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Central limit theorems for multiple Skorohod integrals

Ivan Nourdin* and David Nualart^{†‡}

Abstract: In this paper, we prove a central limit theorem for a sequence of iterated Skorohod integrals using the techniques of Malliavin calculus. The convergence is stable, and the limit is a conditionally Gaussian random variable. Some applications to sequences of multiple stochastic integrals, and renormalized weighted quadratic variation of the fractional Brownian motion are discussed.

Key words: central limit theorem, fractional Brownian motion, Malliavin calculus.

2000 Mathematics Subject Classification: 60F05, 60H05, 60G15, 60H07.

1 Introduction and main results

Let \mathfrak{H} be a real separable Hilbert space and, for $q \geq 1$, let $\mathfrak{H}^{\otimes q}$ (resp. $\mathfrak{H}^{\odot q}$) be the q th tensor product (resp. q th symmetric tensor product) of \mathfrak{H} . In the sequel, we consider

$$X = \{X(h) : h \in \mathfrak{H}\}, \quad (1.1)$$

a centered isonormal Gaussian process on \mathfrak{H} . For every $q \geq 1$, we denote by I_q the isometry between $\mathfrak{H}^{\odot q}$ (equipped with the norm $\sqrt{q!} \|\cdot\|_{\mathfrak{H}^{\otimes q}}$) and the q th Wiener chaos of X . Note that, if \mathfrak{H} is a σ -finite measure space with no atoms, then each random variable $I_q(h)$, $h \in \mathfrak{H}^{\odot q}$, has the form of a multiple Wiener-Itô integral of order q . For $p, q \geq 1$, $f \in \mathfrak{H}^{\odot p}$, $g \in \mathfrak{H}^{\odot q}$ and $r = 0, \dots, p \wedge q$, we denote by $f \otimes_r g \in \mathfrak{H}^{\otimes(p+q-2r)}$ and $f \tilde{\otimes}_r g \in \mathfrak{H}^{\odot(p+q-2r)}$, respectively, the r th contraction and the r th *symmetrized* contraction of f and g . Also in the sequel, as usual, D denotes the Malliavin derivative operator, while δ is the associated divergence operator. For more precise definitions and properties, we refer to Section 2.

In what follows, we are interested in finding conditions ensuring that a (correctly normalized) sequence of nonlinear functionals of X converges in law towards a given random variable. When the limit is Gaussian, we have the following remarkable result (here, and for the rest of the paper, we shall denote by $\mathcal{N}(0, 1)$ the law of a Gaussian random variable with zero mean and unit variance):

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Theorem 1.1 Fix an integer $q \geq 2$ and a sequence $\{f_n\}_{n \geq 1} \subset \mathfrak{H}^{\odot q}$ such that

$$\lim_{n \rightarrow \infty} q! \|f_n\|_{\mathfrak{H}^{\odot q}}^2 = \lim_{n \rightarrow \infty} E[I_q(f_n)^2] = 1. \quad (1.2)$$

Then, the following four conditions are equivalent:

- (i) as $n \rightarrow \infty$, the sequence $\{I_q(f_n)\}_{n \geq 1}$ converges in distribution to $N \sim \mathcal{N}(0, 1)$;
- (ii) $\lim_{n \rightarrow \infty} E[I_q(f_n)^4] = 3$;
- (iii) for every $r = 1, \dots, q-1$, $\lim_{n \rightarrow \infty} \|f_n \otimes_r f_n\|_{\mathfrak{H}^{\otimes 2(q-r)}} = 0$;
- (iv) $\lim_{n \rightarrow \infty} \|D[I_q(f_n)]\|_{\mathfrak{H}}^2 = q$ in L^2 .

Moreover, in this case, we have the following bound:

$$\sup_{z \in \mathbb{R}} |P(I_q(f_n) \leq z) - P(N \leq z)| \leq \sqrt{E(1 - q^{-1} \|D[I_q(f_n)]\|_{\mathfrak{H}}^2)^2}. \quad (1.3)$$

The equivalence between (i), (ii) and (iii) in Theorem 1.1 is proved in Nualart and Peccati [20] by means of the *Dambis, Dubins and Schwarz Theorem*. The equivalence with (iv) was then proved by Nualart and Ortiz-Latorre [19] using exclusively tools from Malliavin calculus. Finally, the Berry-Esseen's type bound (1.3) is proved by Nourdin and Peccati in [15].

In particular, Theorem 1.1 implies that the convergence in distribution of a sequence of multiple stochastic integrals towards a Gaussian random variable is completely determined by the asymptotic behavior of their second and fourth moments. As such, Theorem 1.1 can be seen as a simplification of the classic “method of moments and diagrams” which was, until then, the usual method allowing to prove weak convergence for nonlinear functionals of X (see for instance the works by Breuer and Major [2], Major [9], Giraitis and Surgailis [6] and Chambers and Slud [3]; see also the survey by Surgailis [26] and the references therein). Peccati and Tudor [24] also proved the following multidimensional version of Theorem 1.1:

Theorem 1.2 For $d \geq 2$, fix d integers $1 \leq q_1 \leq \dots \leq q_d$. Consider a sequence of random vectors of the form

$$F_n = (I_{q_1}(f_n^1), \dots, I_{q_d}(f_n^d))$$

where $f_n^i \in \mathfrak{H}^{\odot q_i}$. For every $1 \leq i, j \leq d$, assume that $\lim_{n \rightarrow \infty} E[I_{q_j}(f_n^j) I_{q_i}(f_n^i)] = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Then, the following two conditions are equivalent:

- (i) as $n \rightarrow \infty$, the sequence $\{F_n\}_{n \geq 1}$ converges in distribution to a standard centered Gaussian vector $\mathcal{N}_d(0, I_d)$ (I_d is the $d \times d$ identity matrix),
- (ii) for every $j = 1, \dots, d$, as $n \rightarrow \infty$, the sequence $\{I_{q_j}(f_n^j)\}_{n \geq 1}$ converges in distribution to a standard Gaussian random variable $\mathcal{N}(0, 1)$.

An immediate (but important for applications) consequence of Theorem 1.2 is the following “stable convergence” generalization of Theorem 1.1:

Proposition 1.3 *Let $q \geq 2$ be an integer, and let $\{I_q(f_n)\}_{n \geq 1}$ be a sequence of multiple integrals verifying (1.2) and satisfying either one of conditions (i)-(iv) of Theorem 1.1. Then, as $n \rightarrow \infty$, for any vector $Y = (X(h_1), \dots, X(h_m))$, $h_i \in \mathfrak{H}$, the pair $(I_q(f_n), Y)$ converges in law to (N, Y) , where $N \sim \mathcal{N}(0, 1)$ is independent of X .*

Recent examples of application of Theorems 1.1 and 1.2 are the study of: p -variations of fractional stochastic integrals (Corcuera *et al.* [4]), quadratic functionals of bivariate Gaussian processes (Deheuvels *et al.* [5]), self-intersection local times of fractional Brownian motion (Hu and Nualart [7]), approximation schemes for scalar fractional differential equations (Neuenkirch and Nourdin [12]), high-frequency CLTs for random fields on homogeneous spaces (Marinucci and Peccati [10, 11] and Peccati [21]), needlets analysis on the sphere (Baldi *et al.* [1]), estimation of self-similarity orders (Tudor and Viens [27]), weighted power variations of iterated Brownian motion (Nourdin and Peccati [14]), etc.

