2 - For any disjoint subsets A and B, the random variables Φ_A and Φ_B are independent.

The notion of point process trivially extends to configurations in $\mathbf{R}^k \times X$ where X is a subset of $\mathbf{R}^m$. A configuration is then typically of the form $\{(x_n, y_n), n \geq 1\}$ where for each $n \geq 1$, $x_n \in \mathbf{R}^k$ and $y_n \in X$. We keep writing (x_n, y_n) as a couple, though it could be thought as an element of $\mathbf{R}^{k+m}$, to stress the asymmetry between the spatial coordinate x_n and the so-called mark, y_n. For a marked point process, we denote by Φ the set of locations, i.e., $\Phi(\omega) = \{X_n, n \geq 1\}$ and by $\bar{\Phi}$ the set of both locations and marks, i.e., $\bar{\Phi}(\omega) = \{(X_n, Y_n), n \geq 1\}$. A marked point process with position dependent marking is a marked point process for which the law of Y_n, the mark associated to the atom located at X_n, depends only on X_n through a kernel K:

$$\mathbf{P}(Y_n \in B \,|\, \Phi) = K(X_n, B), \text{ for any } B \subset X.$$

If K is a probability kernel, i.e., if $K(x, X) = 1$ for any $x \in \mathbf{R}^k$ then it is well known that $\bar{\Phi}$ is a Poisson process of intensity $K(x, \mathrm{d}y)\mathrm{d}\Lambda(x)$ on $\mathbf{R}^k \times \mathbf{R}^m$. The Campbell formula is a well known and useful formula

Theorem 1. *Let $\bar{\Phi}$ be a marked Poisson process on $\mathbf{R}^k \times \mathbf{R}^m$. Let Λ be the intensity of the underlying Poisson process and K the kernel of the position dependent marking. For $f : \mathbf{R}^k \times \mathbf{R}^m \to \mathbf{R}$ a measurable non-negative function, let*

$$F = \int f \, d\bar{\Phi} = \sum_{n \geq 1} f(X_n, Y_n).$$

Then,

$$\mathbf{E}\,[F] = \int_{\mathbf{R}^k \times \mathbf{R}^m} f(x, y)\, K(x, dy) d\Lambda(x).$$

Definition 2. *For $F : \Gamma_{\mathbf{R}^k} \to \mathbf{R}$, for any $x \in \mathbf{R}^k$, we define*

$$D_x F(\omega) = F(\omega \cup \{x\}) - F(\omega).$$

Note that for $F = \int f \mathrm{d}\Phi$, $D_x F = f(x)$, for any $x \in \mathbf{R}^k$. We now quote from [3, 10] the main result on which our inequalities are based:

Theorem 2 (Concentration inequality). *Assume that Φ is a Poisson process on $\mathbf{R}^k$ of intensity Λ. Let $f : \mathbf{R}^k \to \mathbf{R}^+$ a measurable non-negative function and let*

$$F(\omega) = \int f \, d\Phi = \sum_{n \geq 1} f(X_n(\omega)).$$

Assume that $|D_x F(\omega)| \leq s$ for any $x \in \mathbf{R}^k$. Let

$$m_F = \mathbf{E}[F] = \int f(x) \, d\Lambda(x)$$

and

$$v_F = \int |D_x F(\omega)|^2 d\Lambda(x) = \int f^2(x) \, d\Lambda(x).$$

Then, for any $t \in \mathbf{R}^+$,

$$\mathbf{P}(F - m_F \geq t) \leq \exp\left(-\frac{v_F}{s^2} \, g\left(\frac{t\, s}{v_F} \right) \right)$$

where $g(t) = (1+t)\ln(1+t) - t$.

3 Deterministic gain

We state the following assumptions:

Assumption 1. *The position of each user is independent on the position of all other. The users are indistinguishable, i.e., the positions are identically distributed.*

Assumption 2. *The time between two consecutive demands of users for service in the system (or interarrival time) is exponentially distributed.*

We define $\rho(x)$ as the surface density of interarrival time in $\mathrm{s}^{-1}\mathrm{m}^{-2}$, constant in time. Hence, for a region $H \subseteq B$, the mean interarrival rate is $h = \int_H \rho(x) dx$ in s^{-1}.

