



HAL
open science

Convergence in law for certain weighted quadratic variations of fractional Brownian motion

Ivan Nourdin, David Nualart

► **To cite this version:**

Ivan Nourdin, David Nualart. Convergence in law for certain weighted quadratic variations of fractional Brownian motion. 2007. hal-00164821v2

HAL Id: hal-00164821

<https://hal.science/hal-00164821v2>

Preprint submitted on 20 Sep 2007 (v2), last revised 2 Sep 2009 (v5)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Convergence in law for certain weighted quadratic variations of fractional Brownian motion

Ivan Nourdin¹ and David Nualart²

Abstract: By means of Malliavin calculus, we prove the convergence in law for certain weighted quadratic variations of a fractional Brownian motion B with Hurst index $H \in (\frac{1}{4}, \frac{1}{2}]$.

Key words: Fractional Brownian motion - weighted quadratic variations - Malliavin calculus - weak convergence.

Actual version: September 2007

1 Introduction

The study of single path behavior of stochastic processes is often based on the study of their power variations and there exists a very extensive literature on the subject. Recall that, a real $\kappa > 1$ being given, the κ -power variation of a stochastic process X , with respect to a subdivision $\pi_n = \{0 = t_{n,0} < t_{n,1} < \dots < t_{n,n} = 1\}$ of $[0, 1]$, is defined to be the sum

$$\sum_{k=0}^{n-1} |X_{t_{n,k+1}} - X_{t_{n,k}}|^{\kappa}.$$

For simplicity, consider from now on the case where $t_{n,k} = k/n$, for $n \in \mathbb{N}$ and $k \in \{0, \dots, n\}$. In the present paper we wish to point out some interesting phenomena when $X = B$ is a fractional Brownian motion of Hurst index $H \in (0, 1)$ and when the value of κ is 2 (or, more generally, *even*) or 3 (or, more generally, *odd*). In fact, we will also drop the absolute value (when κ is odd) and we will introduce some weights. More precisely, we will consider

$$\sum_{k=0}^{n-1} f(B_{k/n})(\Delta B_{k/n})^{\kappa}, \quad \kappa \in \{2, 3, 4, \dots\}, \quad (1.1)$$

where the function $f : \mathbb{R} \rightarrow \mathbb{R}$ is assumed smooth enough and where $\Delta B_{k/n}$ denotes the increment $B_{(k+1)/n} - B_{k/n}$.

The analysis of the asymptotic behavior of quantities of type (1.1) is first motivated by the study of the exact rate of convergence of some approximation schemes of scalar

¹Université Pierre et Marie Curie Paris VI, Laboratoire de Probabilités et Modèles Aléatoires, Boîte courrier 188, 4 place Jussieu, 75252 Paris Cedex 05, France, nourdin@ccr.jussieu.fr

²Department of Mathematics, University of Kansas, 405 Snow Hall, Lawrence, Kansas 66045-2142, nualart@math.ku.edu

stochastic differential equations driven by B (see [5], [10] and [12]). On the other hand, the results and techniques developed in this paper might be applied to establish Itô's type formulas in law for the solution to the stochastic heat equation (see Swanson [14]).

Now, let us recall some known results concerning κ -power variations (for $\kappa = 2, 3, 4, \dots$), which are today more or less classical. First, assume that the Hurst index is $H = 1/2$, that is B is a standard Brownian motion. Let μ_κ denote the κ -moment of a standard Gaussian random variable $G \sim \mathcal{N}(0, 1)$. By the scaling property of the fractional Brownian motion and using the central limit theorem, it is immediate that, as $n \rightarrow \infty$:

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} [n^{\kappa/2} (\Delta B_{k/n})^\kappa - \mu_\kappa] \xrightarrow{\text{Law}} \mathcal{N}(0, \mu_{2\kappa} - \mu_\kappa^2). \quad (1.2)$$

When weights are introduced, an interesting phenomenon appears: instead of Gaussian random variables, we rather obtain mixing random variables as limit in (1.2). Indeed, when κ is even, it is a very particular case of a more general result by Jacod [8] that we have, as $n \rightarrow \infty$:

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(B_{k/n}) [n^{\kappa/2} (\Delta B_{k/n})^\kappa - \mu_\kappa] \xrightarrow{\text{Law}} \sqrt{\mu_{2\kappa} - \mu_\kappa^2} \int_0^1 f(B_s) dW_s. \quad (1.3)$$

Here, W denotes another standard Brownian motion, independent of B . When κ is odd, we have, this time, as $n \rightarrow \infty$:

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(B_{k/n}) [n^{\kappa/2} (\Delta B_{k/n})^\kappa] \xrightarrow{\text{Law}} \int_0^1 f(B_s) (\sqrt{\mu_{2\kappa}} dW_s + \mu_{\kappa+1} dB_s), \quad (1.4)$$

see [5].

Second, assume that $H \neq 1/2$, that is the case where the fractional Brownian motion B has not independent increments anymore. Then (1.2) has been extended by Breuer and Major [1] and Giraitis and Surgailis [4], and two cases are considered according to the evenness of κ :

- if κ is even and if $H \in (0, 3/4)$, as $n \rightarrow \infty$,

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} [n^{\kappa H} (\Delta B_{k/n})^\kappa - \mu_\kappa] \xrightarrow{\text{Law}} \mathcal{N}(0, \sigma_{H,\kappa}^2); \quad (1.5)$$

- if κ is odd and if $H \in (0, 1/2]$, as $n \rightarrow \infty$,

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} n^{\kappa H} (\Delta B_{k/n})^\kappa \xrightarrow{\text{Law}} \mathcal{N}(0, \sigma_{H,\kappa}^2). \quad (1.6)$$

Here $\sigma_{H,\kappa} > 0$ is a constant depending only on H and κ , which can be computed explicitly. In fact, one can relax the conditions made on H in (1.5)–(1.6): in this case, the normalizations are not the same anymore and, for (1.5), one obtains limits which are not Gaussian but the value at time one of the Rosenblatt process (see Taqqu [15] or Dobrushin and Major [3]).