Since the works by Nualart and Peccati [20] and Peccati and Tudor [24], great efforts have been made to find some similar statements, but in the case where the limit is not necessarily a Gaussian random variable. In this direction, let us mention the following recent works:

(1) Peccati and Taqqu have proposed sufficient conditions ensuring that a given sequence of multiple Wiener-Itô integrals converges stably towards *mixtures* of Gaussian random variables. More precisely, the following problem is addressed in [22]. Let $Y \geq 0$ be a non-constant random variable having the (finite) chaotic representation $Y = 1 + I_2(f_2) + \dots + I_{2(q-1)}(f_{2(q-1)})$, let $N \sim \mathcal{N}(0, 1)$ be independent of Y , and suppose that the sequence $I_q(f_n)$ satisfies adequate normalization conditions; is it possible to associate to each f_n and each $r = 1, \dots, q - 1$, two generalized contraction kernels, say $f_n \otimes_{q-r}^* f_n$ and $f_n \otimes_{q-r}^{**} f_n$, in such a way that the two relations

$$f_n \otimes_{q-r}^* f_n \xrightarrow{n \rightarrow \infty} g_{2r} \quad \text{and} \quad f_n \otimes_{q-r}^{**} f_n \xrightarrow{n \rightarrow \infty} 0, \quad \forall r = 1, \dots, q - 1, \quad (1.4)$$

imply that $I_q(f_n)$ converges stably to $\sqrt{Y} \times N$? The work [22] by Peccati and Taqqu provides results in this direction, by constructing the two kernels $f_n \otimes_{q-r}^* f_n$ and $f_n \otimes_{q-r}^{**} f_n$ with the help of “resolutions of the identity” and of a “generalized Clark-Ocone formula”. See also [23] for an alternative proof of these results by using the “principle of conditioning”. Moreover, in this latter reference, general results involving Skorohod integrals on abstract Wiener spaces can also be found.

(2) On the other hand, Nourdin and Peccati [13] proved the following extension of Theorem 1.1:

Theorem 1.4 Fix $\nu > 0$ and let R_ν be a real-valued random variable such that

$$E(e^{i\lambda R_\nu}) = \left(\frac{e^{-i\lambda}}{\sqrt{1-2i\lambda}} \right)^\nu, \quad \lambda \in \mathbb{R}. \quad (1.5)$$

Fix an even integer $q \geq 2$, and define

$$c_q := \frac{1}{(q/2)! \binom{q-1}{q/2-1}^2} = \frac{4}{(q/2)! \binom{q}{q/2}^2}.$$

Then, for any sequence $\{f_n\}_{n \geq 1} \subset \mathfrak{H}^{\odot q}$ verifying

$$\lim_{n \rightarrow \infty} q! \|f_n\|_{\mathfrak{H}^{\otimes q}}^2 = \lim_{n \rightarrow \infty} E[I_q(f_n)^2] = 2\nu,$$

the following six conditions are equivalent:

- (i) as $n \rightarrow \infty$, the sequence $\{I_q(f_n)\}_{n \geq 1}$ converges in distribution to R_ν ;
- (ii) $\lim_{n \rightarrow \infty} E[I_q(f_n)^3] = E[R_\nu^3] = 8\nu$ and $\lim_{n \rightarrow \infty} E[I_q(f_n)^4] = E[R_\nu^4] = 48\nu + 12\nu^2$;
- (iii) $\lim_{n \rightarrow \infty} E[I_q(f_n)^4] - 12E[I_q(f_n)^3] = 12\nu^2 - 48\nu$;
- (iv) $\lim_{n \rightarrow \infty} \|f_n \tilde{\otimes}_{q/2} f_n - c_q \times f_n\|_{\mathfrak{H}^{\otimes q}} = 0$ and $\lim_{n \rightarrow \infty} \|f_n \tilde{\otimes}_r f_n\|_{\mathfrak{H}^{\otimes 2(q-r)}} = 0$, for every $r = 1, \dots, q-1$ such that $r \neq q/2$;
- (v) $\lim_{n \rightarrow \infty} \|f_n \tilde{\otimes}_{q/2} f_n - c_q \times f_n\|_{\mathfrak{H}^{\otimes q}} = 0$ and $\lim_{n \rightarrow \infty} \|f_n \otimes_r f_n\|_{\mathfrak{H}^{\otimes 2(q-r)}} = 0$, for every $r = 1, \dots, q-1$ such that $r \neq q/2$;
- (vi) as $n \rightarrow \infty$, $\|D[I_q(f_n)]\|_{\mathfrak{H}}^2 - 2qI_q(f_n) \rightarrow 2q\nu$ in L^2 .

Observe that the limit random variable R_ν appearing in formula (1.5) is such that $R_\nu \stackrel{\text{Law}}{=} 2G_{\nu/2} - \nu$, where $G_{\nu/2}$ has a Gamma law with parameter $\nu/2$, that is $G_{\nu/2}$ admits

$$g(x) = \frac{x^{\frac{\nu}{2}-1} e^{-x}}{\Gamma(\nu/2)} \mathbf{1}_{(0, \infty)}(x)$$

for density, where Γ is the usual Gamma function. Thus, when $\nu \geq 1$ is an integer, R_ν has a centered χ^2 law with ν degrees of freedom. Let us also mention that, in [15], Nourdin and Peccati associated bounds with the convergence (i) in the previous theorem, see also (1.3).

For a fixed $q \geq 1$, suppose now that F_n is a sequence of random variables of the form $F_n = \delta^q(u_n)$, for some random variables u_n with values in $\mathfrak{H}^{\otimes q}$ and measurable with respect to X . Can we exhibit condition(s) on the Malliavin derivative of F_n ensuring that F_n converges towards mixtures of Gaussian random variables? This question should be compared with the work by Peccati and Taqqu [22], see (1) above. In the present paper, we show the following result:

Theorem 1.5 For a fixed $q \geq 1$, suppose that F_n is a sequence of random variables of the form $F_n = \delta^q(u_n)$, for some random variables u_n with values in $\mathfrak{H}^{\otimes q}$ and measurable with respect to X . Suppose moreover that the sequence F_n is bounded in L^1 , and that:

(i) $\langle u_n, (DF_n)^{\otimes k_1} \tilde{\otimes} \dots \tilde{\otimes} (D^{q-1}F_n)^{\otimes k_{q-1}} \tilde{\otimes} f_1 \tilde{\otimes} \dots \tilde{\otimes} f_r \rangle_{\mathfrak{H}^{\otimes q}}$ converges to zero in L^1 , for all integers $r, k_1, \dots, k_{q-1} \geq 0$ such that $k_1 + 2k_2 + \dots + (q-1)k_{q-1} + r = q$, and all functions $f_1, \dots, f_r \in \mathfrak{H}$;

(ii) $\langle u_n, D^q F_n \rangle_{\mathfrak{H}^{\otimes q}}$ converges in L^1 to a nonnegative random variable S^2 .