Assumption 3. *The service time for every user is exponentially distributed with mean $1/\nu$.*

Assumption 4. *The cell C is circular, with radius R and with the antenna in the center.*

Assumption 5. *The channel gain depends only on the distance from the transmitting antenna.*

Assumption 6. *The surface density of interarrival time is constant.*

These assumptions are commonly done to simplify the mathematical treatment and are quite reasonable. If we show that the point process given by the location of the users is a Poisson process, then it is sufficient to have the two first moments in order to apply theorem 2 and then calculate an upper bound P_{sup} for P_{loss}. To do this, we consider the following lemma:

Lemma 1. *Considering assumptions 1, 2 and 3, the point process Φ of the active users positions is, in equilibrium, a Poisson process with intensity $d\Lambda(x) = \rho(x)\nu^{-1}dx$*

Proof. For a region H, in virtue of assumptions 2 and 3, the number of receiving (i.e., active) customers is the same as the number of customers in an M/M/∞ queue with input rate h and mean service time ν^{-1}. It is known [8] that the distribution of the number of users U in equilibrium is then

$$P(U = u) = \frac{(h/\nu)^u}{u!} e^{-h/\nu}.$$

It follows that the first condition of definition 1 is satisfied with intensity measure $\Lambda(H)$

$$\Lambda(H) = h/\nu = \int_H \frac{\rho(x)}{\nu}\, \mathrm{d}x.$$

Condition 2 of definition 1 follows straightforwardly from assumption 1. □

Without loss of generality, we consider the cell C has its antenna located at the origin. We are looking at evaluating

$$\mathbf{P}(\int N \,\mathrm{d}\Phi \geq N_0),$$

where $N(x)$ is defined by

$$N(x) = \left\lceil \frac{C_0}{W \log_2\left(1 + \dfrac{P_t K \bar{g}}{I x^\gamma}\right)} \right\rceil,$$

where $\bar{g}$ is the mean gain due to shadowing. Note that, with respect to x, N is increasing and piecewise constant. Let R_j, $j = 1, \cdots, N_{max}$ be the values such that $N(x) = j$ for $x \in [R_j, R_{j+1})$. We can easily determine them by

$$R_j = \left(\frac{P_t K \overline{g}}{I(2^{C_0/(jW)} - 1)}\right)^{1/\gamma}.$$

According to Theorem 1, it is then clear that

$$\mathbf{E}\left[\int N \,\mathrm{d}\Phi\right] = \int N \mathrm{d}\Lambda = \frac{\pi\rho}{\nu} \sum_{j=1}^{N_{max}} j(R_j^2 - R_{j-1}^2).$$

We denote by m_N the last quantity. Moreover,

$$\int N^2 \mathrm{d}\Lambda = \frac{\pi\rho}{\nu} \sum_{j=1}^{N_{max}} j^2(R_j^2 - R_{j-1}^2).$$

α	1.5	1.6	1.7	1.8	1.9	2
P_{sup}	0.18	0.1	0.04	0.02	0.008	0.003
Δ	0.98	0.1	1.15	1.3	1.3	1.4

Table 1: Comparison between P_{sup} and P_{loss} for deterministic gain.

We denote by v_N the last quantity. We take N_0 of the form αm_N, so that according to Theorem 2:

$$\mathbf{P}(\int N \, \mathrm{d}\Phi \geq \alpha m_N) \leq P_{sup}(\alpha)$$

where

$$P_{sup}(\alpha) = \exp\left(-\frac{v_N}{N_{max}^2} \, g\left(\frac{(\alpha-1)m_N N_{max}}{v_F}\right)\right).$$

It is then natural to verify how far this bound is from the exact value of the loss probability in simple situations where simulation is available. We used here $\gamma = 2.8$, $C_0 = 200$ kb/s, $W = 250$ kHz and $P_t K/I = 1 \times 10^6$. For the surface density of interarrival time we use $\rho = 0.0006$ min^{-1}m^{-2} and the service time is $1/\nu = 1$ min, so, the mean number of users in the system is $\pi R^2 \rho/\nu = 18.85$ users. If we consider the shadowing with $\sigma = \sqrt{10}$ dB and $\mu = 6$ dB, we can use the mean gain g, giving $\overline{g} = 1/12$. Thus, users in the cell boundary use 3 sub-channels, so $N_{max} = 3$. For α varying from 1 to 2, which corresponds here to loss probabilities about 2% or 0.01%, we computed $\Delta = \log_{10} P_{sup}/P_{loss}$. Though concentration inequalities are usually thought as almost optimal, the results shown in Table 1 seem at first glance disappointing. Remind though that the computation of the bound is immediate whereas the simulation on a fast PC took several hours to get a decent confidence interval. Remind also that the error is about the same order of magnitude as the error made when using a usual trick which consists in replacing infinite buffers by finite ones in Jackson networks (see [2]). The margin provided by the bounds may be viewed as a protection against errors in the modelling or in the estimates of the parameters.