Now, let us proceed with the results concerning the *weighted* power variations in the case where $H \neq 1/2$. In what follows, f denotes a continuously differentiable function such that f and f' have subexponential growth. If κ is even and $H \in (1/2, 3/4)$, then by Theorem 2 in León and Ludeña [9] (see also Corcuera *et al* [2] for related results on the asymptotic behavior of the p -variation of stochastic integrals with respect to B) we have, as $n \rightarrow \infty$:

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(B_{k/n}) [n^{\kappa H} (\Delta B_{k/n})^\kappa - \mu_\kappa] \xrightarrow{\text{Law}} \sigma_{H,\kappa} \int_0^1 f(B_s) dW_s, \quad (1.7)$$

where, once again, W denotes a standard Brownian motion independent of B . Thus, (1.7) shows for (1.1) a similar behavior to that observed in the standard Brownian case, compare with (1.3). In contradistinction, the asymptotic behavior of (1.1) can be completely different of (1.3) or (1.7) when $H \in (0, 1/2)$. The first result in this direction has been observed by Gradinaru *et al* [6] and continued by Gradinaru and Nourdin [5]. Namely, if κ is odd and $H \in (0, 1/2)$, we have, as $n \rightarrow \infty$:

$$n^{H-1} \sum_{k=0}^{n-1} f(B_{k/n}) [n^{\kappa H} (\Delta B_{k/n})^\kappa] \xrightarrow{L^2} -\frac{\mu_{\kappa+1}}{2} \int_0^1 f'(B_s) ds. \quad (1.8)$$

Also, when $\kappa = 2$ and $H \in (0, \frac{1}{4})$, Nourdin [11] proved that we have, as $n \rightarrow \infty$:

$$n^{2H-1} \sum_{k=0}^{n-1} f(B_{k/n}) (n^{2H} (\Delta B_{k/n})^2 - 1) \xrightarrow{L^2} \frac{1}{4} \int_0^1 f''(B_s) ds. \quad (1.9)$$

What happens when $\kappa = 2$ and $H \in [\frac{1}{4}, \frac{1}{2}) \cup [\frac{3}{4}, 1)$, or when $\kappa = 3$ and $H \in (\frac{1}{2}, 1)$? This question is natural in view of the previous convergences. The aim of this note is to answer it when $\kappa = 2$ and $H \in (\frac{1}{4}, \frac{1}{2})$. The case $\kappa = 2$ and $H = \frac{1}{4}$ seems to be more difficult, and other cases such as $\kappa = 3$ and $H \in (\frac{1}{2}, 1)$, or $\kappa = 2$ and $H \in [\frac{3}{4}, 1)$ require different approaches and will be considered in a forthcoming work.

As we said, the main result of this note is to prove that (1.7) is still true when $\kappa = 2$ and $H \in (\frac{1}{4}, \frac{1}{2}]$. To prove this result we introduce an original approach, based on Malliavin calculus, which is completely different from the method used by León and Ludeña in [9]. Let us outline the main ideas of this method. Let

$$G_n(f) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(B_{k/n}) (n^{2H} (\Delta B_{k/n})^2 - 1), \quad n \geq 1,$$

and assume that $H \in (\frac{1}{4}, \frac{1}{2}]$. By means of Malliavin calculus, we will prove, as $n \rightarrow \infty$:

$$E(G_n(f)) \longrightarrow 0 \quad \text{and} \quad E(G_n^2(f)) \longrightarrow \sigma_H^2 \int_0^1 E(f^2(B_s)) ds.$$

In particular, the sequence $(G_n, (B_t)_{t \in [0,1]})$ is tight in $\mathbb{R} \times \mathcal{C}([0,1])$. Let $(G_\infty, (B_t)_{t \in [0,1]})$ denote the limit in law of a certain subsequence of $(G_n, (B_t)_{t \in [0,1]})$. We will show that the conditional characteristic function of G_∞ solves the following linear differential equation:

$$\frac{\partial}{\partial \lambda} E(e^{i\lambda G_\infty} | (B_t)_{t \in [0,1]}) = -\frac{\lambda}{2} \sigma_H^2 \int_0^1 f^2(B_s) ds E(e^{i\lambda G_\infty} | (B_t)_{t \in [0,1]}).$$

The main ingredient in establishing this equation is the study of the asymptotic behavior of $iE(G_n e^{i\lambda G_n} H)$, as n tends to infinity, where H is a smooth and cylindrical functional of the fractional Brownian motion. The limit of this expectation is studied again by means of the duality relationship of Malliavin calculus. Consequently, we deduce

$$E(e^{i\lambda G_\infty} | (B_t)_{t \in [0,1]}) = \exp\left(-\frac{\sigma_H^2 \lambda^2}{2} \int_0^1 f^2(B_s) ds\right),$$

and the conclusion follows.

Our approach could be applied to other types of weighted Riemann sums, replacing $(\Delta B_{k/n})^2$ by higher powers of the increment. For instance, when $H \in (0, \frac{1}{6})$ and $f \in \mathcal{C}_b^3$, one proves in [11] that

$$n^{3H-1} \sum_{k=0}^{n-1} \left(f(B_{k/n}) n^{3H} (\Delta B_{k/n})^3 + \frac{3}{2} f'(B_{k/n}) n^{-H} \right) \xrightarrow{L^2} -\frac{1}{8} \int_0^1 f'''(B_s) ds,$$

as $n \rightarrow \infty$. Using the ideas contained in the current paper, we could prove that the following convergence holds, when $H \in (\frac{1}{6}, \frac{1}{2})$:

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left(f(B_{k/n}) n^{3H} (\Delta B_{k/n})^3 + \frac{3}{2} f'(B_{k/n}) n^{-H} \right) \implies \hat{\sigma}_H \int_0^1 f(B_s) dW_s,$$

as $n \rightarrow \infty$, for a certain constant $\hat{\sigma}_H$ and a standard Brownian motion W independent of B .

The paper is organized as follows. In section 2, we introduce some notations and preliminary results. In section 3, we state and prove the main result of the paper.

2 Preliminaries and notations

We begin by briefly recalling some basic facts about stochastic calculus with respect to a fractional Brownian motion. We refer to [13] for further details. Let $B = (B_t)_{t \in [0,1]}$ be a fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2})$ defined on a probability space

(Ω, \mathcal{A}, P) . We mean that B is a centered Gaussian process with the covariance function $E(B_s B_t) = R_H(s, t)$, where

$$R_H(s, t) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}).$$

We denote by \mathcal{E} the set of step \mathbb{R} -valued functions on $[0, 1]$. Let \mathfrak{H} be the Hilbert space defined as the closure of \mathcal{E} with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathfrak{H}} = R_H(t, s).$$

We denote by $|\cdot|_{\mathfrak{H}}$ the associate norm. The mapping $\mathbf{1}_{[0,t]} \mapsto B_t$ can be extended to an isometry between \mathfrak{H} and the Gaussian space \mathcal{H}_1 associated with B . We denote this isometry by $\varphi \mapsto B(\varphi)$. The space \mathfrak{H} coincides with $I_{0+}^{H-\frac{1}{2}}(L^2([0, 1]))$, where

$$I_{0+}^{H-\frac{1}{2}} f(x) \triangleq \frac{1}{\Gamma(H-\frac{1}{2})} \int_0^x (x-y)^{H-\frac{3}{2}} f(y) dy.$$