Then, F_n converges stably to a random variable with characteristic function $E\left(e^{-\frac{\lambda^2}{2}S^2}\right)$.

Remark 1.6 When $q = 1$, condition (i) of the theorem is that $\langle u_n, h \rangle_{\mathfrak{H}}$ converges to zero in L^1 , for each $h \in \mathfrak{H}$. When $q = 2$, condition (i) means that $\langle u_n, h \otimes g \rangle_{\mathfrak{H}^{\otimes 2}}$, $\langle u_n, h \tilde{\otimes} DF_n \rangle_{\mathfrak{H}^{\otimes 2}}$ and $\langle u_n, DF_n \otimes DF_n \rangle_{\mathfrak{H}^{\otimes 2}}$ converge to zero in L^1 , for each $h, g \in \mathfrak{H}$. And so on.

In the statement of Theorem 1.5, we use the notion of stable convergence for random variables, so let us recall its definition (see for instance, the original paper [25], or the book [8] for an exhaustive discussion of stable convergence). Let $\mathcal{F} \subset \overline{\mathcal{F}}$ be two σ -fields. A sequence of \mathcal{F} -measurable random variables (F_n) converges \mathcal{F} -stably to a $\overline{\mathcal{F}}$ -measurable random variable F if, for any continuous bounded function $f : \mathbb{R} \rightarrow \mathbb{R}$ and any bounded \mathcal{F} -measurable random variable Z , we have, as $n \rightarrow \infty$,

$$E(f(F_n)Z) \rightarrow E(f(F)Z). \quad (1.6)$$

Here, it is implicitly used that \mathcal{F} is the σ -field generated by X , and that the limit is measurable with respect to a certain σ -field $\overline{\mathcal{F}}$ containing \mathcal{F} .

We develop some particular applications of Theorem 1.5 in the following directions:

- (a) Consider first a sequence of multiple stochastic integrals of the form $F_n = I_q(g_n)$, where $g_n \in \mathfrak{H}^{\otimes q}$. In this case we can apply Theorem 1.5 in different ways. We can assume, for instance, that $q = j$, and $u_n = I_{q-j}(g_n)$, for any $j = 1, \dots, q$. Some examples of non Gaussian criteria in the cases $q = 1$, $q = 2$ are given. If we take $j = q$, we recover the implication (iii) \rightarrow (i) in Theorem 1.1.
- (b) We give a new proof of the following stable convergence result, proved by Nourdin, Nualart and Tudor in [16]. Let B be a fractional Brownian motion with Hurst index $H \in (\frac{1}{4}, \frac{1}{2})$, and $f : \mathbb{R} \rightarrow \mathbb{R}$ be a real function verifying hypothesis **(H)** (see below). Then

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(B_{k/n}) \left(n^{2H} (B_{(k+1)/n} - B_{k/n})^2 - 1 \right)$$

converges stably towards $\sigma_H \int_0^1 f(B_s) dW_s$, where W is a standard Brownian motion independent of B and σ_H is an (explicit) constant depending only on H .

The paper is organized as follows. In Section 2 we present some preliminary results about Malliavin calculus and fractional Brownian motion. Section 3 contains the proof of Theorem 1.5, while Section 4 focuses on examples (a) and (b) above.

2 Preliminaries

We follow Nualart [18]. Let \mathfrak{H} be a real separable Hilbert space. As in formula (1.1), we denote by X an isonormal Gaussian process over \mathfrak{H} . Recall that, by definition, X is a collection of centered and jointly Gaussian random variables indexed by the elements of \mathfrak{H} , defined on some probability space (Ω, \mathcal{F}, P) and such that, for every $h, g \in \mathfrak{H}$,

$$E[X(h)X(g)] = \langle h, g \rangle_{\mathfrak{H}}. \quad (2.7)$$

We will systematically assume that \mathcal{F} is generated by X . It is well-known that any random variable F belonging to $L^2(\Omega, \mathcal{F}, P)$ admits the following chaotic expansion:

$$F = \sum_{q=0}^{\infty} I_q(f_q), \quad (2.8)$$

where $I_0(f_0) := E[F]$, the series converges in L^2 and the kernels $f_q \in \mathfrak{H}^{\odot q}$, $q \geq 1$, are uniquely determined by F . As already pointed out, in the particular case where $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$, where (A, \mathcal{A}) is a measurable space and μ is a σ -finite and non-atomic measure, one has that $\mathfrak{H}^{\odot q} = L^2_s(A^q, \mathcal{A}^{\otimes q}, \mu^{\otimes q})$ is the space of symmetric and square integrable functions on A^q . Moreover, for every $f \in \mathfrak{H}^{\odot q}$, $I_q(f)$ coincides with the multiple Wiener-Itô integral (of order q) of f with respect to X . For every $q \geq 0$, we write J_q to indicate the orthogonal projection operator on the q th Wiener chaos associated with X . In particular, if $F \in L^2(\Omega, \mathcal{F}, P)$ is as in (2.8), then $J_q F = I_q(f_q)$ for every $q \geq 0$.

Let $\{e_k, k \geq 1\}$ be a complete orthonormal system in \mathfrak{H} . Given $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, for every $r = 0, \dots, p \wedge q$, the r th contraction of f and g is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined as

$$f \otimes_r g = \sum_{i_1, \dots, i_r=1}^{\infty} \langle f, e_{i_1} \otimes \dots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \dots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}}. \quad (2.9)$$

Note that, in the particular case where $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$ (with μ non-atomic), one has that

$$f \otimes_r g = \int_{A^r} f(t_1, \dots, t_{p-r}, s_1, \dots, s_r) g(t_{p-r+1}, \dots, t_{p+q-2r}, s_1, \dots, s_r) d\mu(s_1) \dots d\mu(s_r).$$

Moreover, $f \otimes_0 g = f \otimes g$ equals the tensor product of f and g while, for $p = q$, $f \otimes_q g = \langle f, g \rangle_{\mathfrak{H}^{\otimes q}}$. Note that, in general (and except for trivial cases), the contraction $f \otimes_r g$ is not a *symmetric* element of $\mathfrak{H}^{\otimes(p+q-2r)}$. As indicated in the Introduction, the canonical symmetrization of $f \otimes_r g$ is written $f \widetilde{\otimes}_r g$.

