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Central limit theorems for multiple Skorohod integrals

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Abstract

In this paper, we prove a central limit theorem for a sequence of iterated Skorohod integrals using the techniques of Malliavin calculus. The convergence is stable, and the limit is a conditionally Gaussian random variable. Some applications to sequences of multiple stochastic integrals, and renormalized weighted quadratic variation of the fractional Brownian motion are discussed.

Key words: central limit theorem, fractional Brownian motion, Malliavin calculus.

2000 Mathematics Subject Classification: 60F05, 60H05, 60G15, 60H07.

1 Introduction

Consider a sequence of random variables $\{F_n, n \geq 1\}$ defined on a complete probability space (Ω, \mathcal{F}, P) . Suppose that the σ -field \mathcal{F} is generated by an

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isonormal Gaussian process $X = \{X(h), h \in \mathfrak{H}\}$ on a separable Hilbert space \mathfrak{H} . This means that X is a centered Gaussian family of random variables indexed by the elements of \mathfrak{H} , and such that for every $h, g \in \mathfrak{H}$,

$$E[X(h)X(g)] = \langle h, g \rangle_{\mathfrak{H}}. \quad (1.1)$$

Suppose that the sequence $\{F_n, n \geq 1\}$ is normalized, that is, $E(F_n) = 0$ and $\lim_{n \rightarrow \infty} E(F_n^2) = 1$. A natural problem is to find suitable conditions ensuring that F_n converges in law towards a given distribution. When the random variables F_n belong to the q th Wiener chaos, $q \geq 2$, then it turns out that the following conditions are equivalent:

- (i) F_n converges in law to $N(0, 1)$;
- (ii) $\lim_{n \rightarrow \infty} E[F_n^4] = 3$;
- (iii) $\lim_{n \rightarrow \infty} f_n \otimes_r f_n = 0$, for all $r = 1, \dots, q - 1$;
- (iv) $\lim_{n \rightarrow \infty} \|DF_n\|_{\mathfrak{H}}^2 = q$ in L^2 ,

where f_n is the element in the symmetric tensor product $\mathfrak{H}^{\odot q}$ associated with F_n , $f_n \otimes_r f_n$ denotes the contraction of order r of f_n and itself, and D is the derivative operator in the sense of Malliavin calculus (see Section 2 for more details).

Moreover, in this case, we have the following bound

$$\sup_{z \in \mathbb{R}} |P(F_n \leq z) - P(N \leq z)| \leq \sqrt{E \left[(1 - q^{-1} \|DF_n\|_{\mathfrak{H}}^2)^2 \right]}, \quad (1.2)$$

where N denotes a random variable with the law $N(0, 1)$.

The equivalence between conditions (i), (ii) and (iii) was proved in Nualart and Peccati [23] by means of the *Dambis, Dubins and Schwarz* Theorem. The equivalence with condition (iv) was proved later by Nualart and Ortiz-Latorre [22] using tools of Malliavin calculus. Finally, the Berry-Esseen's type bound (1.2) has been proved by Nourdin and Peccati in [17].

In particular, the above result implies that the convergence in distribution of a sequence of multiple stochastic integrals towards a Gaussian random variable is completely determined by the asymptotic behavior of their second and fourth moments, and it can be seen as a simplification of the classical "method of moments and diagrams" (see, for instance, the survey by Surgailis [30] and the references therein).

Peccati and Tudor [27] also obtained a multidimensional version of Peccati and Nualart theorem. In particular they proved that given a sequence $\{F_n, n \geq 1\}$ of d -dimensional random vectors, such that F_n^i belongs to the q_i th Wiener chaos for $i = 1, \dots, d$, where $1 \leq q_1 \leq \dots \leq q_d$, then if the covariance matrix of F_n converges to the $d \times d$ identity matrix I_d , the convergence in distribution to each component to the law $N(0, 1)$ implies the convergence in distribution of the whole sequence F_n to the standard centered Gaussian law $N(0, I_d)$.

Recent examples of application of these results are, among others, the study of p -variations of fractional stochastic integrals (Corcuera *et al.* [4]), quadratic functionals of bivariate Gaussian processes (Deheuvels *et al.* [5]), self-intersection local times of fractional Brownian motion (Hu and Nualart [7]), approximation schemes for scalar fractional differential equations (Neuenkirch and Nourdin [13]), high-frequency CLTs for random fields on homogeneous spaces (Marinucci and Peccati [11, 12] and Peccati [24]), needlets analysis on the sphere (Baldi *et al.* [1]), estimation of self-similarity orders (Tudor and Viens [31]) and weighted power variations of iterated Brownian motion (Nourdin and Peccati [16]).

Since the works by Nualart and Peccati [23] and Peccati and Tudor [27], great efforts have been made to find similar statements in the case where the limit is not necessarily Gaussian. In the references [25] and [26] Peccati and Taqqu have proposed sufficient conditions ensuring that a given sequence of multiple Wiener-Itô integrals converges stably towards *mixtures* of Gaussian random variables. On the other hand, Nourdin and Peccati [15] proved an extension of the above equivalence (i) - (iv) for a sequence of random variables $\{F_n, n \geq 1\}$ in a fixed q th Wiener chaos, $q \geq 2$, where the limit law is $2G_{\nu/2} - \nu$, $G_{\nu/2}$ being the Gamma distribution with parameter $\nu/2$. Let us also mention that in [17] Nourdin and Peccati derived bounds for this convergence similar to (1.2).

The purpose of this paper is to study the convergence in distribution of a sequence of random variables of the form $F_n = \delta^q(u_n)$, where u_n are random variables with values in $\mathfrak{H}^{\otimes q}$, and δ^q denotes the iterated divergence operator (also called Skorohod integral), towards a mixture of Gaussian random variables. Our main result, Theorem 3.1, says that under some conditions, if $\langle u_n, D^q F_n \rangle_{\mathfrak{H}^{\otimes q}}$ converges in L^1 to a nonnegative random variable S^2 , then the sequence F_n converges stably to a random variable F with conditional characteristic function $E(e^{i\lambda F} | X) = E\left(e^{-\frac{\lambda^2}{2} S^2}\right)$. Notice that if u_n is deterministic, then F_n belongs to the q th Wiener chaos, and we have a

sequence of the type considered above. In particular, if S^2 is deterministic, we recover the fact that condition (iv) above implies the convergence in distribution to the law $N(0, 1)$.

We develop some particular applications of Theorem 3.1 in the following directions. First we consider a sequence of random variables in a fixed Wiener chaos and we derive new criteria for the convergence to a mixture of Gaussian laws. Second, we show the convergence in law of the sequence $\delta^2(u_n)$, where u_n is a two parameter process of the form

$$u_n = n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} f(B_{k/n}) \mathbf{1}_{(k/n, (k+1)/n]^2},$$

to the random variable $\sigma_H \int_0^1 f(B_s) dW_s$, where B is a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{8}, \frac{1}{2})$, and W is a Brownian motion independent of B . As a consequence of this convergence we establish a new asymptotic result for the behavior of the weighted quadratic variation of the fractional Brownian motion with Hurst parameter $H \in (\frac{1}{8}, \frac{1}{2})$. This result complements and provides a new perspective to the results proved by Nourdin in [14], Nourdin, Nualart and Tudor in [18], and Nourdin and Réveillac in [19].

The paper is organized as follows. In Section 2 we present some preliminary results about Malliavin calculus and fractional Brownian motion. Section 3 contains the proof of the main result. In Section 4, we apply it to sequences of multiple stochastic integrals, while Section 5 focuses on the applications to the quadratic variation of the fractional Brownian motion.

2 Preliminaries

Let \mathfrak{H} be a real separable Hilbert space. For any $q \geq 1$ let $\mathfrak{H}^{\otimes q}$ be the q th tensor product of \mathfrak{H} and denote by $\mathfrak{H}^{\odot q}$ the q th symmetric tensor product.

Suppose that $X = \{X(h), h \in \mathfrak{H}\}$ is an isonormal Gaussian process on \mathfrak{H} , defined on some probability space (Ω, \mathcal{F}, P) . Recall that this means that the covariance of X is given in terms of the scalar product of \mathfrak{H} by (1.1). Suppose that \mathcal{F} is generated by X .

For every $q \geq 1$, let \mathcal{H}_q be the q th Wiener chaos of X , that is, the closed linear subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $\{H_n(X(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where H_n is the n th Hermite polynomial. We denote by \mathcal{H}_0 the space of constant random variables. For any $q \geq 1$,

the mapping $I_q(h^{\otimes q}) = q!H_q(X(h))$ provides a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot q}$ equipped with the modified norm $\sqrt{q!} \|\cdot\|_{\mathfrak{H}^{\otimes q}}$ and the q th Wiener chaos \mathcal{H}_q . For $n = 0$, $\mathcal{H}_0 = \mathbb{R}$, and I_0 is the identity map.

It is well-known (Wiener chaos expansion) that $L^2(\Omega, \mathcal{F}, P)$ can be decomposed into the infinite orthogonal sum of the spaces \mathcal{H}_q . Therefore, any square integrable random variable $F \in L^2(\Omega, \mathcal{F}, P)$ admits the following chaotic expansion

$$F = \sum_{q=0}^{\infty} I_q(f_q), \quad (2.1)$$

where $f_0 = E[F]$, and the $f_q \in \mathfrak{H}^{\odot q}$, $q \geq 1$, are uniquely determined by F . For every $q \geq 0$ we denote by J_q the orthogonal projection operator on the q th Wiener chaos. In particular, if $F \in L^2(\Omega, \mathcal{F}, P)$ is as in (2.1), then $J_q F = I_q(f_q)$ for every $q \geq 0$.