Let \mathcal{S} be the set of all smooth cylindrical random variables, *i.e.* of the form

$$F = \phi(B_{t_1}, \dots, B_{t_m})$$

where $m \geq 1$, $\phi : \mathbb{R}^m \rightarrow \mathbb{R} \in \mathcal{C}_b^\infty$ and $0 \leq t_1 < \dots < t_m \leq 1$. The Malliavin derivative of F with respect to B is the element of $L^2(\Omega, \mathfrak{H})$ defined by

$$D_s F = \sum_{i=1}^m \frac{\partial \phi}{\partial x_i}(B_{t_1}, \dots, B_{t_m}) \mathbf{1}_{[0,t_i]}(s), \quad s \in [0, 1].$$

In particular $D_s B_t = \mathbf{1}_{[0,t]}(s)$. For any integer $k \geq 1$, we denote by $\mathbb{D}^{k,2}$ the closure of the set of smooth random variables with respect to the norm

$$\|F\|_{k,2}^2 = E[F^2] + \sum_{j=1}^k E[|D^j F|_{\mathfrak{H}^{\otimes j}}^2].$$

The Malliavin derivative D verifies the chain rule: if $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ is \mathcal{C}_b^1 and if $(F_i)_{i=1,\dots,n}$ is a sequence of elements of $\mathbb{D}^{1,2}$ then $\varphi(F_1, \dots, F_n) \in \mathbb{D}^{1,2}$ and we have, for any $s \in [0, 1]$:

$$D_s \varphi(F_1, \dots, F_n) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F_1, \dots, F_n) D_s F_i.$$

The divergence operator $I = I_1$ is the adjoint of the derivative operator D . If a random variable $u \in L^2(\Omega, \mathfrak{H})$ belongs to the domain of the divergence operator, that is if it verifies

$$|E\langle DF, u \rangle_{\mathfrak{H}}| \leq c_u \|F\|_{L^2} \quad \text{for any } F \in \mathcal{S},$$

then $I(u)$ is defined by the duality relationship

$$E(FI(u)) = E(\langle DF, u \rangle_{\mathfrak{H}}),$$

for every $F \in \mathbb{D}^{1,2}$.

For every $n \geq 1$, let \mathcal{H}_n be the n th Wiener chaos of B , that is, the closed linear subspace of $L^2(\Omega, \mathcal{A}, P)$ generated by the random variables $\{H_n(B(h)), h \in H, |h|_{\mathfrak{H}} = 1\}$, where H_n is the n th Hermite polynomial. The mapping $I_n(h^{\otimes n}) = n!H_n(B(h))$ provides a linear isometry between the symmetric tensor product $\mathfrak{H}^{\otimes n}$ and \mathcal{H}_n . For $H = \frac{1}{2}$, I_n coincides with the multiple stochastic integral. The following duality formula holds

$$E(FI_n(h)) = E(\langle D^n F, h \rangle_{\mathfrak{H}^{\otimes n}}), \quad (2.10)$$

for any element $h \in \mathfrak{H}^{\otimes n}$ and any random variable $F \in \mathbb{D}^{n,2}$.

3 Main result

In the sequel, we will make use of the following hypothesis on $f : \mathbb{R} \rightarrow \mathbb{R}$:

(H) f belongs to \mathcal{C}^6 and $|f^{(6)}(x)| \leq ke^{cx^2}$ for some constants $k > 0$ and $0 < c < \frac{1}{12}$.

Notice that **(H)** implies

$$\sup_{t \in [0,1]} E(|f^{(i)}(B_t)|^p) < \infty,$$

for all $1 \leq p \leq 6$, and $i = 1, \dots, 6$. We also note G_n instead of $G_n(f)$ for simplicity. Notice that $n^{2H}(\Delta B_{k/n})^2 - 1 = n^{2H}I_2(\delta_{k/n}^{\otimes 2})$, where $\delta_{k/n} = \mathbf{1}_{[k/n, (k+1)/n]}$, and I_2 denotes the double stochastic integral. We will make use of the following technical lemma.

Lemma 1 *Let $s < t$ and r belong to $[0, 1]$. Then*

$$|E(B_r(B_t - B_s))| \leq (t - s)^{2H}.$$

Proof. We have

$$\begin{aligned} E(B_r(B_t - B_s)) &= \frac{1}{2}(r^{2H} + t^{2H} - |t - r|^{2H}) \\ &\quad - \frac{1}{2}(r^{2H} + s^{2H} - |s - r|^{2H}) \\ &= \frac{1}{2}(t^{2H} - s^{2H}) + \frac{1}{2}(|s - r|^{2H} - |t - r|^{2H}). \end{aligned}$$

We have $|b^{2H} - a^{2H}| \leq |b - a|^{2H}$ for any $a, b \in [0, 1]$, because $H < \frac{1}{2}$. The desired result follows. ■

Our aim is to show the following result.

Theorem 2 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfy **(H)**. We have

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(B_{k/n}) (n^{2H} (\Delta B_{k/n})^2 - 1) \implies \sigma_H \int_0^1 f(B_s) dW_s, \quad \text{as } n \rightarrow \infty, \quad (3.11)$$

where W is a Brownian motion independent of B . Moreover $\sigma_{\frac{1}{2}} = \sqrt{2}$ while, for $H \in (\frac{1}{4}, \frac{1}{2})$:

$$\sigma_H = \sqrt{\frac{1}{2} \sum_{p=-\infty}^{\infty} (|p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H})^2}. \quad (3.12)$$

Proof. The proof will be done in several steps. We will denote by C a generic constant that can be different from line to line.

Step 1.- We will first show that $\lim_{n \rightarrow \infty} E(G_n) = 0$. In the case $H = \frac{1}{2}$ we actually have $E(G_n) = 0$. Suppose $H < \frac{1}{2}$. By the duality formula between the iterated derivative and double stochastic integrals we have

$$\begin{aligned} E \left[f(B_{k/n}) I_2(\delta_{k/n}^{\otimes 2}) \right] &= \left\langle E(D^2 f(B_{k/n})), \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} \\ &= E(f''(B_{k/n})) \left\langle \varepsilon_{k/n}^{\otimes 2}, \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}}, \end{aligned}$$

where $\varepsilon_{k/n} = \mathbf{1}_{[0, k/n]}$. Hence, by Lemma 1

$$\begin{aligned} |E(G_n)| &\leq n^{-\frac{1}{2}+2H} \sum_{k=0}^{n-1} |E(f''(B_{k/n}))| \left\langle \varepsilon_{k/n}, \delta_{k/n} \right\rangle_{\mathfrak{H}}^2 \\ &\leq C n^{\frac{1}{2}-2H}, \end{aligned}$$

which converges to zero as n tends to infinity because $H > \frac{1}{4}$.