Let \mathcal{S} be the set of all smooth cylindrical random variables of the form

$$F = g(X(\phi_1), \dots, X(\phi_n))$$

where $n \geq 1$, $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth function with compact support and $\phi_i \in \mathfrak{H}$. The Malliavin derivative of F with respect to X is the element of $L^2(\Omega, \mathfrak{H})$ defined as

$$DF = \sum_{i=1}^n \frac{\partial g}{\partial x_i}(X(\phi_1), \dots, X(\phi_n)) \phi_i.$$

In particular, $DX(h) = h$ for every $h \in \mathfrak{H}$. By iteration, one can define the q th derivative $D^q F$ (which is an element of $L^2(\Omega, \mathfrak{H}^{\otimes q})$) for every $q \geq 2$.

As usual, for $q \geq 1$, $\mathbb{D}^{q,2}$ denotes the closure of \mathcal{S} with respect to the norm $\|\cdot\|_{q,2}$, defined by the relation

$$\|F\|_{q,2}^2 = E[F^2] + \sum_{i=1}^q E[\|D^i F\|_{\mathfrak{H}^{\otimes i}}^2].$$

The Malliavin derivative D verifies the following *chain rule*: if $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ is in \mathcal{C}_b^1 and if $\{F_i\}_{i=1,\dots,n}$ is a vector of elements of $\mathbb{D}^{1,2}$, then $\varphi(F_1, \dots, F_n) \in \mathbb{D}^{1,2}$ and

$$D\varphi(F_1, \dots, F_n) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F_1, \dots, F_n) DF_i.$$

We denote by δ the adjoint of the operator D , also called the *divergence operator*. A random element $u \in L^2(\Omega, \mathfrak{H})$ belongs to the domain of δ , noted $\text{Dom}\delta$, if and only if it verifies

$$|E\langle DF, u \rangle_{\mathfrak{H}}| \leq c_u \|F\|_{L^2} \quad \text{for any } F \in \mathcal{S},$$

where c_u is a constant depending uniquely on u . If $u \in \text{Dom}\delta$, then the random variable $\delta(u)$ is defined by the duality relationship (called ‘‘integration by parts formula’’):

$$E(F\delta(u)) = E\langle DF, u \rangle_{\mathfrak{H}}, \tag{2.10}$$

which holds for every $F \in \mathbb{D}^{1,2}$. We will moreover need the following property: for every $F \in \mathbb{D}^{1,2}$ and every $u \in \text{Dom}\delta$ such that Fu and $F\delta(u) + \langle DF, u \rangle_{\mathfrak{H}}$ are square integrable, one has that $Fu \in \text{Dom}\delta$ and

$$\delta(Fu) = Fu\delta(u) - \langle DF, u \rangle_{\mathfrak{H}}. \tag{2.11}$$

For every $q \geq 1$, let \mathcal{H}_q be the q th Wiener chaos of X , that is, the closed linear subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $\{H_q(X(h)), h \in \mathfrak{H}, |h|_{\mathfrak{H}} = 1\}$, where H_q is the q th Hermite polynomial. The mapping $I_q(h^{\otimes q}) = q!H_q(X(h))$ provides a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot q}$ and \mathcal{H}_q .

The duality formula (2.10) extends to the iterated divergence operator and we have

$$E(F\delta^q(u)) = E(\langle D^q F, h \rangle_{\mathfrak{H}^{\otimes q}}) \quad (2.12)$$

for any element u in the domain of δ^q and any random variable $F \in \mathbb{D}^{q,2}$. Notice that $\delta^q(h) = I_q(h)$ for any $h \in \mathfrak{H}^{\otimes q}$.

The operator L is defined through the projection operators $\{J_q\}_{q \geq 0}$ as $L = \sum_{q=0}^{\infty} -qJ_q$, and is called the *infinitesimal generator of the Ornstein-Uhlenbeck semigroup*. It verifies the following crucial property: a random variable F is an element of $\text{Dom}L$ ($= \mathbb{D}^{2,2}$) if and only if $F \in \text{Dom}\delta D$ (i.e. $F \in \mathbb{D}^{1,2}$ and $DF \in \text{Dom}\delta$), and in this case:

$$\delta DF = -LF.$$

Note that a random variable F as in (2.8) is in $\mathbb{D}^{1,2}$ if, and only if,

$$\sum_{q=1}^{\infty} q \|f_q\|_{\mathfrak{H}^{\otimes q}}^2 < \infty,$$

and, in this case, $E[\|DF\|_{\mathfrak{H}}^2] = \sum_{q \geq 1} q \|f_q\|_{\mathfrak{H}^{\otimes q}}^2$. If $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$ (with μ non-atomic), then the derivative of a random variable F as in (2.8) can be identified with the element of $L^2(A \times \Omega)$ given by

$$D_a F = \sum_{q=1}^{\infty} q I_{q-1}(f_q(\cdot, a)), \quad a \in A. \quad (2.13)$$

3 Proof of Theorem 1.5

We need to show that for any $h_1, \dots, h_m \in \mathfrak{H}$, the sequence $\xi_n := (F_n, X(h_1), \dots, X(h_m))$ converges in distribution to a vector $(F_\infty, X(h_1), \dots, X(h_m))$, where F_∞ satisfies

$$E(e^{i\lambda F_\infty} | X(h_1), \dots, X(h_m)) = e^{-\frac{\lambda^2}{2} S^2} \quad (3.14)$$

for any $\lambda \in \mathbb{R}$. Since the sequence F_n is bounded in L^1 , the sequence ξ_n is tight. Assume that $(F_\infty, X(h_1), \dots, X(h_m))$ denotes the limit in law of a certain subsequence of ξ_n , denoted again by ξ_n .

Let $Y = \phi(X(h_1), \dots, X(h_m))$, with $\phi \in \mathcal{C}_b^\infty(\mathbb{R}^m)$, and consider $\phi_n(\lambda) = E(e^{i\lambda F_n} Y)$ for $\lambda \in \mathbb{R}$. The convergence in law of ξ_n , and the fact that F_n is bounded in L^1 imply that

$$\lim_{n \rightarrow \infty} \phi_n'(\lambda) = \lim_{n \rightarrow \infty} iE(F_n e^{i\lambda F_n} Y) = iE(F_\infty e^{i\lambda F_\infty} Y). \quad (3.15)$$