Let $\{e_k, k \geq 1\}$ be a complete orthonormal system in \mathfrak{H} . Given $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, for every $r = 0, \dots, p \wedge q$, the *contraction* of f and g of order r is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined by

$$f \otimes_r g = \sum_{i_1, \dots, i_r=1}^{\infty} \langle f, e_{i_1} \otimes \dots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \dots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}}. \quad (2.2)$$

Notice that $f \otimes_r g$ is not necessarily symmetric and we denote its symmetrization by $f \tilde{\otimes}_r g \in \mathfrak{H}^{\odot(p+q-2r)}$. Moreover, $f \otimes_0 g = f \otimes g$ equals the tensor product of f and g while, for $p = q$, $f \otimes_q g = \langle f, g \rangle_{\mathfrak{H}^{\otimes q}}$.

In the particular case where $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$, where (A, \mathcal{A}) is a measurable space and μ is a σ -finite and non-atomic measure, one has that $\mathfrak{H}^{\odot q} = L^2_s(A^q, \mathcal{A}^{\otimes q}, \mu^{\otimes q})$ is the space of symmetric and square integrable functions on A^q . Moreover, for every $f \in \mathfrak{H}^{\odot q}$, $I_q(f)$ coincides with the multiple Wiener-Itô integral of order q of f with respect to X introduced by Itô in [8]. In this case, (2.2) can be written as

$$(f \otimes_r g)(t_1, \dots, t_{p+q-2r}) = \int_{A^r} f(t_1, \dots, t_{p-r}, s_1, \dots, s_r) \\ \times g(t_{p-r+1}, \dots, t_{p+q-2r}, s_1, \dots, s_r) d\mu(s_1) \dots d\mu(s_r).$$

Let us introduce some basic elements of the Malliavin calculus with respect to the isonormal Gaussian process X . We refer the reader to Nualart [21] for a more detailed presentation of these notions. Let \mathcal{S} be the set of all smooth and cylindrical random variables of the form

$$F = g(X(\phi_1), \dots, X(\phi_n)), \quad (2.3)$$

where $n \geq 1$, $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is a infinitely differentiable function with compact support and $\phi_i \in \mathfrak{H}$. The Malliavin derivative of F with respect to X is the element of $L^2(\Omega, \mathfrak{H})$ defined as

$$DF = \sum_{i=1}^n \frac{\partial g}{\partial x_i} (X(\phi_1), \dots, X(\phi_n)) \phi_i.$$

In particular, $DX(h) = h$ for every $h \in \mathfrak{H}$. By iteration, one can define the m th derivative $D^m F$, which is an element of $L^2(\Omega, \mathfrak{H}^{\odot m})$, for every $m \geq 2$.

For $m \geq 1$ and $p \geq 1$, $\mathbb{D}^{m,p}$ denotes the closure of \mathcal{S} with respect to the norm $\|\cdot\|_{m,p}$, defined by the relation

$$\|F\|_{m,p}^p = E[|F|^p] + \sum_{i=1}^m E\left(\|D^i F\|_{\mathfrak{H}^{\otimes i}}^p\right).$$

The Malliavin derivative D verifies the following *chain rule*. If $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuously differentiable with bounded partial derivatives and if $F = (F_1, \dots, F_n)$ is a vector of elements of $\mathbb{D}^{1,2}$, then $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$D \varphi(F) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i} (F) DF_i.$$

We denote by δ the adjoint of the operator D , also called the *divergence operator*. A random element $u \in L^2(\Omega, \mathfrak{H})$ belongs to the domain of δ , noted $\text{Dom}\delta$, if and only if it verifies

$$|E\langle DF, u \rangle_{\mathfrak{H}}| \leq c_u \|F\|_{L^2},$$

for any $F \in \mathbb{D}^{1,2}$, where c_u is a constant depending only on u . If $u \in \text{Dom}\delta$, then the random variable $\delta(u)$ is defined by the duality relationship (called “integration by parts formula”)

$$E(F\delta(u)) = E\langle DF, u \rangle_{\mathfrak{H}}, \quad (2.4)$$

which holds for every $F \in \mathbb{D}^{1,2}$. The duality formula (2.4) extends to the iterated divergence operator δ^q , and we have

$$E(F\delta^q(u)) = E\left(\langle D^q F, u \rangle_{\mathfrak{H}^{\otimes q}}\right) \quad (2.5)$$

for any element u in the domain of δ^q and any random variable $F \in \mathbb{D}^{q,2}$. Moreover, $\delta^q(h) = I_q(h)$ for any $h \in \mathfrak{H}^{\odot q}$. The divergence operator δ is also called the Skorohod integral because in the case of the Brownian motion it

coincides with the anticipating stochastic integral introduced by Skorohod in [29]. The iterated divergence δ^q is also called the multiple Skorohod integral.

The following property will be extensively used in the paper (see [21, Proposition 1.3.3]). For every $F \in \mathbb{D}^{1,2}$ and every $u \in \text{Dom}\delta$ such that Fu and $F\delta(u) + \langle DF, u \rangle_{\mathfrak{H}}$ are square integrable, one has that $Fu \in \text{Dom}\delta$ and

$$\delta(Fu) = F\delta(u) - \langle DF, u \rangle_{\mathfrak{H}}. \quad (2.6)$$

Iterating formula (2.6) yields

$$F\delta^2(u) = \delta^2(Fu) + 2\delta(\langle DF, u \rangle_{\mathfrak{H}}) + \langle D^2F, u \rangle_{\mathfrak{H}^{\otimes 2}}, \quad (2.7)$$

for all $F \in \mathbb{D}^{2,2}$ and $u \in \text{Dom}\delta^2$, assuming that all the terms are well defined and square integrable.

For any Hilbert space V , we denote by $\mathbb{D}^{k,p}(V)$ the corresponding Sobolev space of V -valued random variables (see [21, page 31]). The operator δ is continuous from $\mathbb{D}^{k,p}(\mathfrak{H})$ to $\mathbb{D}^{k-1,p}$, for any $k \geq 1$ and $p > 1$, and the operator δ^2 is continuous from $\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes 2})$ to $\mathbb{D}^{k-2,p}$ for any $k \geq 2$, that is, we have

$$\|\delta(u)\|_{k-1,p} \leq c_{k,p} \|u\|_{\mathbb{D}^{k,p}(\mathfrak{H})}, \quad (2.8)$$

$$\|\delta^2(v)\|_{k-2,p} \leq c_{k,p} \|v\|_{\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes 2})}. \quad (2.9)$$

These estimates are consequences of Meyer inequalities (see [21, Proposition 1.5.7]). In particular, these estimates imply that $\mathbb{D}^{1,2}(\mathfrak{H}) \subset \text{Dom}\delta$, and $\mathbb{D}^{2,2}(\mathfrak{H}^{\otimes 2}) \subset \text{Dom}\delta^2$.

We will also use the following commutation relationship between the derivative and divergence operators (see [21, Proposition 1.3.2])

$$D_h(\delta(u)) = \langle u, h \rangle_{\mathfrak{H}} + \delta(D_h u), \quad (2.10)$$

for any $h \in \mathfrak{H}$, and $u \in \mathbb{D}^{2,2}(\mathfrak{H})$, where we use the notation $D_h F = \langle DF, h \rangle_{\mathfrak{H}}$. By iteration we get the formulas

$$D_h(\delta^2(u)) = \delta^2(D_h u) + 2\delta(u \otimes_1 h), \quad (2.11)$$

and

$$\begin{aligned} D_{h \otimes g}^2(\delta^2(u)) &= \delta^2(D_{h \otimes g}^2 u) + 2\delta(D_h u \otimes_1 g) \\ &\quad + 2\delta(D_g u \otimes_1 h) + \langle u, h \otimes g \rangle_{\mathfrak{H}^{\otimes 2}}, \end{aligned} \quad (2.12)$$

provided u is a symmetric element of $\mathbb{D}^{4,2}(\mathfrak{H})$.

The operator L is defined on the Wiener chaos expansion as

$$L = \sum_{q=0}^{\infty} -qJ_q,$$

and is called the infinitesimal generator of the Ornstein-Uhlenbeck semigroup. The domain of this operator in L^2 is the set

$$\text{Dom}L = \{F \in L^2(\Omega) : \sum_{q=1}^{\infty} q^2 \|J_q F\|_{L^2(\Omega)}^2 < \infty\} = \mathbb{D}^{2,2}.$$

There is an important relation between the operators D , δ and L (see [21, Proposition 1.4.3]). A random variable F belongs to the domain of L if and only if $F \in \text{Dom}(\delta D)$ (i.e. $F \in \mathbb{D}^{1,2}$ and $DF \in \text{Dom}\delta$), and in this case

$$\delta DF = -LF. \quad (2.13)$$

Note also that a random variable F as in (2.1) is in $\mathbb{D}^{1,2}$ if, and only if,

$$\sum_{q=1}^{\infty} q \|f_q\|_{\mathfrak{H}^{\odot q}}^2 < \infty,$$

and, in this case, $E(\|DF\|_{\mathfrak{H}}^2) = \sum_{q \geq 1} q \|f_q\|_{\mathfrak{H}^{\odot q}}^2$. If $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$ (with μ non-atomic), then the derivative of a random variable F as in (2.1) can be identified with the element of $L^2(A \times \Omega)$ given by

$$D_a F = \sum_{q=1}^{\infty} q I_{q-1}(f_q(\cdot, a)), \quad a \in A. \quad (2.14)$$

Finally, we need the definition of stable convergence (see for instance, the original paper [28], or the book [9] for an exhaustive discussion of stable convergence).

Definition 2.1 *A sequence of random variables $\{F_n, n \geq 1\}$ converges stably to a random variable F , if F is defined on an enlarged probability space (Ω, \mathcal{G}, P) , where $\mathcal{F} \subseteq \mathcal{G}$, and for any continuous and bounded function $f : \mathbb{R} \rightarrow \mathbb{R}$ and any bounded \mathcal{F} -measurable random variable Z , we have*

$$E(f(F_n)Z) \rightarrow E(f(F)Z),$$

as n tends to infinity.