Step 2.- We claim that $\lim_{n \rightarrow \infty} E(G_n^2) = \sigma_H^2 \int_0^1 E(f(B_s)^2) ds$. We have

$$E(G_n^2) = n^{4H-1} \sum_{j,k=0}^{n-1} E \left(f(B_{j/n}) I_2(\delta_{j/n}^{\otimes 2}) f(B_{k/n}) I_2(\delta_{k/n}^{\otimes 2}) \right).$$

The product formula for multiple stochastic integrals yields

$$\begin{aligned} I_2(\delta_{j/n}^{\otimes 2}) I_2(\delta_{k/n}^{\otimes 2}) &= \sum_{r=0}^2 r! \binom{2}{r}^2 I_{4-2r}(\delta_{j/n}^{\otimes 2} \otimes_r \delta_{k/n}^{\otimes 2}) \\ &= I_4(\delta_{j/n}^{\otimes 2} \tilde{\otimes} \delta_{k/n}^{\otimes 2}) + 4I_2(\delta_{j/n} \tilde{\otimes} \delta_{k/n}) \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\ &\quad + 2 \left\langle \delta_{j/n}^{\otimes 2}, \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}}, \end{aligned}$$

and, as a consequence,

$$\begin{aligned}
E(G_n^2) &= n^{4H-1} \sum_{j,k=0}^{n-1} E \left[f(B_{j/n}) f(B_{k/n}) I_4(\delta_{j/n}^{\otimes 2} \tilde{\otimes} \delta_{k/n}^{\otimes 2}) \right] \\
&\quad + 4n^{4H-1} \sum_{j,k=0}^{n-1} E \left[f(B_{j/n}) f(B_{k/n}) I_2(\delta_{j/n} \tilde{\otimes} \delta_{k/n}) \right] \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\
&\quad + 2n^{4H-1} \sum_{j,k=0}^{n-1} E \left[f(B_{j/n}) f(B_{k/n}) \right] \langle \delta_{j/n}^{\otimes 2}, \delta_{k/n}^{\otimes 2} \rangle_{\mathfrak{H}^{\otimes 2}} \\
&= A_n + B_n + C_n.
\end{aligned}$$

The term A_n makes no contribution in the limit as n tends to infinity because by the duality formula (2.10) we have

$$\begin{aligned}
A_n &= n^{4H-1} \sum_{j,k=0}^{n-1} \left\langle E \left[D^4 (f(B_{j/n}) f(B_{k/n})) \right], \delta_{j/n}^{\otimes 2} \tilde{\otimes} \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 4}} \\
&= 24n^{4H-1} \sum_{j,k=0}^{n-1} \sum_{a+b=4} E (f^{(a)}(B_{j/n}) f^{(b)}(B_{k/n})) \left\langle \varepsilon_{j/n}^{\otimes a} \tilde{\otimes} \varepsilon_{k/n}^{\otimes b}, \delta_{j/n}^{\otimes 2} \tilde{\otimes} \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 4}},
\end{aligned}$$

and by Lemma 1 we obtain

$$|A_n| \leq Cn^{1-4H}.$$

In the same way,

$$\begin{aligned}
B_n &= 4n^{4H-1} \sum_{j,k=0}^{n-1} E \left[\left\langle D^2 (f(B_{j/n}) f(B_{k/n})), \delta_{j/n} \tilde{\otimes} \delta_{k/n} \right\rangle_{\mathfrak{H}^{\otimes 2}} \right] \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\
&= 8n^{4H-1} \sum_{j,k=0}^{n-1} \sum_{a+b=2} E (f^{(a)}(B_{j/n}) f^{(b)}(B_{k/n})) \left\langle \varepsilon_{j/n}^{\otimes a} \tilde{\otimes} \varepsilon_{k/n}^{\otimes b}, \delta_{j/n} \tilde{\otimes} \delta_{k/n} \right\rangle_{\mathfrak{H}^{\otimes 2}} \\
&\quad \times \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}},
\end{aligned}$$

and by Lemma 1 we obtain

$$|B_n| \leq Cn^{-1} \sum_{j,k=0}^{n-1} \left| \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right|,$$

which tends to zero as n tends to infinity. In fact, this is clear for $H = \frac{1}{2}$, and for $H < \frac{1}{2}$ we have

$$\begin{aligned}
n^{-1} \sum_{j,k=0}^{n-1} \left| \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right| &= \frac{n^{-1-2H}}{2} \sum_{j,k=0}^{n-1} \left| |k-j+1|^{2H} + |k-j-1|^{2H} - 2|k-j|^{2H} \right| \\
&\leq Cn^{-2H} \sum_{p=-\infty}^{\infty} \left| |p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H} \right|,
\end{aligned}$$

and this series is convergent. Finally,

$$\begin{aligned}
C_n &= 2n^{4H-1} \sum_{j,k=0}^{n-1} E [f(B_{j/n})f(B_{k/n})] \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \\
&= \frac{1}{2n} \sum_{j,k=0}^{n-1} E [f(B_{j/n})f(B_{k/n})] (|k-j+1|^{2H} + |k-j-1|^{2H} - 2|k-j|^{2H})^2 \\
&= \frac{1}{2n} \sum_{p=-\infty}^{\infty} \sum_{j=0 \vee -p}^{(n-1) \wedge (n-1-p)} E [f(B_{j/n})f(B_{(j+p)/n})] \\
&\quad \times (|p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H})^2.
\end{aligned}$$

This converges to

$$\frac{1}{2} \left(\sum_{p=-\infty}^{\infty} (|p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H})^2 \right) \int_0^1 E [f(B_s)^2] ds.$$

This gives us the value (3.12).

Step 3.- By step 2, the sequence $(G_n, (B_t)_{t \in [0,1]})$ is tight in $\mathbb{R} \times \mathcal{C}([0,1])$. Assume that $(G_\infty, (B_t)_{t \in [0,1]})$ denotes the limit in law of a certain subsequence of $(G_n, (B_t)_{t \in [0,1]})$. In order to show that (3.11) holds, we have to prove that

$$G_\infty \stackrel{\mathcal{L}}{=} \sigma_H \int_0^1 f(B_s) dW_s$$

for a standard Brownian motion W independent of B . Let $H = \phi(B_{t_1}, \dots, B_{t_m})$, with $\phi \in \mathcal{C}_b^\infty(\mathbb{R}^m)$ and $0 \leq t_1 < \dots < t_m \leq 1$, be a smooth random variable and consider $\phi_n(\lambda) \triangleq E(e^{i\lambda G_n} H)$ for $\lambda \in \mathbb{R}$. We have

$$\phi'_n(\lambda) = iE(G_n e^{i\lambda G_n} H) = in^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} E \left(f(B_{k/n}) I_2(\delta_{k/n}^{\otimes 2}) e^{i\lambda G_n} H \right). \quad (3.13)$$

Our next step is to analyze the asymptotic behavior of $\phi'_n(\lambda)$, as n tends to infinity. To do this we will compute the right-hand side of Equation (3.13) using the duality relationship (2.10).