On the other hand, by the duality relationship (2.12) between the iterated derivative and divergence operators we obtain

$$\begin{aligned}
\phi'_n(\lambda) &= iE(F_n e^{i\lambda F_n} Y) = iE(\delta^q(u_n) e^{i\lambda F_n} Y) = iE\left(\langle u_n, D^q(e^{i\lambda F_n} Y) \rangle_{\mathfrak{H}^{\otimes q}}\right) \\
&= i \sum_{a=0}^q \binom{q}{a} E\left(\langle u_n, D^a(e^{i\lambda F_n}) \tilde{\otimes} D^{q-a} Y \rangle_{\mathfrak{H}^{\otimes q}}\right) \\
&= i \sum_{a=0}^q \binom{q}{a} \sum \frac{a!}{k_1! \dots k_a!} (i\lambda)^{k_1 + \dots + k_a} \\
&\quad \times E\left(e^{i\lambda F_n} \langle u_n, (DF_n)^{\otimes k_1} \tilde{\otimes} \dots \tilde{\otimes} (D^a F_n)^{\otimes k_a} \tilde{\otimes} D^{q-a} Y \rangle_{\mathfrak{H}^{\otimes q}}\right),
\end{aligned}$$

where the second sum in the last equality runs over all sequences of integers (k_1, \dots, k_a) such that $k_1 + 2k_2 + \dots + ak_a = a$ by the Faà di Bruno's formula. By condition (i), this yields that

$$\phi'_n(\lambda) = -\lambda E(e^{i\lambda F_n} \langle u_n, D^q F_n \rangle_{\mathfrak{H}^{\otimes q}} Y) + R_n,$$

with R_n converging to zero as $n \rightarrow \infty$. Using condition (ii) and (3.15), we obtain that

$$iE(F_\infty e^{i\lambda F_\infty} Y) = -\lambda E(e^{i\lambda F_\infty} S^2 Y).$$

Since S^2 is defined through condition (ii), it is in particular measurable with respect to B . Thus, the following linear differential equation verified by the conditional characteristic function of F_∞ holds:

$$\frac{\partial}{\partial \lambda} E(e^{i\lambda F_\infty} | X(h_1), \dots, X(h_m)) = -\lambda S^2 E(e^{i\lambda F_\infty} | X(h_1), \dots, X(h_m)).$$

By solving it, we obtain (3.14), which yields the desired conclusion. \blacksquare

The next corollary provides stronger but easier conditions for the convergence in law.

Corollary 3.1 *For a fixed $q \geq 1$, suppose that F_n is a sequence of random variables of the form $F_n = \delta^q(u_n)$, for some random variables u_n with values in $\mathfrak{H}^{\otimes q}$ and measurable with respect to X . Suppose moreover that the sequence F_n is bounded in $\mathbb{D}^{q,p}$, for all $p \geq 2$, and that:*

- (i') $\langle u_n, h \rangle_{\mathfrak{H}^{\otimes q}}$ converges to zero in L^1 , for all $h \in \mathfrak{H}^{\otimes q}$; and $u_n \otimes_\ell D^\ell F_n$ converges to zero in $L^2(\Omega; \mathfrak{H}^{\otimes(q-\ell)})$, for all $\ell = 1, \dots, q-1$;
- (ii) $\langle u_n, D^q F_n \rangle_{\mathfrak{H}^{\otimes q}}$ converges in L^1 to a nonnegative random variable S^2 .

Then, F_n converges stably to a random variable with characteristic function $E\left(e^{-\frac{\lambda^2}{2} S^2}\right)$.

Proof It suffices to show that condition (i') implies condition (i) in Theorem 1.5. When $k_a \neq 0$ we have

$$\begin{aligned} & \left| \langle u_n, (DF_n)^{\otimes k_1} \tilde{\otimes} \dots \tilde{\otimes} (D^a F_n)^{\otimes k_a} \tilde{\otimes} h \rangle_{\mathfrak{H}^{\otimes q}} \right| \\ & \leq \|u_n \otimes_a D^a F_n\|_{\mathfrak{H}^{\otimes(q-a)}} \left\| (DF_n)^{\otimes k_1} \otimes \dots \otimes (D^a F_n)^{\otimes(k_a-1)} \otimes h \right\|_{\mathfrak{H}^{\otimes(q-a)}}. \end{aligned}$$

The second factor is bounded in L^2 , and the first factor converges to zero in L^2 , for all $a = 1, \dots, q-1$. In the case $a = 0$ we have that $\langle u_n, h \rangle_{\mathfrak{H}^{\otimes q}}$ converges to zero in L^1 , for all $h \in \mathfrak{H}^{\otimes q}$, by condition (i'). This completes the proof. \blacksquare

4 Examples and applications

In this section, we will consider several examples of application of Theorem 1.5.

4.1 Multiple stochastic integrals

Fix an integer $q \geq 2$, and consider the sequence of multiple stochastic integrals $F_n = I_q(g_n)$ with $g_n \in \mathfrak{H}^{\odot q}$. We have $DF_n = qI_{q-1}(g_n)$, and $F_n = -\frac{1}{q}LF_n = \frac{1}{q}\delta(DF_n)$. Applying Theorem 1.5 with $q = 1$ and $u_n = \frac{1}{q}DF_n = I_{q-1}(g_n)$ yields:

Proposition 4.1 *For a fixed $q \geq 2$, suppose that F_n is a sequence of random variables of the form $F_n = I_q(g_n)$, with $g_n \in \mathfrak{H}^{\odot q}$. Suppose moreover that the sequence g_n is bounded in $\mathfrak{H}^{\odot q}$, and that:*

- (a) $\langle g_n \otimes_{q-1} g_n, h^{\otimes 2} \rangle_{\mathfrak{H}^{\otimes 2}}$ converges to zero for all $h \in \mathfrak{H}$;
- (b) $\frac{1}{q}\|DF_n\|_{\mathfrak{H}}^2$ converges in L^1 to a nonnegative random variable S^2 .

Then, F_n converges stably to a random variable with characteristic function $E\left(e^{-\frac{\lambda^2}{2}S^2}\right)$.

Proof It suffices to apply Theorem 1.5 to $u_n = I_{q-1}(g_n)$. We have

$$\begin{aligned} E(\langle u_n, h \rangle_{\mathfrak{H}}^2) &= E(\langle I_{q-1}(g_n), h \rangle_{\mathfrak{H}}^2) = E(I_{q-1}(g_n \otimes_1 h)^2) = (q-1)! \|g_n \otimes_1 h\|_{\mathfrak{H}^{\otimes(q-1)}}^2 \\ &= (q-1)! (g_n \otimes_1 h) \otimes_{q-1} (g_n \otimes_1 h) \\ &= (q-1)! \langle g_n \otimes_{q-1} g_n, h^{\otimes 2} \rangle_{\mathfrak{H}^{\otimes 2}} \rightarrow 0, \end{aligned}$$

which implies condition (i) in Theorem 1.5, see Remark 1.6. Condition (ii) in Theorem 1.5 follows from (b) and the equality $\langle u_n, DF_n \rangle_{\mathfrak{H}} = \frac{1}{q}\|DF_n\|_{\mathfrak{H}}^2$. \blacksquare

If S^2 is deterministic, equal to 1, then condition (b) coincides with condition (iv) in Theorem 1.1. In this case, Theorem 1.1 combined with Lemma 6 in [19] tells us that if the variances converge to one, then, condition (a) is automatically satisfied.