3 Convergence in law of multiple Skorohod integrals

As in the previous section, let $X = \{X(h), h \in \mathfrak{H}\}$ be an isonormal Gaussian process associated with a separable Hilbert space \mathfrak{H} . The next theorem is the main result of this paper.

Theorem 3.1 *For a fixed $q \geq 1$, suppose that F_n is a sequence of random variables of the form $F_n = \delta^q(u_n)$, for some random variables u_n in $\mathbb{D}^{2q, 2q}(\mathfrak{H}^{\odot q})$. Suppose moreover that the sequence F_n is bounded in L^1 , and that:*

- (i) $\langle u_n, (DF_n)^{\otimes k_1} \otimes \dots \otimes (D^{q-1}F_n)^{\otimes k_{q-1}} \otimes h \rangle_{\mathfrak{H}^{\otimes q}}$ converges to zero in L^1 , for all integers $r, k_1, \dots, k_{q-1} \geq 0$ such that $k_1 + 2k_2 + \dots + (q-1)k_{q-1} + r = q$, and all $h \in \mathfrak{H}^{\otimes r}$;
- (ii) $\langle u_n, D^q F_n \rangle_{\mathfrak{H}^{\otimes q}}$ converges in L^1 to a nonnegative random variable S^2 .

Then, F_n converges stably to a random variable with conditional Gaussian law $N(0, S^2)$ given X .

Remark 3.2 *When $q = 1$, condition (i) of the theorem is that $\langle u_n, h \rangle_{\mathfrak{H}}$ converges to zero in L^1 , for each $h \in \mathfrak{H}$. When $q = 2$, condition (i) means that $\langle u_n, h \otimes g \rangle_{\mathfrak{H}^{\otimes 2}}$, $\langle u_n, DF_n \otimes h \rangle_{\mathfrak{H}^{\otimes 2}}$ and $\langle u_n, DF_n \otimes DF_n \rangle_{\mathfrak{H}^{\otimes 2}}$ converge to zero in L^1 , for each $h, g \in \mathfrak{H}$. And so on.*

Proof of Theorem 3.1. Taking into account Definition 2.1, it suffices to show that for any $h_1, \dots, h_m \in \mathfrak{H}$, the sequence $\xi_n := (F_n, X(h_1), \dots, X(h_m))$ converges in distribution to a vector

$$(F_\infty, X(h_1), \dots, X(h_m)),$$

where F_∞ satisfies

$$E(e^{i\lambda F_\infty} | X(h_1), \dots, X(h_m)) = e^{-\frac{\lambda^2}{2} S^2} \quad (3.1)$$

for any $\lambda \in \mathbb{R}$. Since the sequence F_n is bounded in L^1 , the sequence ξ_n is tight. Assume that $(F_\infty, X(h_1), \dots, X(h_m))$ denotes the limit in law of a certain subsequence of ξ_n , denoted again by ξ_n .

Let $Y = \phi(X(h_1), \dots, X(h_m))$, with $\phi \in C_b^\infty(\mathbb{R}^m)$ (ϕ is infinitely differentiable, bounded, with bounded partial derivatives of all orders), and

consider $\phi_n(\lambda) = E(e^{i\lambda F_n} Y)$ for $\lambda \in \mathbb{R}$. The convergence in law of ξ_n , and the fact that F_n is bounded in L^1 imply that

$$\lim_{n \rightarrow \infty} \phi'_n(\lambda) = \lim_{n \rightarrow \infty} iE(F_n e^{i\lambda F_n} Y) = iE(F_\infty e^{i\lambda F_\infty} Y). \quad (3.2)$$

On the other hand, by the duality relationship (2.5) between the iterated derivative and divergence operators we obtain

$$\begin{aligned} \phi'_n(\lambda) &= iE(F_n e^{i\lambda F_n} Y) = iE\left(\delta^q(u_n) e^{i\lambda F_n} Y\right) = iE\left(\left\langle u_n, D^q \left(e^{i\lambda F_n} Y\right) \right\rangle_{\mathfrak{H}^{\otimes q}}\right) \\ &= i \sum_{a=0}^q \binom{q}{a} E\left(\left\langle u_n, D^a \left(e^{i\lambda F_n}\right) \tilde{\otimes} D^{q-a} Y \right\rangle_{\mathfrak{H}^{\otimes q}}\right) \\ &= i \sum_{a=0}^q \binom{q}{a} \sum \frac{a!}{k_1! \dots k_a!} (i\lambda)^{k_1 + \dots + k_a} \\ &\quad \times E\left(e^{i\lambda F_n} \left\langle u_n, (DF_n)^{\otimes k_1} \tilde{\otimes} \dots \tilde{\otimes} (D^a F_n)^{\otimes k_a} \tilde{\otimes} D^{q-a} Y \right\rangle_{\mathfrak{H}^{\otimes q}}\right) \\ &= i \sum_{a=0}^q \binom{q}{a} \sum \frac{a!}{k_1! \dots k_a!} (i\lambda)^{k_1 + \dots + k_a} \\ &\quad \times E\left(e^{i\lambda F_n} \left\langle u_n, (DF_n)^{\otimes k_1} \otimes \dots \otimes (D^a F_n)^{\otimes k_a} \otimes D^{q-a} Y \right\rangle_{\mathfrak{H}^{\otimes q}}\right), \end{aligned}$$

where the second sum in the last equality runs over all sequences of integers (k_1, \dots, k_a) such that $k_1 + 2k_2 + \dots + ak_a = a$ by the Faà di Bruno's formula. By condition (i), this yields that

$$\phi'_n(\lambda) = -\lambda E\left(e^{i\lambda F_n} \langle u_n, D^q F_n \rangle_{\mathfrak{H}^{\otimes q}} Y\right) + R_n,$$

with R_n converging to zero as $n \rightarrow \infty$. Using condition (ii) and (3.2), we obtain that

$$iE(F_\infty e^{i\lambda F_\infty} Y) = -\lambda E\left(e^{i\lambda F_\infty} S^2 Y\right).$$

Since S^2 is defined through condition (ii), it is in particular measurable with respect to X . Thus, the following linear differential equation verified by the conditional characteristic function of F_∞ holds:

$$\frac{\partial}{\partial \lambda} E(e^{i\lambda F_\infty} | X(h_1), \dots, X(h_m)) = -\lambda S^2 E(e^{i\lambda F_\infty} | X(h_1), \dots, X(h_m)).$$

By solving it we obtain (3.1), which yields the desired conclusion. \blacksquare

The next corollary provides stronger but easier conditions for the convergence in law.

Corollary 3.3 For a fixed $q \geq 1$, suppose that F_n is a sequence of random variables of the form $F_n = \delta^q(u_n)$, for some random variables u_n in $\mathbb{D}^{2q,2q}(\mathfrak{H}^{\odot q})$. Suppose moreover that the sequence F_n is bounded in $\mathbb{D}^{q,p}$, for all $p \geq 2$, and that:

- (i') $\langle u_n, h \rangle_{\mathfrak{H}^{\otimes q}}$ converges to zero in L^1 for all $h \in \mathfrak{H}^{\otimes q}$; and $u_n \otimes_l D^l F_n$ converges to zero in $L^2(\Omega; \mathfrak{H}^{\otimes(q-l)})$ for all $l = 1, \dots, q-1$;
- (ii) $\langle u_n, D^q F_n \rangle_{\mathfrak{H}^{\otimes q}}$ converges in L^1 to a nonnegative random variable S^2 .

Then, F_n converges stably to a random variable with conditional Gaussian law $N(0, S^2)$ given X .

Proof. It suffices to show that condition (i') implies condition (i) in Theorem 3.1. When $k_a \neq 0$ for $1 \leq a \leq q-1$, we have, for all $h \in \mathfrak{H}^{\otimes r}$ (with $r = q - k_1 - 2k_2 - \dots - ak_a$)

$$\begin{aligned}
& \left| \left\langle u_n, (DF_n)^{\otimes k_1} \otimes \dots \otimes (D^a F_n)^{\otimes k_a} \otimes h \right\rangle_{\mathfrak{H}^{\otimes q}} \right| \\
&= \left| \left\langle u_n \otimes_a D^a F_n, \right. \right. \\
&\quad \left. \left. (DF_n)^{\otimes k_1} \otimes \dots \otimes (D^{a-1} F_n)^{\otimes k_{a-1}} \otimes (D^a F_n)^{\otimes(k_a-1)} \otimes h \right\rangle_{\mathfrak{H}^{\otimes(q-a)}} \right| \\
&\leq \|u_n \otimes_a D^a F_n\|_{\mathfrak{H}^{\otimes(q-a)}} \\
&\quad \times \left\| (DF_n)^{\otimes k_1} \otimes \dots \otimes (D^{a-1} F_n)^{\otimes k_{a-1}} \otimes (D^a F_n)^{\otimes(k_a-1)} \otimes h \right\|_{\mathfrak{H}^{\otimes(q-a)}}.
\end{aligned}$$

The second factor is bounded in L^2 , and the first factor converges to zero in L^2 , for all $a = 1, \dots, q-1$. In the case $a = 0$ we have that $\langle u_n, h \rangle_{\mathfrak{H}^{\otimes q}}$ converges to zero in L^1 , for all $h \in \mathfrak{H}^{\otimes q}$, by condition (i'). This completes the proof. ■

4 Multiple stochastic integrals

Suppose that \mathfrak{H} is a Hilbert space $L^2(A, \mathcal{A}, \mu)$, where μ is a σ -finite measure without atoms.

