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# Convergence in law for certain weighted quadratic variations of fractional Brownian motion

Ivan Nourdin<sup>1</sup> and David Nualart<sup>2</sup>

**Abstract:** By means of Malliavin calculus, we prove the convergence in law for certain weighted quadratic variations of a fractional Brownian motion  $B$  with Hurst index  $H \in (\frac{1}{4}, \frac{1}{2}]$ .

*Key words:* Fractional Brownian motion - weighted quadratic variations - Malliavin calculus - weak convergence.

*Actual version:* July 2007

## 1 Introduction

The purpose of this paper is to study the asymptotic behavior of the sequence of random variables

$$G_n(f) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(B_{k/n}) (n^{2H} (\Delta B_{k/n})^2 - 1), \quad n \geq 1,$$

where  $B$  is a fractional Brownian motion of index  $H \in (0, 1)$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $\Delta B_{k/n}$  denotes the increment  $B_{(k+1)/n} - B_{k/n}$ .

The case  $f \equiv 1$  is classical (see, e.g., Breuer and Major [1] or Giraitis and Surgailis [4]), and for any  $H \in (0, \frac{3}{4})$  we have

$$G_n(\mathbf{1}) \implies \mathcal{N}(0, \sigma_H^2), \quad \text{as } n \rightarrow \infty, \quad (1.1)$$

for a certain (explicit) constant  $\sigma_H$ , and, where, as usual, we denote by " $\implies$ " the convergence in law. When  $H \in [\frac{3}{4}, 1)$ , non-central limit theorems appear (see Taqqu [10] or Dobrushin and Major [3]).

If  $H \in (\frac{1}{2}, \frac{3}{4})$ , and  $f$  is a continuously differentiable function such that  $f$  and  $f'$  have subexponential growth, then by Theorem 2 in León and Ludeña [7] we have that

$$G_n(f) \implies \sigma_H \int_0^1 f(B_s) dW_s, \quad (1.2)$$

as  $n$  tends to infinity, where  $W$  is a standard Brownian motion independent of  $B$ . We refer to [2] for related results on the asymptotic behavior of the  $p$ -variation of stochastic

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integrals with respect to the fractional Brownian motion. When  $H = \frac{1}{2}$ , (1.2) is still true, and is in fact a very particular case of a more general result by Jacod [6]. On the other hand, when  $H \in (0, \frac{1}{4})$ , the first-named author [8] proved that if  $f$  belongs to  $\mathcal{C}_b^2$ , then

$$n^{2H-\frac{1}{2}}G_n(f) \xrightarrow{L^2} \frac{1}{4} \int_0^1 f''(B_s)ds, \quad \text{as } n \rightarrow \infty. \quad (1.3)$$

In (1.3), the condition  $H \in (0, \frac{1}{4})$  is necessary: compare indeed with (1.1).

What happens for  $H$  between  $\frac{1}{4}$  and  $\frac{1}{2}$ ? This question is natural in view of (1.2) and (1.3). In [8], it was conjectured that we still have (1.2) and not (1.3). The aim of this note is to prove this last fact.

Before going into details, let us describe briefly our approach. Assume that  $H \in (\frac{1}{4}, \frac{1}{2}]$ . By means of Malliavin calculus, we will prove, as  $n \rightarrow \infty$ :

$$E(G_n(f)) \longrightarrow 0 \quad \text{and} \quad E(G_n^2(f)) \longrightarrow \sigma_H^2 \int_0^1 E(f^2(B_s))ds.$$

In particular, the sequence  $(G_n, (B_t)_{t \in [0,1]})$  is tight in  $\mathbb{R} \times \mathcal{C}([0,1])$ . Let  $(G_\infty, (B_t)_{t \in [0,1]})$  denote the limit in law of a certain subsequence of  $(G_n, (B_t)_{t \in [0,1]})$ . We will show that the conditional characteristic function of  $G_\infty$  solves the following linear differential equation:

$$\frac{\partial}{\partial \lambda} E(e^{i\lambda G_\infty} | (B_t)_{t \in [0,1]}) = -\frac{\lambda}{2} \sigma_H^2 \int_0^1 f^2(B_s)ds E(e^{i\lambda G_\infty} | (B_t)_{t \in [0,1]}).$$

Consequently, we deduce

$$E(e^{i\lambda G_\infty} | (B_t)_{t \in [0,1]}) = \exp\left(-\frac{\sigma_H^2 \lambda^2}{2} \int_0^1 f^2(B_s)ds\right),$$

and (1.2) follows.

The case  $H = \frac{1}{4}$  is not covered by our method, and the asymptotic behavior of  $G_n(f)$  in this case is unknown. Our approach could be applied to other types of weighted Riemann sums, replacing  $(\Delta B_{k/n})^2$  by higher powers of the increment. For instance, when  $H \in (0, \frac{1}{6})$  and  $f \in \mathcal{C}_b^3$ , one proves in [8] that

$$n^{3H-1} \sum_{k=0}^{n-1} \left( f(B_{k/n}) n^{3H} (\Delta B_{k/n})^3 + \frac{3}{2} f'(B_{k/n}) n^{-H} \right) \xrightarrow{L^2} -\frac{1}{8} \int_0^1 f'''(B_s)ds,$$

as  $n \rightarrow \infty$ . Using the ideas contained in the current paper, we could prove that the following convergence holds, when  $H \in (\frac{1}{6}, \frac{1}{2})$ :

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left( f(B_{k/n}) n^{3H} (\Delta B_{k/n})^3 + \frac{3}{2} f'(B_{k/n}) n^{-H} \right) \implies \hat{\sigma}_H \int_0^1 f(B_s) dW_s,$$

as  $n \rightarrow \infty$ , for a certain constant  $\hat{\sigma}_H$  and a standard Brownian motion  $W$  independent of  $B$ .