On the other hand, we can also apply Theorem 1.5 with $u_n = g_n$. In this way, applying Corollary 3.1, we obtain that the following conditions imply that F_n converges to a normal random variable $N(0, 1)$ independent of X :

- (α) g_n is bounded in $\mathfrak{H}^{\odot q}$ and it converges weakly to zero,
- (β) $\|g_n \otimes_\ell g_n\|_{\mathfrak{H}^{\otimes 2(q-\ell)}}$ converges to zero for all $\ell = 1, \dots, q-1$;
- (γ) $q! \|g_n\|_{\mathfrak{H}^{\otimes q}}^2$ converges to 1.

In fact, notice that whenever F_n is bounded in all the Sobolev spaces $\mathbb{D}^{q,p}$, condition (ii) in Corollary 3.1 follows from (γ) and the equality $D^q(I_q(g_n)) = q!g_n$, and, finally, condition (i') in Corollary 3.1 follows from (ii) and

$$\begin{aligned} E \left[\|g_n \otimes_\ell D^\ell F_n\|_{\mathfrak{H}^{\otimes(q-\ell)}}^2 \right] &= \frac{q!^2}{(q-\ell)!^2} E \left[\|g_n \otimes_\ell I_{q-\ell}(g_n)\|_{\mathfrak{H}^{\otimes(q-\ell)}}^2 \right] \\ &= \frac{q!^2}{(q-\ell)!^2} E \left[\|I_{q-\ell}(g_n \otimes_\ell g_n)\|_{\mathfrak{H}^{\otimes(q-\ell)}}^2 \right] \\ &= \frac{q!^2}{(q-\ell)!} \|g_n \otimes_\ell g_n\|_{\mathfrak{H}^{\otimes 2(q-\ell)}}^2. \end{aligned}$$

In this way we recover the implication “(iii) implies (i)” in Theorem 1.1.

4.2 Weighted quadratic variation of the fractional Brownian motion

Let $B = (B_t)_{t \in [0,1]}$ be a fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2})$ defined on a probability space (Ω, \mathcal{A}, P) . We mean that B is a centered Gaussian process with the covariance function $E(B_s B_t) = R_H(s, t)$, where

$$R_H(s, t) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}).$$

We denote by \mathcal{E} the set of step functions on $[0, T]$. Then \mathfrak{H} is the Hilbert space defined as the closure of \mathcal{E} with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathfrak{H}} = R_H(t, s).$$

It coincides with $I_{0+}^{H-\frac{1}{2}}(L^2[0, T])$, where

$$I_{0+}^{H-\frac{1}{2}} f(x) \triangleq \frac{1}{\Gamma(H-\frac{1}{2})} \int_0^x (x-y)^{H-\frac{3}{2}} f(y) dy.$$

We refer the reader to [17] for additional details.

Consider the sequence

$$G_n = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(B_{k/n}) (n^{2H} (\Delta B_{k/n})^2 - 1), \quad n \geq 1.$$

Notice that $n^{2H} (\Delta B_{k/n})^2 - 1 = n^{2H} I_2(\partial_{k/n}^{\otimes 2})$, where $\partial_{k/n} = \mathbf{1}_{[k/n, (k+1)/n]}$. We will make use of the following technical lemma.

Lemma 4.2 *Let $s < t$ and r belong to $[0, 1]$. Then*

$$|E(B_r(B_t - B_s))| \leq (t - s)^{2H}.$$

Proof We have

$$\begin{aligned} E(B_r(B_t - B_s)) &= \frac{1}{2} (r^{2H} + t^{2H} - |t - r|^{2H}) - \frac{1}{2} (r^{2H} + s^{2H} - |s - r|^{2H}) \\ &= \frac{1}{2} (t^{2H} - s^{2H}) + \frac{1}{2} (|s - r|^{2H} - |t - r|^{2H}). \end{aligned}$$

We have $|b^{2H} - a^{2H}| \leq |b - a|^{2H}$ for any $a, b \in [0, 1]$, because $H < \frac{1}{2}$. The desired result follows. \blacksquare

In the sequel, we will also make use of the following hypothesis on $f : \mathbb{R} \rightarrow \mathbb{R}$:

(H) f belongs to \mathcal{C}^6 and $|f^{(6)}(x)| \leq ke^{cx^2}$ for some constants $k > 0$ and $0 < c < \frac{1}{12}$.

Notice that **(H)** implies

$$\sup_{t \in [0, 1]} E(|f^{(i)}(B_t)|^p) < \infty,$$

for all $1 \leq p \leq 6$, and $i = 1, \dots, 6$. The following result holds.

Theorem 4.3 *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfy **(H)**. Suppose that $H \in (\frac{1}{4}, \frac{1}{2})$. We have*

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(B_{k/n}) (n^{2H} (\Delta B_{k/n})^2 - 1) \xrightarrow{\text{Stably}} \sigma_H \int_0^1 f(B_s) dW_s, \quad (4.16)$$

as n tends to infinity, where W is a Brownian motion independent of B , and

$$\sigma_H = \sqrt{\frac{1}{2} \sum_{p=-\infty}^{\infty} (|p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H})^2}. \quad (4.17)$$

Proof We can write

$$G_n = n^{2H - \frac{1}{2}} \sum_{k=0}^{n-1} f(B_{k/n}) I_2(\partial_{k/n}^{\otimes 2}), \quad n \geq 1.$$

The proof will be done in several steps. We denote by C a generic constant that may vary from one formula to another one.

Step 1. We are going to decompose G_n as the sum of a sequence of the form $\delta^2(u_n)$ plus other terms which converge to zero in L^2 as n tends to zero. To do this we write,

using (2.11) repeatedly,

$$\begin{aligned}
f(B_{k/n})I_2(\partial_{k/n}^{\otimes 2}) &= f(B_{k/n})\delta(\Delta B_{k/n}\partial_{k/n}) \\
&= \delta(f(B_{k/n})\Delta B_{k/n}\partial_{k/n}) + f'(B_{k/n})\Delta B_{k/n} \langle \varepsilon_{k/n}, \partial_{k/n} \rangle_{\mathfrak{H}} \\
&= \delta(f(B_{k/n})\delta(\partial_{k/n})\partial_{k/n}) + f'(B_{k/n})\Delta B_{k/n} \langle \varepsilon_{k/n}, \partial_{k/n} \rangle_{\mathfrak{H}} \\
&= \delta^2(f(B_{k/n})\partial_{k/n}^{\otimes 2}) + \delta(f'(B_{k/n})\partial_{k/n}) \langle \varepsilon_{k/n}, \partial_{k/n} \rangle_{\mathfrak{H}} \\
&\quad + f'(B_{k/n})\Delta B_{k/n} \langle \varepsilon_{k/n}, \partial_{k/n} \rangle_{\mathfrak{H}} \\
&= \delta^2(f(B_{k/n})\partial_{k/n}^{\otimes 2}) + 2f'(B_{k/n})\Delta B_{k/n} \langle \varepsilon_{k/n}, \partial_{k/n} \rangle_{\mathfrak{H}} \\
&\quad - f''(B_{k/n}) \langle \varepsilon_{k/n}, \partial_{k/n} \rangle_{\mathfrak{H}}^2,
\end{aligned}$$

where $\varepsilon_{k/n} = \mathbf{1}_{[0, k/n]}$. As a consequence,

$$G_n = \delta^2(u_n) - A_n - B_n,$$

where

$$u_n = n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} f(B_{k/n})\partial_{k/n}^{\otimes 2},$$

$$A_n = 2n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} f'(B_{k/n})\Delta B_{k/n} \langle \varepsilon_{k/n}, \partial_{k/n} \rangle_{\mathfrak{H}},$$

and

$$B_n = -n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} f''(B_{k/n}) \langle \varepsilon_{k/n}, \partial_{k/n} \rangle_{\mathfrak{H}}^2.$$