Fix an integer $m \geq 2$, and consider the sequence of multiple stochastic integrals $\{F_n = I_m(g_n), n \geq 1\}$ with $g_n \in \mathfrak{H}^{\odot m}$. From (2.14) we have $DF_n = mI_{m-1}(g_n)$. Using (2.13) we can write $F_n = -\frac{1}{m}LF_n = \frac{1}{m}\delta(DF_n)$. Applying Theorem 3.1 with $q = 1$ and $u_n = \frac{1}{m}DF_n$ yields:

Proposition 4.1 *For a fixed $m \geq 2$, suppose that F_n is a sequence of random variables of the form $F_n = I_m(g_n)$, with $g_n \in \mathfrak{H}^{\odot m}$. Suppose moreover that the sequence g_n is bounded in $\mathfrak{H}^{\odot m}$, and that:*

- (a) $\langle g_n \otimes_{m-1} g_n, h^{\otimes 2} \rangle_{\mathfrak{H}^{\otimes 2}}$ converges to zero for all $h \in \mathfrak{H}$;
- (b) $\frac{1}{m} \|DF_n\|_{\mathfrak{H}}^2$ converges in L^1 to a nonnegative random variable S^2 .

Then, F_n converges stably to a random variable with conditional Gaussian law $N(0, S^2)$ given X .

Proof. It suffices to apply Theorem 3.1 to $u_n = I_{m-1}(g_n)$ and $q = 1$. We have

$$\begin{aligned} E \left(\langle u_n, h \rangle_{\mathfrak{H}}^2 \right) &= E \left(\langle I_{m-1}(g_n), h \rangle_{\mathfrak{H}}^2 \right) = E \left(I_{m-1}(g_n \otimes_1 h)^2 \right) \\ &= (m-1)! \|g_n \otimes_1 h\|_{\mathfrak{H}^{\otimes(m-1)}}^2 \\ &= (m-1)! \langle g_n \otimes_{m-1} g_n, h^{\otimes 2} \rangle_{\mathfrak{H}^{\otimes 2}} \rightarrow 0, \end{aligned}$$

which implies condition (i) in Theorem 3.1, see Remark 3.2. Condition (ii) in Theorem 3.1 follows from (b) and the equality $\langle u_n, DF_n \rangle_{\mathfrak{H}} = \frac{1}{m} \|DF_n\|_{\mathfrak{H}}^2$. \blacksquare

If $S^2 = 1$, then condition (b) coincides with condition (iv) in the introduction. In this case, Nualart and Peccati Theorem combined with Lemma 6 in [22] tells us that if the variances converge to one condition (a) is automatically satisfied.

On the other hand, we can also apply Theorem 3.1 with $u_n = g_n$. In this way, applying Corollary 3.3, we obtain that the following conditions imply that F_n converges to a normal random variable $N(0, 1)$ independent of X :

- (α) g_n is bounded in $\mathfrak{H}^{\odot q}$ and it converges weakly to zero,
- (β) $\|g_n \otimes_l g_n\|_{\mathfrak{H}^{\otimes 2(q-l)}}$ converges to zero for all $l = 1, \dots, q-1$;
- (γ) $q! \|g_n\|_{\mathfrak{H}^{\otimes q}}^2$ converges to 1.

In fact, notice first that if g_n is bounded in $\mathfrak{H}^{\odot q}$, F_n is bounded in all the Sobolev spaces $\mathbb{D}^{q,p}$, $p \geq 2$. Then, condition (ii) in Corollary 3.3 follows from (γ) and the equality $D^q(I_q(g_n)) = q!g_n$, and, finally, condition (i') in

Corollary 3.3 follows from (ii) and

$$\begin{aligned}
E \left[\left\| g_n \otimes_l D^l F_n \right\|_{\mathfrak{H}^{\otimes(q-l)}}^2 \right] &= \frac{q!^2}{(q-l)!^2} E \left[\|g_n \otimes_l I_{q-l}(g_n)\|_{\mathfrak{H}^{\otimes(q-l)}}^2 \right] \\
&= \frac{q!^2}{(q-l)!^2} E \left[\|I_{q-l}(g_n \otimes_l g_n)\|_{\mathfrak{H}^{\otimes(q-l)}}^2 \right] \\
&= \frac{q!^2}{(q-l)!} \|g_n \otimes_l g_n\|_{\mathfrak{H}^{\otimes 2(q-l)}}^2.
\end{aligned}$$

In this way we recover the fact that condition (iii) in the introduction implies the normal convergence.

5 Weighted quadratic variation of the fractional Brownian motion

The fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$ is a centered Gaussian process $B = \{B_t, t \geq 0\}$ with the covariance function

$$E(B_s B_t) = R_H(s, t) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}). \quad (5.1)$$

From (5.1) it follows that $E(|B_t - B_s|^2) = (t - s)^{2H}$, for all $0 \leq s < t$. On the other hand, for each $a > 0$ the process $\{a^{-H} B_{at}, t \geq 0\}$ is also a fBm with Hurst parameter H (*self-similarity* property).

As a consequence, the sequence $\{B_j - B_{j-1}, j = 1, 2, \dots\}$ is stationary, Gaussian and ergodic, with correlation given by

$$\rho_H(n) = \frac{1}{2} [(n+1)^{2H} - 2n^{2H} + (n-1)^{2H}], \quad (5.2)$$

which behaves as $H(2H-1)n^{2H-2}$ as n tends to infinity.

Set $\Delta B_k = B_{(k+1)/n} - B_{k/n}$, where $k = 0, 1, \dots, n$, and $n \geq 1$. The Ergodic Theorem and the self-similarity property imply that, for any $p > 0$, the sequence $n^{pH-1} \sum_{k=0}^{n-1} |\Delta B_k|^p$ converges almost surely and in L^1 to $c_p = E(|B_1|^p)$. In this situation, it is well-known (see, for instance [2], [4]), that a central limit theorem holds, and the sequence

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (n^{pH} |\Delta B_k|^p - c_p) \quad (5.3)$$

converges in law to the law $N(0, \sigma_{p,H}^2)$, for some constant $\sigma_{p,H} > 0$, provided $H \in (0, \frac{3}{4})$.

There has been some recent interest in the study of the asymptotic behavior of the weighted p -variation of the fBm (see [18] and the references therein). Consider the case of the quadratic variation, that is, $p = 2$. In this case, when we introduce a weight of the form $f(B_{k/n})$ some unexpected results happen, and the critical value $H = \frac{1}{4}$ plays an important role. We are interested in the asymptotic behavior of the sequence of random variables

$$G_n = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(B_{k/n}) \left(n^{2H} (\Delta B_k)^2 - 1 \right). \quad (5.4)$$

Assuming that the function f satisfies some regularity and growth conditions, for any $H \in (\frac{1}{4}, \frac{3}{4})$ we have

$$G_n \xrightarrow{\text{Stably}} \sigma_H \int_0^1 f(B_s) dW_s, \quad (5.5)$$

where W is a Brownian motion independent of B , and

$$\sigma_H^2 = \frac{1}{2} \sum_{r \in \mathbb{Z}} \rho_H(r)^2 < \infty, \quad (5.6)$$

whereas for $H \in (0, \frac{1}{4})$

$$n^{2H} G_n \xrightarrow{L^2} \frac{1}{4} \int_0^1 f''(B_s) ds. \quad (5.7)$$

The convergence (5.5) follows easily from (5.3) in the case $H \in (\frac{1}{2}, \frac{3}{4})$ (see [4]), and in the case $H \in (\frac{1}{2}, \frac{3}{4})$ this convergence is a particular case of the general asymptotic results for the p -variation obtained by Nourdin, Nualart and Tudor in [18]. The method of proof in these papers consists on considering two different levels of discretization $n \leq m$, and use the convergence to a normal law of (5.3) (case $f = 1$).

The convergence (5.7) was obtained by Nourdin [14] using the techniques of Malliavin calculus. The critical case $H = \frac{1}{4}$ has been recently solved by Nourdin and Réveillac [19]: the convergence is in law and the limit is equal to a linear combination of the limits in (5.5) and (5.7). The proof of this result follows an approach similar to the proof of Theorem 3.1.

We are going to show that Theorem 3.1 provides a proof of the following new result for $H \in (\frac{1}{8}, \frac{1}{2})$

$$G_n - \frac{1}{4} n^{\frac{1}{2}-2H} \int_0^1 f''(B_s) ds \xrightarrow{\text{Stably}} \sigma_H \int_0^1 f(B_s) dW_s. \quad (5.8)$$

Notice that (5.8) provides the asymptotic behavior of the fluctuations associated with the convergence (5.7) in the case $H \in (\frac{1}{8}, \frac{1}{4})$. In this way we get also a proof of (5.5) in the case $H \in (\frac{1}{4}, \frac{1}{2})$ without using the two steps of discretization as in [18], and a new proof of the convergence in the critical case $H = \frac{1}{4}$, obtained by Nourdin and Réveillac in [19].

The main idea of the proof is a decomposition of the random variable G_n into the sum of three terms, using Equation (2.7). The first one is a double divergence and it converges in law for any $H \in (\frac{1}{8}, \frac{1}{2})$ by Theorem 5.2 below, the second one is a divergence which converges to zero in L^2 , and the third one behaves as $-\frac{1}{4}n^{\frac{1}{2}-2H} \int_0^1 f''(B_s) ds$.

Before stating and proving our results, we need some preliminaries on the Malliavin calculus associated with the fBm and some technical results (see [21, Chapter 5]).

In the following we assume $H \in (0, \frac{1}{2})$. We denote by \mathcal{E} the set of step functions on $[0, T]$. Let \mathfrak{H} be the Hilbert space defined as the closure of \mathcal{E} with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathfrak{H}} = R_H(t, s).$$

The mapping $\mathbf{1}_{[0,t]} \rightarrow B_t$ can be extended to a linear isometry between the Hilbert space \mathfrak{H} and the Gaussian space spanned by B . We denote this isometry by $\phi \rightarrow B(\phi)$. In this way $\{B(\phi), \phi \in \mathfrak{H}\}$ is an isonormal Gaussian space.