4 Random gain

Let us determine now the upper bound probability P_{sup} for P_{loss} without assumption 5 but holding all other assumptions of the preceding section. Lemma 1 still holds, since it is a consequence of assumptions 1, 2 and 3. We also state two other natural assumptions:

Assumption 7. *The random gain is totally described by the log-normal shadowing, with mean μ and standard deviation σ, both in dB.*

For a user at distance d from the origin, the gain is $G = 1/S$, where S follows a log-normal distribution:

$$p_S(y) = \frac{\xi}{\sqrt{2\pi}\sigma y} \exp\left[-\frac{(10\log_{10} y - \mu)^2}{2\sigma^2}\right],$$

where $\xi = 10/\ln 10$.

Assumption 8. *A user is able to receive the signal only if the signal-to-interference ratio is above some constant β_{min}.*

This means, in particular, that the number of subcarriers needed by a transmitting user is surely bounded by

$$N_{max} = \left\lceil \frac{C_0}{W \log_2(1+\beta_{min})} \right\rceil .$$

The situation is slightly different from that of Section 3, since the functional depends on two aleas: positions and gains. Consider now that our configurations are of the form (x, s) where $x \in \mathbf{R}^2$ is still a position and $s \in \mathbf{R}$ is a gain. Since gain and positions are independent, we then have a Poisson process on $\mathbf{R}^3$ of intensity measure $d\Lambda(x) \otimes p_S(y)\mathrm{d}y$. Thus we want to evaluate an upper bound of

$$\mathbf{P}(\int N \mathrm{d}\Phi \geq \mathbf{N}_0)$$

where

$$N(x,\, y) = \left\lceil \frac{C_0}{W \log_2\left(1 + \dfrac{P_t K}{I y x^\gamma}\right)} \right\rceil .$$

According to Theorem 2, we must compute

$$m_N = \int N(x,\ y) p_S(y) \mathrm{d}y \ \mathrm{d}\Lambda(x)$$

and

$$v_N = \sup_\omega \int |D_{x,y} F(\omega)|^2 p_S(y) \mathrm{d}y \ \mathrm{d}\Lambda(x) = \int N^2(x,\ y) p_S(y) \mathrm{d}y \ \mathrm{d}\Lambda(x).$$

Let $\beta_0 = \infty$ and $\beta_j = 2^{C_0/(Wj)} - 1$ for $j = 1, \cdots, N_{max} - 1$. For $j = 1, \cdots, N_{max} - 1$, let

$$A_j = \int_{C \times \mathbf{R}^+} \mathbf{1}_{\{y \|x\|^\gamma \leq P_t K / I \beta_j\}} p_S(y) \, \mathrm{d}y \, \mathrm{d}x$$

and $A_0 = 0$.

Lemma 2. *For $j = 1, \cdots, N_{max} - 1$,*

$$A_j = \pi R^2 Q(\alpha_j - \zeta \ln R) + \pi e^{2/\zeta^2 + 2\alpha_j/\zeta} Q(\zeta \ln R - 2/\zeta - \alpha_j),$$

where

$$\alpha_j = \frac{1}{\sigma}(10 \log_{10}(P_t K / I \beta_j) - \mu) \ \textit{and} \ \zeta = \frac{10\gamma}{\sigma \ln 10}.$$

Proof. We can write

$$A_j = \int_C \mathbf{P}(S\|x\|^\gamma \leq \tilde{\beta}_j)\,\mathrm{d}x$$

where $\tilde{\beta}_j = P_t K/I\beta_j$. Remind that S is equal in distribution to $\exp(\mathcal{N}(\mu,\sigma^2)\xi)$ with $\xi = \ln(10)/10$. Thus after a few manipulations, we get

$$A_j = 2\pi \int_0^R r\,Q(\alpha_j - \zeta \ln r)\,\mathrm{d}r,$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\frac{u^2}{2})\,\mathrm{d}u.$$

The final result follows by a tedious but straightforward integration by parts. □

Theorem 3. *For any function $\theta : \mathbf{R} \to \mathbf{R}$,*

$$\int \theta(N(x,\ y))p_S(y)dy\ d\Lambda(x) = \sum_{j=1}^{N_{max}-1} \theta(j)(A_j - A_{j-1}) + \theta(N_{max})(\pi R^2 - A_{N_{max}-1}).$$