Notice that

$$E[A_n^2] = 4n^{4H-1} \sum_{k,l=0}^{n-1} E[f'(B_{k/n})f'(B_{l/n})\Delta B_{k/n}\Delta B_{l/n}] \langle \varepsilon_{k/n}, \partial_{k/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{l/n}, \partial_{l/n} \rangle_{\mathfrak{H}},$$

which converges to zero as n tends to infinity. Indeed, using the notation

$$\alpha_{k,l} = \langle \varepsilon_{k/n}, \partial_{l/n} \rangle_{\mathfrak{H}},$$

and

$$\beta_{k,l} = \langle \partial_{k/n}, \partial_{l/n} \rangle_{\mathfrak{H}},$$

we can write

$$\begin{aligned}
E[A_n^2] &= 4n^{4H-1} \sum_{k,l=0}^{n-1} E[f'(B_{k/n})f'(B_{l/n})] \beta_{k,l} \alpha_{k,k} \alpha_{l,l} \\
&\quad + 4n^{4H-1} \sum_{k,l=0}^{n-1} E[f''(B_{k/n})f''(B_{l/n})] \alpha_{k,l} \alpha_{l,k} \alpha_{k,k} \alpha_{l,l} \\
&\quad + 4n^{4H-1} \sum_{k,l=0}^{n-1} E[f''(B_{k/n})f''(B_{l/n})] \alpha_{k,l}^2 \alpha_{l,l}^2 \\
&\quad + 4n^{4H-1} \sum_{k,l=0}^{n-1} E[f'''(B_{k/n})f'(B_{l/n})] \alpha_{k,k}^2 \alpha_{l,l} \alpha_{k,l} \\
&\quad + 4n^{4H-1} \sum_{k,l=0}^{n-1} E[f'(B_{k/n})f'''(B_{l/n})] \alpha_{l,l}^2 \alpha_{k,k} \alpha_{l,k}.
\end{aligned}$$

Hence, by Lemma 4.2 and hypothesis **(H)**:

$$E[A_n^2] \leq Cn^{-1} \sum_{k,l=0}^{n-1} |\beta_{k,l}| + Cn^{1-4H},$$

which converges to zero as n tends to infinity, because we have

$$\begin{aligned}
n^{-1} \sum_{k,l=0}^{n-1} |\beta_{k,l}| &= \frac{n^{-1-2H}}{2} \sum_{k,l=0}^{n-1} \left| |k-l+1|^{2H} + |k-l-1|^{2H} - 2|k-l|^{2H} \right| \\
&\leq Cn^{-2H} \sum_{p=-\infty}^{\infty} \left| |p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H} \right|,
\end{aligned}$$

and this series is convergent.

On the other hand, still by Lemma 4.2 and hypothesis **(H)**:

$$E[B_n^2] \leq Cn^{1-4H},$$

which also converges to zero as n tends to infinity.

Step 2. It suffices to show that the sequence $F_n = \delta^2(u_n)$ satisfies the conditions of Theorem 1.5 with $q = 2$. Let us show first that F_n is bounded in L^2 . This follows from the fact that u_n is bounded in $\mathbb{D}^{2,2}$. In fact, we have

$$\begin{aligned}
Du_n &= n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} f'(B_{k/n}) \varepsilon_{k/n} \otimes \partial_{k/n}^{\otimes 2}, \\
D^2u_n &= n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} f''(B_{k/n}) \varepsilon_{k/n}^{\otimes 2} \otimes \partial_{k/n}^{\otimes 2},
\end{aligned}$$

and

$$\begin{aligned}
E \left(\|Du_n\|_{\mathfrak{H}^{\otimes 3}}^2 \right) &= n^{4H-1} \sum_{k,l=0}^{n-1} E \left(f'(B_{k/n}) f'(B_{l/n}) \langle \varepsilon_{k/n}, \varepsilon_{l/n} \rangle \langle \partial_{k/n}, \partial_{l/n} \rangle^2 \right) \\
&\leq C n^{4H-1} \sum_{k,l=0}^{n-1} \langle \partial_{k/n}, \partial_{k/n} \rangle^2 \\
&\leq C \sum_{p=-\infty}^{\infty} (|p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H})^2 < \infty.
\end{aligned}$$

In the same way, we can show that $\sup_n E \left(\|D^2u_n\|_{\mathfrak{H}^{\otimes 4}}^2 \right) < \infty$.

Step 3. Let us show condition (i) (see also Remark 1.6). Notice that it suffices to show this condition for elements f_j belonging to a total subset of \mathfrak{H} . Taking $f_1 = \mathbf{1}_{[0,t]}$, and $f_2 = \mathbf{1}_{[0,s]}$ we have

$$\begin{aligned}
E \left(\left| \langle u_n, f_1 \otimes f_2 \rangle_{\mathfrak{H}^{\otimes 2}} \right| \right) &\leq n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} E \left(|f(B_{k/n})| \left| \langle \partial_{k/n}, f_1 \rangle_{\mathfrak{H}} \langle \partial_{k/n}, f_2 \rangle_{\mathfrak{H}} \right| \right) \\
&\leq C n^{\frac{1}{2}-2H}, \quad \text{by Lemma 4.2,}
\end{aligned}$$

which converges to zero as n tends to infinity. On the other hand,

$$\begin{aligned}
E \left| \langle u_n, f_1 \otimes DF_n \rangle_{\mathfrak{H}^{\otimes 2}} \right| &\leq n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} E \left(\left| f(B_{k/n}) \langle \partial_{k/n}, DF_n \rangle_{\mathfrak{H}} \right| \left| \langle \partial_{k/n}, f_1 \rangle_{\mathfrak{H}} \right| \right) \\
&\leq C n^{\frac{1}{2}} \sup_{k=0, \dots, n-1} \left[E \left(\langle \partial_{k/n}, DF_n \rangle_{\mathfrak{H}}^2 \right) \right]^{\frac{1}{2}}, \quad (4.18)
\end{aligned}$$