We know that the space \mathfrak{H} coincides with $I_{0+}^{H-\frac{1}{2}}(L^2[0, T])$, where

$$I_{0+}^{H-\frac{1}{2}} f(x) = \frac{1}{\Gamma(H-\frac{1}{2})} \int_0^x (x-y)^{H-\frac{3}{2}} f(y) dy$$

is the left-sided Liouville fractional integral of order $H - \frac{1}{2}$ (see [6]).

We make use of the notation

$$\begin{aligned} \varepsilon_t &= \mathbf{1}_{[0,t]}, \\ \partial_{k/n} &= \varepsilon_{(k+1)/n} - \varepsilon_{k/n} = \mathbf{1}_{((k+1)/n - k/n)}, \end{aligned}$$

for $t \in [0, 1]$, $n \geq 1$, and $k = 0, \dots, n-1$. Notice that $n^{2H}(\Delta B_k)^2 - 1 = n^{2H} I_2(\partial_{k/n}^{\otimes 2})$.

We need the following technical lemma.

Lemma 5.1 *Suppose that $H < \frac{1}{2}$. Let $n \geq 1$ and $k = 0, \dots, n-1$. We have*

$$(a) \quad |E(B_r(B_t - B_s))| \leq (t-s)^{2H} \text{ for any } r \in [0, 1] \text{ and } 0 \leq s < t \leq 1.$$

(b) $\left| \langle \varepsilon_t, \partial_{k/n} \rangle_{\mathfrak{H}} \right| \leq n^{-2H}$ for any $t \in [0, 1]$.

(c) $\sup_{t \in [0, 1]} \sum_{k=0}^{n-1} \left| \langle \varepsilon_t, \partial_{k/n} \rangle_{\mathfrak{H}} \right| = O(1)$, as n tends to infinity.

(d) $\left| \langle \varepsilon_{k/n}, \partial_{k/n} \rangle_{\mathfrak{H}}^2 - \frac{1}{4n^{4H}} \right| \leq \frac{3}{4n^{4H}} [(k+1)^{2H} - k^{2H}]$, which implies

$$\sum_{k=0}^{n-1} \left| \langle \varepsilon_{k/n}, \partial_{k/n} \rangle_{\mathfrak{H}}^2 - \frac{1}{4n^{4H}} \right| \leq Cn^{-2H}, \quad (5.9)$$

for some constant C .

(e) We have $\langle \partial_{j/n}, \partial_{k/n} \rangle_{\mathfrak{H}} = n^{-2H} \rho_H(k-j)$. Consequently,

$$\sum_{k,j=0}^{n-1} \left| \langle \partial_{j/n}, \partial_{k/n} \rangle_{\mathfrak{H}} \right| = O(n^{1-2H}), \quad (5.10)$$

and

$$\sum_{k,j=0}^{n-1} \langle \partial_{j/n}, \partial_{k/n} \rangle_{\mathfrak{H}}^2 = O(n^{1-4H}). \quad (5.11)$$

Proof. We have

$$\begin{aligned} E(B_r(B_t - B_s)) &= \frac{1}{2} (r^{2H} + t^{2H} - |t-r|^{2H}) - \frac{1}{2} (r^{2H} + s^{2H} - |s-r|^{2H}) \\ &= \frac{1}{2} (t^{2H} - s^{2H}) + \frac{1}{2} (|s-r|^{2H} - |t-r|^{2H}). \end{aligned}$$

Using the inequality $|b^{2H} - a^{2H}| \leq |b-a|^{2H}$ for any $a, b \in [0, 1]$, because $H < \frac{1}{2}$, we deduce (a). Property (b) is an immediate consequence of (a). To show property (c) we use

$$\langle \varepsilon_t, \partial_{k/n} \rangle_{\mathfrak{H}} = \frac{1}{2n^{2H}} [(k+1)^{2H} - k^{2H} - |k+1-nt|^{2H} + |k-nt|^{2H}].$$

Property (d) follows from

$$\langle \varepsilon_{k/n}, \partial_{k/n} \rangle_{\mathfrak{H}} = \frac{1}{2n^{2H}} [(k+1)^{2H} - k^{2H} - 1],$$

and

$$\left| \langle \varepsilon_{k/n}, \partial_{k/n} \rangle_{\mathfrak{H}}^2 - \frac{1}{4n^{4H}} \right| = \frac{1}{4n^{4H}} [(k+1)^{2H} - k^{2H}] |(k+1)^{2H} - k^{2H} - 2|.$$

Finally, property (e) follows from

$$\sum_{k,j=0}^{n-1} \left| \langle \partial_{j/n}, \partial_{k/n} \rangle_{\mathfrak{H}} \right| \leq n^{-2H} \sum_{k,j=0}^{n-1} |\rho_H(j-k)| \leq n^{1-2H} \sum_{r \in \mathbb{Z}} |\rho_H(r)|.$$

■

In the sequel, we will also make use of the following hypotheses on $f : \mathbb{R} \rightarrow \mathbb{R}$:

(H) f belongs to \mathcal{C}^4 and

$$\sup_{t \in [0,1]} E(|f^{(i)}(B_t)|^p) < \infty, \quad (5.12)$$

for any $p \geq 2$ and $i = 0, 1, 2, 3, 4$.

Notice that a sufficient condition for (5.12) to hold is that f satisfies an exponential growth condition of the form $|f^{(4)}(x)| \leq ke^{cx^{|p|}}$ for some constants $c, k > 0$ and $0 < p < 2$.

Our main result is the following theorem on the convergence of double Skorohod integrals.

Theorem 5.2 *Suppose $H \in (\frac{1}{8}, \frac{1}{2})$, and let f be a function satisfying Hypothesis (H). Consider the sequence of two-parameter step processes defined by*

$$u_n = n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} f(B_{k/n}) \partial_{k/n}^{\otimes 2}. \quad (5.13)$$

Then $\delta^2(u_n)$ converges stably to $\sigma_H \int_0^1 f(B_s) dW_s$, where W is a Brownian motion independent of B , and σ_H has been defined in (5.6).

Proof. We are going to show that the sequence $F_n = \delta^2(u_n)$ satisfies the conditions of Theorem 3.1 with $q = 2$. We make use of the notation

$$\alpha_{k,j} = \langle \varepsilon_{k/n}, \partial_{j/n} \rangle_{\mathfrak{H}}, \quad \beta_{k,j} = \langle \partial_{k/n}, \partial_{j/n} \rangle_{\mathfrak{H}},$$

for $k, j = 0, \dots, n-1$ and $n \geq 1$. Also C will denote a generic constant.

Step 1. Let us show first that F_n is bounded in L^2 . Taking into account the continuity of the divergence operator (see (2.9)), it suffices to show that u_n is bounded in $\mathbb{D}^{2,2}(\mathfrak{H}^{\otimes 2})$. Using (H), the estimate (5.11) we obtain

$$E \left(\|u_n\|_{\mathfrak{H}^{\otimes 2}}^2 \right) = n^{4H-1} \sum_{k,j=0}^{n-1} E \left(f(B_{k/n}) f(B_{j/n}) \right) \beta_{k,j}^2 \leq C,$$

and from

$$\begin{aligned} Du_n &= n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} f'(B_{k/n}) \varepsilon_{k/n} \otimes \partial_{k/n}^{\otimes 2}, \\ D^2u_n &= n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} f''(B_{k/n}) \varepsilon_{k/n}^{\otimes 2} \otimes \partial_{k/n}^{\otimes 2}, \end{aligned}$$

we obtain in the same way

$$\begin{aligned} E\left(\|Du_n\|_{\mathfrak{H}^{\otimes 3}}^2\right) &= n^{4H-1} \sum_{k,j=0}^{n-1} E\left(f'(B_{k/n})f'(B_{j/n})\right) \langle \varepsilon_{k/n}, \varepsilon_{j/n} \rangle \beta_{k,j}^2 \leq C, \\ E\left(\|D^2u_n\|_{\mathfrak{H}^{\otimes 4}}^2\right) &= n^{4H-1} \sum_{k,j=0}^{n-1} E\left(f''(B_{k/n})f''(B_{j/n})\right) \langle \varepsilon_{k/n}, \varepsilon_{j/n} \rangle^2 \beta_{k,j}^2 \leq C. \end{aligned}$$

Notice that by the same method we can show that the sequence u_n is bounded in $\mathbb{D}^{3,2}(\mathfrak{H}^{\otimes 2})$.

Step 2. Let us show condition (i) of Theorem 3.1. In view of Remark 3.2, it suffices to show that for any $s, t \in [0, 1]$ the random variables $\langle u_n, \varepsilon_t \otimes \varepsilon_s \rangle_{\mathfrak{H}^{\otimes 2}}$, $\langle u_n, \varepsilon_t \otimes DF_n \rangle_{\mathfrak{H}^{\otimes 2}}$ and $\langle u_n, DF_n \otimes DF_n \rangle_{\mathfrak{H}^{\otimes 2}}$ converges to zero in L^1 as n tends to infinity. From conditions (b) and (c) in Lemma 5.1 we obtain

$$\begin{aligned} E\left(\left|\langle u_n, \varepsilon_t \otimes \varepsilon_s \rangle_{\mathfrak{H}^{\otimes 2}}\right|\right) &\leq n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} E\left(|f(B_{k/n})|\right) \left|\langle \partial_{k/n}, \varepsilon_t \rangle_{\mathfrak{H}} \langle \partial_{k/n}, \varepsilon_s \rangle_{\mathfrak{H}}\right| \\ &\leq C n^{-\frac{1}{2}}, \end{aligned} \tag{5.14}$$

which converges to zero as n tends to infinity. Consider now the term

$$\langle u_n, \varepsilon_t \otimes DF_n \rangle_{\mathfrak{H}^{\otimes 2}} = n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} f(B_{k/n}) \langle \partial_{k/n}, DF_n \rangle_{\mathfrak{H}} \langle \partial_{k/n}, \varepsilon_t \rangle_{\mathfrak{H}}.$$