Step 4.- We claim that

$$\begin{aligned}
\phi'_n(\lambda) &= -2\lambda n^{4H-1} \sum_{k,\ell=0}^{n-1} E (f(B_{k/n})f(B_{\ell/n})e^{i\lambda G_n} H) \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \\
&\quad + in^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} r_{k,n},
\end{aligned} \quad (3.14)$$

where

$$\sup_k |r_{k,n}| \leq Cn^{-4H}. \quad (3.15)$$

The proof of the equality (3.14) and the estimate (3.15) is rather involved. By the duality between the derivative and divergence operators we have

$$E \left(f(B_{k/n}) I_2(\delta_{k/n}^{\otimes 2}) e^{i\lambda G_n} H \right) = \left\langle E \left(D^2 \left(f(B_{k/n}) e^{i\lambda G_n} H \right) \right), \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}}. \quad (3.16)$$

The first and second derivatives of $f(B_{k/n}) e^{i\lambda G_n} H$ are given by

$$\begin{aligned} D \left(f(B_{k/n}) e^{i\lambda G_n} H \right) &= f'(B_{k/n}) e^{i\lambda G_n} H \varepsilon_{k/n} \\ &\quad + i\lambda f(B_{k/n}) e^{i\lambda G_n} H D G_n + f(B_{k/n}) e^{i\lambda G_n} D H \end{aligned}$$

and

$$\begin{aligned} D^2 \left(f(B_{k/n}) e^{i\lambda G_n} H \right) &= f''(B_{k/n}) e^{i\lambda G_n} H (\varepsilon_{k/n} \otimes \varepsilon_{k/n}) \\ &\quad + 2i\lambda f'(B_{k/n}) e^{i\lambda G_n} H (\varepsilon_{k/n} \tilde{\otimes} D G_n) \\ &\quad + 2f'(B_{k/n}) e^{i\lambda G_n} (\varepsilon_{k/n} \tilde{\otimes} D H) \\ &\quad - \lambda^2 f(B_{k/n}) e^{i\lambda G_n} H (D G_n \otimes D G_n) \\ &\quad + 2i\lambda f(B_{k/n}) e^{i\lambda G_n} (D G_n \tilde{\otimes} D H) \\ &\quad + i\lambda f(B_{k/n}) e^{i\lambda G_n} H D^2 G_n + f(B_{k/n}) e^{i\lambda G_n} D^2 H. \end{aligned}$$

Hence, taking expectation and multiplying by $\delta_{k/n}^{\otimes 2}$ yields

$$\begin{aligned} &\left\langle E \left(D^2 \left(f(B_{k/n}) e^{i\lambda G_n} H \right) \right), \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} \\ &= E \left(f''(B_{k/n}) e^{i\lambda G_n} H \right) \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \\ &\quad + 2i\lambda E \left(f'(B_{k/n}) e^{i\lambda G_n} H \langle D G_n, \delta_{k/n} \rangle_{\mathfrak{H}} \right) \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\ &\quad + 2E \left(f'(B_{k/n}) e^{i\lambda G_n} \langle D H, \delta_{k/n} \rangle_{\mathfrak{H}} \right) \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\ &\quad - \lambda^2 E \left(f(B_{k/n}) e^{i\lambda G_n} H \langle D G_n, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \right) \\ &\quad + 2i\lambda E \left(f(B_{k/n}) e^{i\lambda G_n} \langle D H, \delta_{k/n} \rangle_{\mathfrak{H}} \langle D G_n, \delta_{k/n} \rangle_{\mathfrak{H}} \right) \\ &\quad + i\lambda E \left(f(B_{k/n}) e^{i\lambda G_n} H \left\langle D^2 G_n, \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} \right) \\ &\quad + E \left(f(B_{k/n}) e^{i\lambda G_n} \left\langle D^2 H, \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} \right). \end{aligned} \quad (3.17)$$

We also need explicit expressions for $\langle D G_n, \delta_{k/n} \rangle_{\mathfrak{H}}$ and for $\left\langle D^2 G_n, \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}}$. Differentiating G_n we obtain

$$D G_n = n^{2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} \left[f'(B_{\ell/n}) I_2(\delta_{\ell/n}^{\otimes 2}) \varepsilon_{\ell/n} + 2f(B_{\ell/n}) \Delta B_{\ell/n} \delta_{\ell/n} \right]$$

and, as a consequence,

$$\begin{aligned} \langle DG_n, \delta_{k/n} \rangle_{\mathfrak{H}} &= n^{2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} f'(B_{\ell/n}) I_2(\delta_{\ell/n}^{\otimes 2}) \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\ &\quad + 2n^{2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} f(B_{\ell/n}) \Delta B_{\ell/n} \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}. \end{aligned} \quad (3.18)$$

Also

$$\begin{aligned} D^2 G_n &= n^{2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} \left[f''(B_{\ell/n}) I_2(\delta_{\ell/n}^{\otimes 2}) \varepsilon_{\ell/n}^{\otimes 2} \right. \\ &\quad \left. + 4f'(B_{\ell/n}) \Delta B_{\ell/n} (\varepsilon_{\ell/n} \tilde{\otimes} \delta_{\ell/n}) + 2f(B_{\ell/n}) \delta_{\ell/n}^{\otimes 2} \right], \end{aligned}$$

and, as a consequence,

$$\begin{aligned} \langle D^2 G_n, \delta_{k/n}^{\otimes 2} \rangle_{\mathfrak{H}^{\otimes 2}} &= n^{2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} \left[f''(B_{\ell/n}) I_2(\delta_{\ell/n}^{\otimes 2}) \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \right. \\ &\quad \left. + 4f'(B_{\ell/n}) \Delta B_{\ell/n} \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right. \\ &\quad \left. + 2f(B_{\ell/n}) \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \right]. \end{aligned} \quad (3.19)$$