and

$$E \left(\left| \langle u_n, DF_n \otimes DF_n \rangle_{\mathfrak{H}^{\otimes 2}} \right| \right) \leq n^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} E \left((1 + f^2(B_{k/n})) \langle \partial_{k/n}, DF_n \rangle_{\mathfrak{H}}^2 \right), \quad (4.19)$$

because $|x| \leq 1 + x^2$. Set $H_{k/n} = 1 + f^2(B_{k/n})$. We claim that

$$E \left(H_{k/n} \langle DF_n, \partial_{k/n} \rangle_H^2 \right) \leq C n^{-4H}, \quad (4.20)$$

and this would imply that (4.18) and (4.19) tend to zero as n tends to infinity. Using $D\delta(u) = u + \delta(Du)$ repeatedly, we have

$$\langle DF_n, \partial_{k/n} \rangle_{\mathfrak{H}} = 2\delta(\langle u_n, \partial_{k/n} \rangle_{\mathfrak{H}}) + \delta^2(D \langle u_n, \partial_{k/n} \rangle_{\mathfrak{H}}).$$

We need to show that

$$E \left(H_{k/n} \left[\delta(\langle u_n, \partial_{k/n} \rangle_{\mathfrak{H}}) \right]^2 \right) \leq Cn^{-4H}, \quad (4.21)$$

$$E \left(H_{k/n} \left[\delta^2(D \langle u_n, \partial_{k/n} \rangle_{\mathfrak{H}}) \right]^2 \right) \leq Cn^{-4H}. \quad (4.22)$$

We will only prove (4.21), the proof of (4.22) being similar. Set

$$\begin{aligned} \mu_{k,n} &= E \left(H_{k/n} \left[\delta(\langle u_n, \partial_{k/n} \rangle) \right]^2 \right), \\ \nu_{k,n} &= \langle u_n, \partial_{k/n} \rangle_{\mathfrak{H}}. \end{aligned}$$

Applying (2.11) yields

$$\begin{aligned} \mu_{k,n} &= 2 \langle \nu_{k,n} \otimes DH_{k/n}, D\nu_{k,n} \rangle_{\mathfrak{H}^{\otimes 2}} + \langle \nu_{k,n} \otimes \nu_{k,n}, D^2 H_{k/n} \rangle_{\mathfrak{H}^{\otimes 2}} \\ &\quad + H_{k/n} \langle \nu_{k,n}, \nu_{k,n} \rangle_{\mathfrak{H}} + H_{k/n} \text{Tr} \left(D\nu_{k,n} \circ (D\nu_{k,n})^T \right). \end{aligned}$$

This implies

$$\begin{aligned} \mu_{k,n} &= 2n^{4H-1} \sum_{h,l=0}^{n-1} E \left(\langle DH_{k/n}, \partial_{l/n} \rangle f(B_{h/n}) f'(B_{l/n}) \beta_{k,l} \beta_{k,h} \alpha_{l,h} \right) \\ &\quad + n^{4H-1} \sum_{h,l=0}^{n-1} E \left(f(B_{h/n}) f(B_{l/n}) \beta_{k,l} \beta_{k,h} \langle \partial_{h/n} \otimes \partial_{l/n}, D^2 H_{k/n} \rangle_{\mathfrak{H}^{\otimes 2}} \right) \\ &\quad + n^{4H-1} \sum_{h,l=0}^{n-1} E \left(H_{k/n} f(B_{h/n}) f(B_{l/n}) \right) \beta_{k,l} \beta_{k,h} \beta_{h,l} \\ &\quad + n^{4H-1} \sum_{h,l=0}^{n-1} E \left(H_{k/n} f'(B_{h/n}) f'(B_{l/n}) \right) \beta_{k,l} \beta_{k,h} \alpha_{l,h} \alpha_{h,l}. \end{aligned}$$

which converges to zero as n tends to infinity.

Step 5. Let us show condition (ii). Using $D\delta(u) = u + \delta(Du)$ repeatedly, we have

$$D^2 F_n = 2u_n + 4\delta(Du_n) + \delta^2(D^2 u_n).$$

Hence,

$$\begin{aligned} \langle u_n, D^2 F_n \rangle_{\mathfrak{H}^{\otimes 2}} &= 2n^{4H-1} \sum_{k,l=0}^{n-1} f(B_{k/n}) f(B_{l/n}) \langle \partial_{k/n}, \partial_{l/n} \rangle_{\mathfrak{H}}^2 \\ &\quad + 4 \langle u_n, \delta(Du_n) \rangle_{\mathfrak{H}^{\otimes 2}} + \langle u_n, \delta^2(D^2 u_n) \rangle_{\mathfrak{H}^{\otimes 2}}. \end{aligned}$$

The first term converges to a nonnegative square integrable random variable:

$$\begin{aligned}
& 2n^{4H-1} \sum_{k,l=0}^{n-1} f(B_{k/n}) f(B_{l/n}) \langle \partial_{k/n}, \partial_{l/n} \rangle_{\mathfrak{H}}^2 \\
&= \frac{1}{2n} \sum_{k,l=0}^{n-1} f(B_{k/n}) f(B_{l/n}) (|k-l+1|^{2H} + |k-l-1|^{2H} - 2|k-l|^{2H})^2 \\
&= \frac{1}{2n} \sum_{p=-\infty}^{\infty} \sum_{j=0 \vee -p}^{(n-1) \wedge (n-1-p)} f(B_{j/n}) f(B_{(j+p)/n}) (|p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H})^2,
\end{aligned}$$

which converges to

$$\frac{1}{2} \left(\sum_{p=-\infty}^{\infty} (|p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H})^2 \right) \int_0^1 f(B_s)^2 ds.$$

The remaining terms converges to zero. We will discuss only the first one:

$$\begin{aligned}
\langle u_n, \delta(Du_n) \rangle_{\mathfrak{H}^{\otimes 2}} &= n^{4H-1} \sum_{k,l=0}^{n-1} \delta(f(B_{k/n}) f'(B_{l/n}) \partial_{l/n}) \langle \partial_{k/n}, \partial_{l/n} \rangle_{\mathfrak{H}} \langle \partial_{k/n}, \varepsilon_{l/n} \rangle_{\mathfrak{H}} \\
&= n^{4H-1} \sum_{k,l=0}^{n-1} f(B_{k/n}) f'(B_{l/n}) \Delta B_{l,n} \beta_{k,l} \alpha_{l,k} \\
&\quad - n^{4H-1} \sum_{k,l=0}^{n-1} f'(B_{k/n}) f'(B_{l/n}) \beta_{k,l}^2 \alpha_{l,k} \\
&\quad - n^{4H-1} \sum_{k,l=0}^{n-1} f'(B_{k/n}) f''(B_{l/n}) \beta_{l,k} \beta_{l,t} \alpha_{l,k},
\end{aligned}$$

and this tends to zero as n tends to infinity. ■

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