We have, using again condition (c) in Lemma 5.1

$$\begin{aligned} E\left(\left|\langle u_n, \varepsilon_t \otimes DF_n \rangle_{\mathfrak{H}^{\otimes 2}}\right|\right) &\leq n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} E\left(|f(B_{k/n}) \langle \partial_{k/n}, DF_n \rangle_{\mathfrak{H}}|\right) \left|\langle \partial_{k/n}, \varepsilon_t \rangle_{\mathfrak{H}}\right| \\ &\leq C n^{2H-\frac{1}{2}} \sup_{0 \leq k \leq n-1} \left[E\left(\langle \partial_{k/n}, DF_n \rangle_{\mathfrak{H}}^2\right)\right]^{\frac{1}{2}} \\ &\leq C n^{H-\frac{1}{2}} \left[E\left(\|DF_n\|_{\mathfrak{H}}^2\right)\right]^{\frac{1}{2}}. \end{aligned} \tag{5.15}$$

Then, applying the continuity of the operator δ^2 from $\mathbb{D}^{3,2}(\mathfrak{H}^{\otimes 2})$ into $\mathbb{D}^{1,2}$ (see (2.9)) yields

$$\left[E \left(\|DF_n\|_{\mathfrak{H}}^2 \right) \right]^{\frac{1}{2}} \leq \|\delta^2(u_n)\|_{\mathbb{D}^{1,2}} \leq C \|u_n\|_{\mathbb{D}^{3,2}(\mathfrak{H}^{\otimes 2})} \leq C'. \quad (5.16)$$

Therefore, (5.16) implies that (5.15) converges to zero as n tends to infinity because $H < \frac{1}{2}$.

Consider finally the term

$$E \left(|\langle u_n, DF_n \otimes DF_n \rangle_{\mathfrak{H}^{\otimes 2}}| \right) \leq n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} E \left(|f(B_{k/n})| \langle \partial_{k/n}, DF_n \rangle_{\mathfrak{H}}^2 \right). \quad (5.17)$$

Using the inequality $|x| \leq 1 + x^2$ we can estimate right-side of (5.17) by

$$M_n := n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} E \left(h(B_{k/n}) \langle \partial_{k/n}, DF_n \rangle_{\mathfrak{H}}^2 \right),$$

where $h(x) = 1 + f(x)^2$.

Using the commutation relationship (2.11) we obtain

$$\left\langle \partial_{k/n}, D \left(\delta^2(f(B_{j/n})\partial_{j/n}^{\otimes 2}) \right) \right\rangle_{\mathfrak{H}} = 2\delta(f(B_{j/n})\partial_{j/n})\beta_{j,k} + \delta^2(f'(B_{j/n})\partial_{j/n}^{\otimes 2})\alpha_{j,k}, \quad (5.18)$$

which implies

$$\langle \partial_{k/n}, DF_n \rangle_{\mathfrak{H}} = 2n^{2H-\frac{1}{2}} \sum_{j=0}^{n-1} \delta(f(B_{j/n})\partial_{j/n})\beta_{j,k} + n^{2H-\frac{1}{2}} \sum_{j=0}^{n-1} \delta^2(f'(B_{j/n})\partial_{j/n}^{\otimes 2})\alpha_{j,k}.$$

As a consequence

$$\begin{aligned} & E \left(h(B_{k/n}) \langle \partial_{k/n}, DF_n \rangle_{\mathfrak{H}}^2 \right) \\ & \leq 8n^{4H-1} \sum_{j,l=0}^{n-1} \beta_{j,k}\beta_{l,k} E \left(h(B_{k/n}) \delta(f(B_{j/n})\partial_{j/n}) \delta(f(B_{l/n})\partial_{l/n}) \right) \\ & \quad + 2n^{4H-1} \sum_{j,l=0}^{n-1} \alpha_{j,k}\alpha_{l,k} E \left(h(B_{k/n}) \delta^2(f'(B_{j/n})\partial_{j/n}^{\otimes 2}) \delta^2(f'(B_{l/n})\partial_{l/n}^{\otimes 2}) \right) \\ & = A_{k,n} + B_{k,n}. \end{aligned}$$

This leads to the decomposition

$$M_n \leq n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} A_{k,n} + n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} B_{k,n}.$$

For the term $A_{k,n}$, using $\sum_{j,l=0}^{n-1} |\beta_{jk}\beta_{lk}| \leq n^{-4H} (\sum_{r \in \mathbb{Z}} |\rho_H(r)|)^2$ and $\sum_{j=0}^{n-1} |\beta_{jk}| \leq n^{-2H} \sum_{r \in \mathbb{Z}} |\rho_H(r)|$ (which follows from Lemma 5.1 (e)), we obtain

$$\begin{aligned} |A_{k,n}| &\leq Cn^{-1} \sup_{0 \leq k,j,l \leq n-1} |E(h(B_{k/n})\delta(f(B_{j/n})\partial_{j/n})\delta(f(B_{l/n})\partial_{l/n}))| \\ &\leq Cn^{-1} \sup_{0 \leq j \leq n-1} \left[E\left(|\delta(f(B_{j/n})\partial_{j/n})|^4\right) \right]^{\frac{1}{2}}. \end{aligned}$$

Using $\delta(f(B_{j/n})\partial_{j/n}) = f(B_{j/n})\Delta B_j - f'(B_{j/n})\alpha_{j,j}$ we get the estimate

$$|A_{k,n}| \leq Cn^{-2H-1}. \quad (5.19)$$

Therefore,

$$n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} |A_{k,n}| \leq Cn^{-\frac{1}{2}}$$

converges to zero as n tends to infinity.

The term $B_{k,n}$ requires a finer analysis. Set

$$J_{k,j,l} = E\left(h(B_{k/n})\delta^2(f'(B_{j/n})\partial_{j/n}^{\otimes 2})\delta^2(f'(B_{l/n})\partial_{l/n}^{\otimes 2})\right).$$

Then,

$$|B_{k,n}| \leq 2n^{4H-1} \sum_{j,l=0}^{n-1} |\alpha_{j,k}\alpha_{l,k}| |J_{k,j,l}|. \quad (5.20)$$

Applying the integration by parts formula (2.5) yields

$$\begin{aligned} J_{k,j,l} &= E\left(f'(B_{j/n}) \left\langle D^2 \left[h(B_{k/n})\delta^2(f'(B_{l/n})\partial_{l/n}^{\otimes 2}) \right], \partial_{j/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}}\right) \\ &= E\left(f'(B_{j/n})h''(B_{k/n})\delta^2(f'(B_{l/n})\partial_{l/n}^{\otimes 2})\alpha_{k,j}^2\right) \\ &\quad + 2E\left(f'(B_{j/n})h'(B_{k/n}) \left\langle D \left(\delta^2(f'(B_{l/n})\partial_{l/n}^{\otimes 2}) \right), \partial_{j/n} \right\rangle_{\mathfrak{H}} \alpha_{k,j}\right) \\ &\quad + E\left(f'(B_{j/n})h(B_{k/n}) \left\langle D^2 \left[\delta^2(f'(B_{l/n})\partial_{l/n}^{\otimes 2}) \right], \partial_{j/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}}\right) \\ &= J_{k,j,l}^{(1)} + J_{k,j,l}^{(2)} + J_{k,j,l}^{(3)}. \end{aligned} \quad (5.21)$$

For the first term we have, using again (2.5),

$$\begin{aligned} J_{k,j,l}^{(1)} &= E\left(f'(B_{l/n})f^{(3)}(B_{j/n})h''(B_{k/n})\alpha_{j,l}^2\alpha_{k,j}^2\right) \\ &\quad + 2E\left(f'(B_{j/n})f''(B_{l/n})h^{(3)}(B_{k/n})\alpha_{j,l}\alpha_{k,j}^2\alpha_{k,l}\right) \\ &\quad + E\left(f'(B_{j/n})f'(B_{l/n})h^{(4)}(B_{k/n})\alpha_{k,l}^2\alpha_{k,j}^2\right). \end{aligned}$$

Hence, using Lemma 5.1 (b), we get

$$\left| J_{k,j,l}^{(1)} \right| \leq C n^{-8H}. \quad (5.22)$$

In order to handle the term $J^{(2)}$ using (5.18) we can write

$$\begin{aligned} J_{k,j,l}^{(2)} &= 4E \left(f'(B_{j/n}) h'(B_{k/n}) \delta(f'(B_{l/n}) \partial_{l/n}) \beta_{l,j} \alpha_{k,j} \right) \\ &\quad + 2E \left(f'(B_{j/n}) h'(B_{k/n}) \delta^2(f''(B_{l/n}) \partial_{l/n}^{\otimes 2}) \alpha_{l,j} \alpha_{k,j} \right) \end{aligned}$$

Applying the integration by parts formulas (2.4) and (2.5) we obtain

$$\begin{aligned} J_{k,j,l}^{(2)} &= 4E \left(f'(B_{l/n}) f''(B_{j/n}) h'(B_{k/n}) \alpha_{j,l} \beta_{l,j} \alpha_{k,j} \right) \\ &\quad + 4E \left(f'(B_{l/n}) f'(B_{j/n}) h''(B_{k/n}) \beta_{l,j} \alpha_{k,l} \alpha_{k,j} \right) \\ &\quad + 2E \left(f''(B_{l/n}) f^{(3)}(B_{j/n}) h'(B_{k/n}) \alpha_{j,l}^2 \alpha_{l,j} \alpha_{k,j} \right) \\ &\quad + 4E \left(f''(B_{l/n}) f''(B_{j/n}) h''(B_{k/n}) \alpha_{j,l} \alpha_{l,j} \alpha_{k,l} \alpha_{k,j} \right) \\ &\quad + 2E \left(f''(B_{l/n}) f'(B_{j/n}) h^{(3)}(B_{k/n}) \alpha_{l,j} \alpha_{k,l}^2 \alpha_{k,j} \right). \end{aligned}$$