Substituting (3.19) into (3.17) yields

$$\begin{aligned} &\langle E(D^2(f(B_{k/n})e^{i\lambda G_n} H)), \delta_{k/n}^{\otimes 2} \rangle_{\mathfrak{H}^{\otimes 2}} \\ &= 2i\lambda n^{2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} E(f(B_{k/n})f(B_{\ell/n})e^{i\lambda G_n} H) \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 + r_{k,n}, \end{aligned} \quad (3.20)$$

where

$$\begin{aligned}
r_{k,n} &= E \left(f''(B_{k/n}) e^{i\lambda G_n} H \right) \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \\
&\quad + 2i\lambda E \left(f'(B_{k/n}) e^{i\lambda G_n} H \langle DG_n, \delta_{k/n} \rangle_{\mathfrak{H}} \right) \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\
&\quad + 2E \left(f'(B_{k/n}) e^{i\lambda G_n} \langle DH, \delta_{k/n} \rangle_{\mathfrak{H}} \right) \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\
&\quad - \lambda^2 E \left(f(B_{k/n}) e^{i\lambda G_n} H \langle DG_n, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \right) \\
&\quad + 2i\lambda E \left(f(B_{k/n}) e^{i\lambda G_n} \langle DH, \delta_{k/n} \rangle_{\mathfrak{H}} \langle DG_n, \delta_{k/n} \rangle_{\mathfrak{H}} \right) \\
&\quad + i\lambda n^{2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} E \left(f(B_{k/n}) e^{i\lambda G_n} H f''(B_{\ell/n}) I_2(\delta_{\ell/n}^{\otimes 2}) \right) \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \\
&\quad + 4i\lambda n^{2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} E \left(f(B_{k/n}) e^{i\lambda G_n} H f'(B_{\ell/n}) \Delta B_{\ell/n} \right) \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\
&\quad + E \left(f(B_{k/n}) e^{i\lambda G_n} \langle D^2 H, \delta_{k/n}^{\otimes 2} \rangle_{\mathfrak{H}^{\otimes 2}} \right) \\
&= \sum_{j=1}^8 R_{k,n}^j.
\end{aligned}$$

Then, substituting (3.20) into (3.16) and using (3.13) yields (3.14). To complete the proof of this step we need to show the estimate (3.15). For this we will get estimates of the form Cn^{-4H} for each of the eight terms in the above expression. Clearly, by Lemma 1 we obtain $|R_{k,n}^1| \leq Cn^{-4H}$. On the other hand, since H is a random variable of the form $\phi(B_{t_1}, \dots, B_{t_m})$ with $\phi \in \mathcal{C}_b^\infty(\mathbb{R}^m)$, we also have $|R_{k,n}^8| \leq Cn^{-4H}$, and $|R_{k,n}^3| \leq Cn^{-4H}$. For the other terms we have

$$\begin{aligned}
|R_{k,n}^2| + |R_{k,n}^5| &\leq Cn^{-2H} \left\| \langle DG_n, \delta_{k/n} \rangle_{\mathfrak{H}} \right\|_2, \tag{3.21} \\
|R_{k,n}^4| &\leq CE \left((1 + f^2(B_{k/n})) \langle DG_n, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \right), \\
|R_{k,n}^6| &\leq Cn^{-2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} \left| E \left(f(B_{k/n}) e^{i\lambda G_n} H f''(B_{\ell/n}) I_2(\delta_{\ell/n}^{\otimes 2}) \right) \right|, \\
|R_{k,n}^7| &\leq Cn^{-\frac{1}{2}} \sum_{\ell=0}^{n-1} \left| E \left(f(B_{k/n}) e^{i\lambda G_n} H f'(B_{\ell/n}) \Delta B_{\ell/n} \right) \right| \left| \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right|.
\end{aligned}$$

Step 5.- Set $H_{k/n} = 1 + f^2(B_{k/n})$. We claim that

$$E \left(H_{k/n} \langle DG_n, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \right) \leq Cn^{-4H},$$

and substituting this estimate in (3.21) we will get the desired estimate for $R_{k,n}^2$, $R_{k,n}^4$ and $R_{k,n}^5$. We have

$$E \left(H_{k/n} \langle DG_n, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \right) \leq M_{k,n} + N_{k,n},$$

where

$$M_{k,n} = 2n^{4H-1} \sum_{\ell,j=0}^{n-1} E(H_{k/n} f'(B_{\ell/n}) I_2(\delta_{\ell/n}^{\otimes 2}) f'(B_{j/n}) I_2(\delta_{j/n}^{\otimes 2}) \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}})$$

and

$$N_{k,n} = 8n^{4H-1} \sum_{\ell,j=0}^{n-1} E(H_{k/n} f(B_{\ell/n}) \Delta B_{\ell/n} f(B_{j/n}) \Delta B_{j/n}) \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}}.$$

Notice that $M_{k,n}$ has an expression similar to that of $E(G_n^2)$, but replacing f by f' , and adding the factors $H_{k/n}$ and $\langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}}$, and we have

$$\left| \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right| \leq n^{-4H}.$$

As a consequence, estimations similar to those of Step 2 lead to

$$|M_{k,n}| \leq Cn^{-4H}.$$

The term $N_{k,n}$ is treated in a similar way. In fact, by the duality relation

$$\begin{aligned} & \left| E(H_{k/n} f(B_{\ell/n}) f(B_{j/n}) \Delta B_{\ell/n} \Delta B_{j/n}) \right| \\ & \leq \left| E(H_{k/n} f''(B_{\ell/n}) f(B_{j/n}) \langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{\ell/n}, \delta_{j/n} \rangle_{\mathfrak{H}}) \right| \\ & \quad + 2 \left| E(H_{k/n} f'(B_{\ell/n}) f'(B_{j/n}) \langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{j/n}, \delta_{j/n} \rangle_{\mathfrak{H}}) \right| \\ & \quad + \left| E(H_{k/n} f(B_{\ell/n}) f''(B_{j/n}) \langle \varepsilon_{j/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{j/n}, \delta_{j/n} \rangle_{\mathfrak{H}}) \right| \\ & \quad + \left| E(H_{k/n} f(B_{\ell/n}) f(B_{j/n}) \langle \delta_{\ell/n}, \delta_{j/n} \rangle_{\mathfrak{H}}) \right| \\ & \quad + 2E(\langle D^2 H_{k/n}, \delta_{\ell/n} \otimes \delta_{j/n} \rangle_{\mathfrak{H}^{\otimes 2}} f(B_{\ell/n}) f(B_{j/n})) \\ & \leq Cn^{-2H} \end{aligned} \tag{3.22}$$

Thus

$$\begin{aligned} N_{k,n} & \leq Cn^{-1+2H} \left(\sum_{\ell=0}^{n-1} |\langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}| \right)^2 \\ & \leq Cn^{-1-2H} \left(\sum_{p=-\infty}^{\infty} \left| |p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H} \right| \right)^2 = Cn^{-1-2H} \leq Cn^{-4H}, \end{aligned} \tag{3.23}$$

since $H < \frac{1}{2}$, and we get the desired estimate.