Proof. Since N can take only a finite number of values, we have

$$\int \theta(N(x,\ y))p_S(y)\mathrm{d}y\ \mathrm{d}\Lambda(x) = \frac{\rho}{\nu} \sum_{j=1}^{N_{max}} \theta(j) \int_{C\times\mathbf{R}^+} \mathbf{1}_{\{(x,y),\ N(x,y)=j\}} p_S(y)\mathrm{d}y\ \mathrm{d}x.$$

Now we see that

$$N(x,\,y) = j \Longleftrightarrow \tilde{\beta}_{j-1} < y\|x\|^\gamma \leq \tilde{\beta}_j,$$

for $j = 1, \cdots, N_{max} - 1$ and $N(x,\,y) = N_{max}$ when $y\|x\|^\gamma > \tilde{\beta}_{N_{max}-1}$. The proof is thus complete. □

We used the same set of values as for the simulation of Section 3 together with assumptions 8 and 7 with $\beta_{min} = 0.2$. Results of Table 2 show that the theoretical bound is rather stable when gains become stochastic.

α	1.5	1.6	1.7	1.8	1.9	2
P_{sup}	0.2	0.1	0.05	0.02	0.01	0.004
Δ	1.7	1.8	2.1	2.3	2.4	2.6

Table 2: Comparison between P_{sup} and P_{loss} for random gain.

5 General case

Actually, the method can be applied to more general situations as we illustrate now. We consider only assumptions 1, 2, 3 and 8, a non-frequency selective random gain G with distribution p_G, a finite number of antennas with a deterministic pattern and the assumption that the user will receive the signal from the antenna which can provide a better signal-to-interference ratio.

Now C is the Borel set where it is possible to find users. Possibly, $C = \mathbf{R}^2$. In this region, we have a finite number of antennas $J+1$ with the j-th located at y_j, $j = \overline{1, J}$, and the one we observe located at y_0. This means that for each user, there is a vector $\mathbf{G} = (G, G^1, ..., G^J)$ where G is the gain due the antenna at y_0 and G^j due to antenna located at y_j. We then define the Poisson point process Ψ in $C \times \mathbf{R}_+^{J+1}$, representing the user positions and the gain of each one due to each antenna. Again, since gains from different antennas and positions are independent altogether, Ψ has intensity λ_m:

$$\lambda_m(\mathbf{g}, x) = p_G(g)p_G(g^1)...p_G(g^J)\frac{\rho(x)}{\nu}$$

We define the sets

$$A' = \left\{ \bigcup_{j=1}^{J} \left((g, g^1, \ldots, x), g^j > \frac{\|x - y_0\|^\gamma}{\|x - y_j\|^\gamma} g \right) \right\}$$

and

$$B = \left((s, x),\ s \leq R(x) \right),$$

where

$$R(x) = \frac{P_t K}{\beta_{min} \|x - y_0\|^\gamma}.$$

The event $((\mathbf{G}, X) \in A')$ means that the antenna at y_0 provides the highest signal-to-interference ratio to a point X. The event $((S, X) \in B)$ means the signal-to-interference ratio provided by the antenna at y_0 to a point X is higher than β_{min}. By Theorem 2, we are thus led to compute

$$\iint_{A \cap B} N(\|x\|, g)^k \mathrm{d}\lambda_m(x, g),$$

for $k = 1, 2$. There is no longer a closed form formula for these integrals but they can be easily and quickly computed by numerical methods. This yields to an upper bound of P_{loss}. We simulated in this section the loss probability for an antenna placed at the origin and six other antennas placed at the points $y_1 = (2R, 0)$, $y_2 = (R, R\sqrt{3})$, $y_3 = (-R, R\sqrt{3})$, $y_4 = (-2R, 0)$, $y_5 = (-R, -R\sqrt{3})$ and $y_6 = (R, -R\sqrt{3})$, representing an hexagonal arrangement. All other parameters are the same as the ones in previous sections. We find a mean $m_N = 21.60$ and the second moment $v_N = 26.81$. It turns out that the results are close to the results in Section 4, suggesting that the approach of Section 4 is satisfactory enough with our physical assumptions.

6 Concluding remarks

Using the concentration and deviation inequalities and the difference operator on Poisson space, we have calculated the upper bound probability of overloading the system by high demand of subcarriers, over path loss and shadow fading. To do this we have found the first and second moment of the marked Poisson point process of users. We conclude that it is possible to find an upper bound for the overloading probability, even in a relatively complex system, which is analytically computable in a very simple fashion. The method works for any functional of the configurations, possibly enriched by marks, which depends only on the positions of each user. It does not work for functionals involving relative distance between two or more users. Actually, for such a functional F, there is no bound on $D_x F(\omega)$ valid for all x and ω.

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