As a consequence, using Lemma 5.1 (b)

$$\left| J_{k,j,l}^{(2)} \right| \leq C \left(n^{-4H} \beta_{l,j} + n^{-8H} \right). \quad (5.23)$$

To handle the term $J_{k,j,l}^{(3)}$, using (2.12) we can write

$$\begin{aligned} \left\langle D^2 \left[\delta^2(f'(B_{l/n}) \partial_{l/n}^{\otimes 2}) \right], \partial_{j/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} &= 2f'(B_{l/n}) \beta_{l,j}^2 \\ &\quad + 4\delta \left(f''(B_{l/n}) \partial_{l/n} \right) \beta_{l,j} \alpha_{l,j} + \delta^2 \left(f^{(3)}(B_{l/n}) \partial_{l/n}^{\otimes 2} \right) \alpha_{l,j}^2. \end{aligned}$$

As a consequence

$$\begin{aligned} J_{k,j,l}^{(3)} &= 2E \left(f'(B_{j/n}) h(B_{k/n}) f'(B_{l/n}) \beta_{l,j}^2 \right) \\ &\quad + 4E \left(f'(B_{j/n}) h(B_{k/n}) \delta \left(f''(B_{l/n}) \partial_{l/n} \right) \beta_{j,l} \alpha_{l,j} \right) \\ &\quad + E \left(f'(B_{j/n}) h(B_{k/n}) \delta^2 \left(f^{(3)}(B_{l/n}) \partial_{l/n}^{\otimes 2} \right) \alpha_{l,j}^2 \right). \end{aligned}$$

Applying the integration by parts formula (2.4) and (2.5) we obtain

$$\begin{aligned}
J_{k,j,l}^{(3)} &= 2E(f'(B_{j/n})h(B_{k/n})f'(B_{l/n})\beta_{l,j}^2) \\
&\quad + 4E(f''(B_{l/n})f''(B_{j/n})h(B_{k/n})\beta_{l,j}\alpha_{j,l}\alpha_{l,j}) \\
&\quad + 4E(f''(B_{l/n})f'(B_{j/n})h'(B_{k/n})\beta_{l,j}\alpha_{l,j}\alpha_{k,l}) \\
&\quad + E\left(f^{(3)}(B_{l/n})f^{(3)}(B_{j/n})h(B_{k/n})\alpha_{j,l}^2\alpha_{l,j}^2\right) \\
&\quad + 2E\left(f^{(3)}(B_{l/n})f''(B_{j/n})h'(B_{k/n})\alpha_{j,l}\alpha_{l,j}^2\alpha_{k,l}\right) \\
&\quad + E\left(f^{(3)}(B_{l/n})f'(B_{j/n})h''(B_{k/n})\alpha_{l,j}^2\alpha_{k,l}^2\right)
\end{aligned}$$

As a consequence, from Lemma 5.1 (b),

$$|J_{k,j,l}^{(3)}| \leq C(\beta_{j,l}^2 + n^{-4H}|\beta_{l,j}| + n^{-8H}). \quad (5.24)$$

Substituting the estimates (5.22), (5.23) and (5.24) in (5.21) we obtain

$$|J_{k,j,l}| \leq C(\beta_{j,l}^2 + n^{-4H}|\beta_{l,j}| + n^{-8H}). \quad (5.25)$$

From Lemma 5.1 (b) and (c) we get

$$\sum_{k=0}^{n-1} |\alpha_{j,k}\alpha_{l,k}| \leq Cn^{-2H},$$

and, as a consequence, from (5.20) and (5.25) we get

$$\begin{aligned}
n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} |B_{k,n}| &\leq Cn^{6H-\frac{3}{2}} \sum_{k,j,l=0}^{n-1} |\alpha_{j,k}\alpha_{l,k}| (\beta_{j,l}^2 + n^{-4H}|\beta_{l,j}| + n^{-8H}) \\
&\leq Cn^{4H-\frac{3}{2}} (n^{2-8H} + n^{-4H}n^{1-2H} + n^{1-4H}) \\
&\leq C(n^{\frac{1}{2}-4H} + n^{-\frac{1}{2}}),
\end{aligned}$$

which converges to zero if $H > \frac{1}{8}$.

Step 3. Let us show condition (ii). We have

$$\langle u_n, D^2 F_n \rangle_{\mathfrak{S}^{\otimes 2}} = n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} f(B_{k/n}) \langle \partial_{k/n}, D^2 F_n \rangle_{\mathfrak{S}^{\otimes 2}}.$$

Using (2.12) this yields

$$\begin{aligned}
\langle u_n, D^2 F_n \rangle_{\mathfrak{H}^{\otimes 2}} &= n^{4H-1} \sum_{k,j=0}^{n-1} f(B_{k/n}) (2f(B_{j/n}) \beta_{j,k}^2 \\
&\quad + 4\delta (f'(B_{j/n}) \partial_{j/n}) \beta_{j,k} \alpha_{j,k} + \delta^2 (f''(B_{j/n}) \partial_{j/n}^{\otimes 2}) \alpha_{j,k}^2) \\
&= A_n + B_n + C_n,
\end{aligned}$$

The first term A_n converges to a nonnegative square integrable random variable. Indeed,

$$\begin{aligned}
A_n &= \frac{1}{2n} \sum_{k,j=0}^{n-1} f(B_{k/n}) f(B_{j/n}) (|k-j+1|^{2H} + |k-j-1|^{2H} - 2|k-j|^{2H})^2 \\
&= \frac{1}{2n} \sum_{p=-\infty}^{\infty} \sum_{j=0 \vee -p}^{(n-1) \wedge (n-1-p)} f(B_{j/n}) f(B_{(j+p)/n}) (|p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H})^2,
\end{aligned}$$

which converges in L^1 to

$$\frac{1}{2} \left(\sum_{k \in \mathbb{Z}} \rho_H(k) \right) \int_0^1 f(B_s)^2 ds.$$

Then, it suffices to show that the terms B_n and C_n converge to zero in L^2 . Consider first B_n . Using (2.10) we can write

$$\begin{aligned}
B_n &= 4n^{4H-1} \sum_{k,j=0}^{n-1} \delta(f'(B_{k/n}) f'(B_{j/n}) \partial_{j/n}) \beta_{k,j} \alpha_{j,k} \\
&\quad + 4n^{4H-1} \sum_{k,j=0}^{n-1} f'(B_{k/n}) f''(B_{j/n}) \beta_{k,j} \alpha_{j,k} \alpha_{j,k} \\
&= B_n^{(1)} + B_n^{(2)}.
\end{aligned}$$

We have, using Lemma 5.1 (b) and (5.10)

$$\|B_n^{(2)}\|_{L^2} \leq Cn^{-2H}.$$

On the other hand, the term $B_n^{(1)}$ is the divergence of

$$w_n = 4n^{4H-1} \sum_{k,j=0}^{n-1} f'(B_{k/n}) f'(B_{j/n}) \partial_{j/n} \beta_{k,j} \alpha_{j,k}.$$

Then it suffices to show that w_n tends to zero in the norm of the space $\mathbb{D}^{1,2}(\mathfrak{H})$, which follows from

$$\|w_n\|_{\mathbb{D}^{1,2}(\mathfrak{H})} \leq Cn^{3H-1} \sum_{k,j=0}^{n-1} |\beta_{k,j}\alpha_{j,k}| \leq Cn^{-H}.$$

Finally, using Equation (2.7) we can write

$$\begin{aligned} f'(B_{k/n})\delta^2(f''(B_{j/n})\partial_{j/n}^{\otimes 2}) &= \delta^2(f'(B_{k/n})f''(B_{j/n})\partial_{j/n}^{\otimes 2}) \\ &\quad + 2\delta(f''(B_{k/n})f''(B_{j/n})\partial_{j/n})\alpha_{k,j} \\ &\quad + f^{(3)}(B_{k/n})f''(B_{j/n})\alpha_{k,j}^2, \end{aligned}$$

and this leads to the following decomposition for the term C_n

$$C_n = \delta^2(z_n) + 2\delta(v_n) + F_n,$$

where

$$\begin{aligned} z_n &= n^{4H-1} \sum_{k,j=0}^{n-1} f'(B_{k/n})f''(B_{j/n})\alpha_{j,k}^2\partial_{j/n}^{\otimes 2}, \\ v_n &= n^{4H-1} \sum_{k,j=0}^{n-1} f'(B_{k/n})f''(B_{j/n})\partial_{j/n}\alpha_{k,j}\alpha_{j,k}^2, \\ F_n &= n^{4H-1} \sum_{k,j=0}^{n-1} f^{(3)}(B_{k/n})f''(B_{j/n})\alpha_{k,j}^2\alpha_{j,k}^2 \end{aligned}$$

We have the following estimates, using Lemma 5.1 (b), (c), (5.10), and (5.11)

$$\begin{aligned} \|z_n\|_{\mathbb{D}^{2,2}(\mathfrak{H}^{\otimes 2})}^2 &\leq Cn^{8H-2} \sum_{k,j,k',j'=0}^{n-1} \alpha_{j,k}^2\alpha_{j',k'}^2\beta_{j,j'}^2 \leq Cn^{-1}, \\ \|v_n\|_{\mathbb{D}^{1,2}(\mathfrak{H})}^2 &\leq Cn^{8H-2} \sum_{k,j,k',j'=0}^{n-1} |\beta_{j,j'}\alpha_{k,j}\alpha_{k',j'}\alpha_{j,k}^2\alpha_{j',k'}^2| \leq Cn^{-2H-1}, \\ \|F_n\|_{L^2(\Omega)} &\leq Cn^{4H-1} \sum_{k,j=0}^{n-1} \alpha_{k,j}^2\alpha_{j,k}^2 \leq Cn^{-1}. \end{aligned}$$

This completes the proof. \blacksquare

As a consequence of Theorem 5.2 we can derive the following result.