Step 6.- Estimation of the term $|R_{k,n}^6|$. Applying the duality formula yields

$$\begin{aligned}
& E \left(f(B_{k/n}) e^{i\lambda G_n} H f''(B_{\ell/n}) I_2(\delta_{\ell/n}^{\otimes 2}) \right) \\
&= \left\langle E \left(D^2 \left(f(B_{k/n}) e^{i\lambda G_n} H f''(B_{\ell/n}) \right) \right), \delta_{\ell/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} \\
&= E \left(f''(B_{k/n}) e^{i\lambda G_n} H f''(B_{\ell/n}) \langle \varepsilon_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}}^2 \right. \\
&\quad + 2i\lambda E \left(f'(B_{k/n}) e^{i\lambda G_n} H f''(B_{\ell/n}) \langle DG_n, \delta_{\ell/n} \rangle_{\mathfrak{H}} \right) \langle \varepsilon_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \\
&\quad + 2E \left(f'(B_{k/n}) e^{i\lambda G_n} f''(B_{\ell/n}) \langle DH, \delta_{\ell/n} \rangle_{\mathfrak{H}} \right) \langle \varepsilon_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \\
&\quad + 2E \left(f'(B_{k/n}) e^{i\lambda G_n} H f'''(B_{\ell/n}) \langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \right) \langle \varepsilon_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \\
&\quad - \lambda^2 E \left(f(B_{k/n}) e^{i\lambda G_n} H f''(B_{\ell/n}) \langle DG_n, \delta_{\ell/n} \rangle_{\mathfrak{H}}^2 \right) \\
&\quad + 2i\lambda E \left(f(B_{k/n}) e^{i\lambda G_n} f''(B_{\ell/n}) \langle DH, \delta_{\ell/n} \rangle_{\mathfrak{H}} \langle DG_n, \delta_{\ell/n} \rangle_{\mathfrak{H}} \right) \\
&\quad + 2i\lambda E \left(f(B_{k/n}) e^{i\lambda G_n} f'''(B_{\ell/n}) \langle DG_n, \delta_{\ell/n} \rangle_{\mathfrak{H}} \right) \langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \\
&\quad + i\lambda E \left(f(B_{k/n}) e^{i\lambda G_n} H f''(B_{\ell/n}) \left\langle D^2 G_n, \delta_{\ell/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} \right) \\
&\quad + 2\lambda E \left(f(B_{k/n}) e^{i\lambda G_n} f'''(B_{\ell/n}) \langle DH, \delta_{\ell/n} \rangle_{\mathfrak{H}} \right) \langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \\
&\quad + E \left(f(B_{k/n}) e^{i\lambda G_n} f''(B_{\ell/n}) \left\langle D^2 H, \delta_{\ell/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} \right) \\
&\quad + E \left(f(B_{k/n}) e^{i\lambda G_n} H f^{(4)}(B_{\ell/n}) \langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}}^2 \right).
\end{aligned}$$

By Lemma 1 we have that $|\langle \varepsilon_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}}|$, $|\langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}}|$, and $\left\| \langle DH, \delta_{\ell/n} \rangle_{\mathfrak{H}} \right\|_2$ are bounded by Cn^{-2H} while $\left\| \left\langle D^2 H, \delta_{\ell/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} \right\|_2$ is bounded by Cn^{-4H} . On the other hand, as in Step 5, and taking into account hypothesis **(H)**, we can show that

$$E \left(\widehat{H}_{k/n} \langle DG_n, \delta_{k/n} \rangle_{\mathfrak{H}} \right) \leq Cn^{-4H},$$

where $\widehat{H}_{k/n} = 1 + (f(B_{k/n}) f''(B_{\ell/n}))^2$. As a consequence,

$$|R_{k,n}^6| \leq Cn^{-2H-\frac{1}{2}} \left[n^{1-4H} + \sum_{\ell=0}^{n-1} \left\| \left\langle D^2 G_n, \delta_{\ell/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} \right\|_2 \right].$$

Then, since $H > \frac{1}{4}$, it suffices to show that

$$\left\| \left\langle D^2 G_n, \delta_{\ell/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} \right\|_2 \leq Cn^{-2H-\frac{1}{2}}. \quad (3.24)$$

We have

$$\begin{aligned}
\left\langle D^2 G_n, \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} &= n^{2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} \left[f''(B_{\ell/n}) I_2(\delta_{\ell/n}^{\otimes 2}) \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \right. \\
&\quad + 4f'(B_{\ell/n}) \Delta B_{\ell/n} \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\
&\quad \left. + 2f(B_{\ell/n}) \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \right]. \tag{3.25}
\end{aligned}$$

$$\begin{aligned}
&E \left(\left\langle D^2 G_n, \delta_{\ell/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}}^2 \right) \\
&\leq 3n^{4H-1} \sum_{\ell,j=0}^{n-1} E \left(f''(B_{\ell/n}) I_2(\delta_{\ell/n}^{\otimes 2}) f''(B_{j/n}) I_2(\delta_{j/n}^{\otimes 2}) \right) \\
&\quad \times \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \langle \varepsilon_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \\
&\quad + 48n^{4H-1} \sum_{\ell,j=0}^{n-1} E \left(f'(B_{\ell/n}) \Delta B_{\ell/n} f'(B_{j/n}) \Delta B_{j/n} \right) \\
&\quad \times \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\
&\quad + 12n^{4H-1} \sum_{\ell,j=0}^{n-1} E \left(f(B_{\ell/n}) f(B_{j/n}) \right) \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \\
&= R_{1,n} + S_{1,n} + T_{1,n}.
\end{aligned}$$

The term $R_{1,n}$ has an expression similar to that of $E(G_n^2)$, but replacing f by f'' , and adding the factor $\langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \langle \varepsilon_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2$. As a consequence, we obtain

$$R_{1,n} \leq Cn^{-8H}.$$

For the term $S_{1,n}$, using (3.22) yields

$$\begin{aligned}
S_{1,n} &\leq Cn^{-2H-1} \left(\sum_{\ell=0}^{n-1} \left| \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right| \right)^2 \\
&\leq Cn^{-6H-1} \left(\sum_{p=-\infty}^{\infty} \left| |p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H} \right| \right)^2.
\end{aligned}$$