Theorem 5.3 Suppose that f is a function satisfying Hypothesis **(H)**. Let G_n be the sequence of random variables defined in (5.4). Then, if $H \in (\frac{1}{8}, \frac{1}{2})$

$$G_n - \frac{1}{4}n^{\frac{1}{2}-2H} \int_0^1 f''(B_s)ds \xrightarrow{\text{Stably}} \sigma_H \int_0^1 f(B_s)dW_s, \quad (5.26)$$

where W is a Brownian motion independent of B and σ_H is defined in (5.6).

Proof. We recall first that $n^{2H}(\Delta B_{k/n})^2 - 1 = n^{2H}I_2(\partial_{k/n}^{\otimes 2})$. Then, using (2.7) yields

$$f(B_{k/n})I_2(\partial_{k/n}^{\otimes 2}) = \delta^2(f(B_{k/n})\partial_{k/n}^{\otimes 2}) + 2\delta(f'(B_{k/n})\partial_{k/n})\alpha_{k,k} + f''(B_{k/n})\alpha_{k,k}^2.$$

As a consequence,

$$G_n = \delta^2(u_n) + 2\delta(v_n) + R_n, \quad (5.27)$$

where u_n is defined in (5.13),

$$v_n = n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} f'(B_{k/n})\partial_{k/n} \alpha_{k,k},$$

and

$$R_n = n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} f''(B_{k/n})\alpha_{k,k}^2$$

The proof will be done in two steps.

Step 1 We first show that if $H \in (0, \frac{1}{2})$, $\delta(v_n)$ converges to zero in L^2 as n tends to infinity. It suffices to show that v_n converges to zero in the norm of the space $\mathbb{D}^{1,2}(\mathfrak{H})$. We can write

$$E \left(\|v_n\|_{\mathfrak{H}}^2 \right) = n^{4H-1} \sum_{k,l=0}^{n-1} E \left(f'(B_{k/n})f'(B_{l/n}) \right) \alpha_{k,k}\alpha_{l,l}\beta_{k,l},$$

and

$$E \left(\|Dv_n\|_{\mathfrak{H}^{\otimes 2}}^2 \right) = n^{4H-1} \sum_{k,l=0}^{n-1} E \left(f''(B_{k/n})f''(B_{l/n}) \right) \alpha_{k,k}\alpha_{l,l}\beta_{k,l} \langle \varepsilon_{k/n}, \varepsilon_{l/n} \rangle_{\mathfrak{H}}.$$

Hence, using Lemma 5.1 (b) and (5.10) yields

$$\|v_n\|_{\mathbb{D}^{1,2}(\mathfrak{H})}^2 \leq Cn^{-2H},$$

which converges to zero as n tends to infinity.

Step 2 To complete the proof it suffices to check that $R_n - n^{\frac{1}{2}-2H} \frac{1}{4} \int_0^1 f''(B_s) ds$ converges to zero in L^2 as n tends to infinity. This follows from (5.9) and the estimates

$$\begin{aligned} & \left\| n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} f''(B_{k/n}) \alpha_{k,k}^2 - \frac{1}{4} n^{-\frac{1}{2}-2H} \sum_{k=0}^{n-1} f''(B_{k/n}) \right\|_2 \\ & \leq C n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} \left| \alpha_{k,k}^2 - \frac{1}{4n^{4H}} \right| \leq C n^{-\frac{1}{2}}. \end{aligned}$$

■

We can make the following remarks:

1. If $H \in (\frac{1}{4}, \frac{1}{2})$ then (5.26) implies that

$$G_n \xrightarrow{\text{Stably}} \sigma_H \int_0^1 f(B_s) dW_s,$$

which was proved in [18].

2. If $H \in (0, \frac{1}{4})$, then (5.26) together with the fact that the sequence $\delta^2(u_n)$ is bounded in L^2 (which holds actually for any $H \in (0, \frac{1}{2})$ by Step 1 in the proof of Theorem 5.2) imply that $n^{2H-\frac{1}{2}} G_n$ converges in L^2 to $\frac{1}{4} \int_0^1 f''(B_s) ds$. This was proved by Nourdin in [14]. On the other hand, for $H \in (\frac{1}{8}, \frac{1}{4})$, (5.26) can also be written as

$$n^{\frac{1}{2}-2H} \left[n^{2H-\frac{1}{2}} G_n - \frac{1}{4} \int_0^1 f''(B_s) ds \right] \xrightarrow{\text{Stably}} \sigma_H \int_0^1 f(B_s) dW_s,$$

which is the asymptotic behavior of the fluctuations associated to the result in [14].

3. In the critical case $H = \frac{1}{4}$, (5.26) implies that

$$G_n \xrightarrow{\text{Stably}} \frac{1}{4} \int_0^1 f''(B_s) ds + \sigma_{1/4} \int_0^1 f(B_s) dW_s,$$

which is a result obtained by Nourdin and Réveillac in [19].

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References

- [1] P. Baldi, G. Kerkycharian, D. Marinucci and D. Picard (2007). Sub-sampling Spherical Needlets. Preprint.
- [2] P. Breuer and P. Major (1983). Central limit theorems for nonlinear functionals of Gaussian fields. *J. Multivariate Anal.* **13** (3), 425-441.
- [3] D. Chambers and E. Slud (1989). Central limit theorems for nonlinear functionals of stationary Gaussian processes. *Probab. Th. Related Fields* **80**, 323-349.
- [4] J.M. Corcuera, D. Nualart and J.H.C. Woerner (2006). Power variation of some integral fractional processes. *Bernoulli* **12**, 713-735.
- [5] P. Deheuvels, G. Peccati and M. Yor (2006). On quadratic functionals of the Brownian sheet and related processes. *Stoch. Proc. Appl.* **116**, 493-538.
- [6] L. Decreusefond and A. S. Üstünel (1998). Stochastic analysis of the fractional Brownian motion. *Potential Analysis* **10**, 177-214.
- [7] Y. Hu and D. Nualart (2005). Renormalized self-intersection local time for fractional Brownian motion. *Ann. Probab.* **33** (3), 948-983.
- [8] K. Itô (1951). Multiple Wiener integral. *J. Math. Soc. Japan* **3**, 157-169.
- [9] J. Jacod and A.N. Shiriyayev (1987). *Limit Theorems for Stochastic Processes*. Springer-Verlag, Berlin, Heidelberg, New York.
- [10] P. Major (1981). Multiple Wiener-Itô integrals. *Lecture Notes in Math.* **849**. Springer-Verlag, Berlin, Heidelberg, New York.
- [11] D. Marinucci and G. Peccati (2007a). High-frequency asymptotics for subordinated stationary fields on an Abelian compact group. To appear in: *Stoch. Proc. Appl.*
- [12] D. Marinucci and G. Peccati (2007b). Group representations and high-resolution central limit theorems for subordinated spherical random fields. Preprint.
- [13] A. Neuenkirch and I. Nourdin (2007). Exact rate of convergence of some approximation schemes associated to SDEs driven by fractional Brownian motion. *J. Theoret. Probab.* **20**, no. 4, 871-899.

- [14] I. Nourdin (2008). Asymptotic behavior of weighted quadratic and cubic variations of fractional Brownian motion. *Ann. Probab.* To appear.
- [15] I. Nourdin and G. Peccati (2007). Non-central convergence of multiple integrals. Preprint.
- [16] I. Nourdin and G. Peccati (2007). Weighted power variations of iterated Brownian motion. *Electron. J. Probab.* **13**, 1229-1256.
- [17] I. Nourdin and G. Peccati (2007). Stein's method on Wiener chaos. *Probab. Theory Rel. Fields*, to appear.
- [18] I. Nourdin, D. Nualart and C.A. Tudor (2007). Central and non-central limit theorems for weighted power variations of fractional Brownian motion. Preprint.
- [19] I. Nourdin and A. Réveillac (2008). Asymptotic behavior of weighted quadratic variations of fractoinal Brownian motion: the critical case $H = 1/4$. Preprint.
- [20] D. Nualart (2003). Stochastic calculus with respect to the fractional Brownian motion and applications. *Contemp. Math.* **336**, 3-39.
- [21] D. Nualart (2006). The Malliavin calculus and related topics. Springer-Verlag, Berlin, 2nd edition.
- [22] D. Nualart and S. Ortiz-Latorre (2007). Central limit theorems for multiple stochastic integrals and Malliavin calculus. To appear in: *Stoch. Proc. Appl.*
- [23] D. Nualart and G. Peccati (2005). Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.* **33** (1), 177-193.
- [24] G. Peccati (2007). Gaussian approximations of multiple integrals. *Electron. Comm. Probab.* **12**, 350-364.
- [25] G. Peccati and M.S. Taqqu (2006). Stable convergence of multiple Wiener-Itô integrals. To appear in: *J. Theoret. Probab.*
- [26] G. Peccati and M.S. Taqqu (2007). Stable convergence of generalized L^2 stochastic integrals and the principle of conditioning. *Electron. J. Probab.* **12**, no. 15, 447-480.

- [27] G. Peccati and C.A. Tudor (2005). Gaussian limits for vector-valued multiple stochastic integrals. In: Séminaire de Probabilités XXXVIII , 247-262. Lecture Notes in Math. **1857**, Springer-Verlag, Berlin.
- [28] A. Rényi (1963). On stable sequences of events. *Sankhya* **25 A**, 293-302.
- [29] A. V. Skorohod (1975). On a generalization of a stochastic integral. *Theory Probab. Appl.* **20**, 219-233.
- [30] D. Surgailis (2003). CLTs for Polynomials of Linear Sequences: Diagram Formulae with Applications. In: *Long Range Dependence*. Birkhäuser, Basel, 111-128.
- [31] C.A. Tudor and F. Viens (2007). Variations and estimators for the selfsimilarity order through Malliavin calculus. Preprint.