Finally,

$$\begin{aligned}
T_{1,n} &\leq Cn^{4H-1} \left(\sum_{\ell=0}^{n-1} \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \right)^2 \\
&\leq Cn^{-4H-1} \left(\sum_{p=-\infty}^{\infty} \left| |p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H} \right|^2 \right)^2.
\end{aligned}$$

Step 7.- Estimation of the term $|R_{k,n}^7|$. We have

$$\begin{aligned}
& E \left(f(B_{k/n}) e^{i\lambda G_n} H f'(B_{\ell/n}) \Delta B_{\ell/n} \right) \\
= & E \left(f'(B_{k/n}) e^{i\lambda G_n} H f'(B_{\ell/n}) \right) \langle \varepsilon_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \\
& + i\lambda E \left(f(B_{k/n}) e^{i\lambda G_n} H f'(B_{\ell/n}) \langle DG_n, \delta_{\ell/n} \rangle_{\mathfrak{H}} \right) \\
& + E \left(f(B_{k/n}) e^{i\lambda G_n} f'(B_{\ell/n}) \langle DH, \delta_{\ell/n} \rangle_{\mathfrak{H}} \right) \\
& + E \left(f(B_{k/n}) e^{i\lambda G_n} H f''(B_{\ell/n}) \right) \langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}}.
\end{aligned}$$

We deduce, using also Step 5 and (3.23):

$$|R_{k,n}^7| \leq C n^{2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} \left| \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right| \leq C n^{-4H-\frac{1}{2}}.$$

Step 8.- From (3.14), taking the limit as $n \rightarrow \infty$, we obtain

$$\begin{aligned}
\lim_{n \rightarrow \infty} \phi'_n(\lambda) &= -2\lambda \lim_{n \rightarrow \infty} n^{4H-1} \sum_{k,\ell=0}^{n-1} E \left(f(B_{k/n}) f(B_{\ell/n}) e^{i\lambda G_n} H \right) \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \\
&= -\lambda 2\sigma_H^2 \int_0^1 E \left(f^2(B_s) H e^{i\lambda G_\infty} \right) ds.
\end{aligned}$$

This limit requires some justification. A way to prove it consists to use the Skorohod almost sure representation theorem (see, *e.g.*, [7, p. 281]). Indeed, by using the fact that, on an auxiliary space, there exists a fractional Brownian motion \overline{B} such that, with obvious notations, $\overline{G}_n \xrightarrow{\text{a.s.}} \overline{G}_\infty$, we easily obtain the desired convergence by bounded convergence.

On the other hand, due to the definition of ϕ_n , we also have:

$$\phi'_n(\lambda) \longrightarrow iE \left(G_\infty H e^{i\lambda G_\infty} \right) = \frac{\partial}{\partial \lambda} E \left(H e^{i\lambda G_\infty} \right).$$

Since this result is valid for any smooth (with respect to the σ -field generated by $(B_t)_{t \in [0,1]}$) random variable H , we have proved:

$$\frac{\partial}{\partial \lambda} E \left(e^{i\lambda G_\infty} \mid (B_t)_{t \in [0,1]} \right) = -\frac{1}{2} \sigma_H^2 \lambda \int_0^1 f^2(B_s) ds E \left(e^{i\lambda G_\infty} \mid (B_t)_{t \in [0,1]} \right).$$

In other words, $\lambda \mapsto E \left(e^{i\lambda G_\infty} \mid (B_t)_{t \in [0,1]} \right)$ solves a linear differential equation, and we deduce:

$$E \left(e^{i\lambda G_\infty} \mid (B_t)_{t \in [0,1]} \right) = e^{-\frac{\sigma_H^2 \lambda^2}{2} \int_0^1 f^2(B_s) ds},$$

which proves the desired result.

■

Acknowledgment. This work was initiated while both authors were attending the conference "Stochastic Analysis and Related Fields" during June 20-21, 2007, at Toulouse, France. We thank the organizers, especially Fabrice Baudoin, for their invitation and hospitality. The work of David Nualart has been supported by the NSF Grant No. DMS-0604207.

References

- [1] P. Breuer and P. Major (1983): *Central limit theorems for nonlinear functionals of Gaussian fields*. J. Multivariate Anal. **13** (3), 425-441.
- [2] J.M. Corcuera, D. Nualart and J.H.C. Woerner (2006): *Power variation of some integral fractional processes*. Bernoulli **12**, 713-735.
- [3] R.L. Dobrushin and P. Major (1979): *Non-central limit theorems for nonlinear functionals of Gaussian fields*. Z. Wahrscheinlichkeitstheorie verw. Gebiete **50**, 27-52.
- [4] L. Giraitis and D. Surgailis (1985): *CLT and other limit theorems for functionals of Gaussian processes*. Z. Wahrsch. verw. Geb. **70**, 191-212.
- [5] M. Gradinaru and I. Nourdin (2007): *Convergence of weighted power variations of fractional Brownian motion*. Preprint 2007/05, University of Nancy.
- [6] M. Gradinaru, F. Russo and P. Vallois (2001): *Generalized covariations, local time and Stratonovich Itô's formula for fractional Brownian motion with Hurst index $H \geq \frac{1}{4}$* . Ann. Probab. **31**, 1772-1820.
- [7] G.R. Grimmet and D.R. Stirzaker (1992): *Probability and Random Processes*. Oxford Science Publications. Oxford.
- [8] J. Jacod (1994): *Limit of random measures associated with the increments of a Brownian semimartingale*. Preprint University Paris VI (revised version, unpublished work).
- [9] J. León and C. Ludeña (2006): *Limits for weighted p -variations and likewise functionals of fractional diffusions with drift*. Stoch. Proc. Appl. **117** (3), 271-296.
- [10] A. Neuenkirch and I. Nourdin (2007): *Exact rate of convergence of some approximation schemes associated to SDEs driven by a fractional Brownian motion*. J. Theor. Probab., to appear.
- [11] I. Nourdin (2007): *Asymptotic behavior of some weighted quadratic and cubic variations of the fractional Brownian motion*. Preprint University Paris VI.
- [12] I. Nourdin (2007): *A simple theory for the study of SDEs driven by a fractional Brownian motion, in dimension one*. Séminaire de Probabilités **XLI**, to appear.

- [13] D. Nualart (2003): *Stochastic calculus with respect to the fractional Brownian motion and applications*. Contemp. Math. **336**, 3-39.
- [14] J. Swanson (2007): *Variations of the solution to a stochastic heat equation*. Ann. Probab., to appear.
- [15] M. Taqqu (1979): *Convergence of integrated processes of arbitrary Hermite rank*. Z. Wahrscheinlichkeitstheorie verw. Gebiete **50**, 53-83.