The paper is organized as follows. In section 2, we introduce some notations and preliminary results. In section 3, we state and prove the main result of the paper.

## 2 Preliminaries and notations

We begin by briefly recalling some basic facts about stochastic calculus with respect to a fractional Brownian motion. We refer to [9] for further details. Let  $B = (B_t)_{t \in [0,1]}$  be a fractional Brownian motion with Hurst parameter  $H \in (0, \frac{1}{2})$  defined on a probability space  $(\Omega, \mathcal{A}, P)$ . We mean that  $B$  is a centered Gaussian process with the covariance function  $E(B_s B_t) = R_H(s, t)$ , where

$$R_H(s, t) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}).$$

We denote by  $\mathcal{E}$  the set of step  $\mathbb{R}$ -valued functions on  $[0, 1]$ . Let  $\mathfrak{H}$  be the Hilbert space defined as the closure of  $\mathcal{E}$  with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathfrak{H}} = R_H(t, s).$$

We denote by  $|\cdot|_{\mathfrak{H}}$  the associate norm. The mapping  $\mathbf{1}_{[0,t]} \mapsto B_t$  can be extended to an isometry between  $\mathfrak{H}$  and the Gaussian space  $\mathcal{H}_1$  associated with  $B$ . We denote this isometry by  $\varphi \mapsto B(\varphi)$ . The space  $\mathfrak{H}$  coincides with  $I_{0+}^{H-\frac{1}{2}}(L^2([0, 1]))$ , where

$$I_{0+}^{H-\frac{1}{2}} f(x) \triangleq \frac{1}{\Gamma(H - \frac{1}{2})} \int_0^x (x - y)^{H-\frac{3}{2}} f(y) dy.$$

Let  $\mathcal{S}$  be the set of all smooth cylindrical random variables, *i.e.* of the form

$$F = \phi(B_{t_1}, \dots, B_{t_m})$$

where  $m \geq 1$ ,  $\phi : \mathbb{R}^m \rightarrow \mathbb{R} \in \mathcal{C}_b^\infty$  and  $0 \leq t_1 < \dots < t_m \leq 1$ . The Malliavin derivative of  $F$  with respect to  $B$  is the element of  $L^2(\Omega, \mathfrak{H})$  defined by

$$D_s F = \sum_{i=1}^m \frac{\partial \phi}{\partial x_i}(B_{t_1}, \dots, B_{t_m}) \mathbf{1}_{[0, t_i]}(s), \quad s \in [0, 1].$$

In particular  $D_s B_t = \mathbf{1}_{[0,t]}(s)$ . For any integer  $k \geq 1$ , we denote by  $\mathbb{D}^{k,2}$  the closure of the set of smooth random variables with respect to the norm

$$\|F\|_{k,2}^2 = E[F^2] + \sum_{j=1}^k E[|D^j F|_{\mathfrak{H}^{\otimes j}}^2].$$

The Malliavin derivative  $D$  verifies the chain rule: if  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  is  $\mathcal{C}_b^1$  and if  $(F_i)_{i=1,\dots,n}$  is a sequence of elements of  $\mathbb{D}^{1,2}$  then  $\varphi(F_1, \dots, F_n) \in \mathbb{D}^{1,2}$  and we have, for any  $s \in [0, 1]$ :

$$D_s \varphi(F_1, \dots, F_n) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F_1, \dots, F_n) D_s F_i.$$

The divergence operator  $I = I_1$  is the adjoint of the derivative operator  $D$ . If a random variable  $u \in L^2(\Omega, \mathfrak{H})$  belongs to the domain of the divergence operator, that is if it verifies

$$|\mathbb{E}\langle DF, u \rangle_{\mathfrak{H}}| \leq c_u \|F\|_{L^2} \quad \text{for any } F \in \mathcal{S},$$

then  $I(u)$  is defined by the duality relationship

$$\mathbb{E}(FI(u)) = \mathbb{E}(\langle DF, u \rangle_{\mathfrak{H}}),$$

for every  $F \in \mathbb{D}^{1,2}$ .

For every  $n \geq 1$ , let  $\mathcal{H}_n$  be the  $n$ th Wiener chaos of  $B$ , that is, the closed linear subspace of  $L^2(\Omega, \mathcal{A}, P)$  generated by the random variables  $\{H_n(B(h)), h \in H, |h|_{\mathfrak{H}} = 1\}$ , where  $H_n$  is the  $n$ th Hermite polynomial. The mapping  $I_n(h^{\otimes n}) = n!H_n(B(h))$  provides a linear isometry between the symmetric tensor product  $\mathfrak{H}^{\odot n}$  and  $\mathcal{H}_n$ . For  $H = \frac{1}{2}$ ,  $I_n$  coincides with the multiple stochastic integral. The following duality formula holds

$$E(FI_n(h)) = E(\langle D^n F, h \rangle_{\mathfrak{H}^{\otimes n}}), \quad (2.4)$$

for any element  $h \in \mathfrak{H}^{\odot n}$  and any random variable  $F \in \mathbb{D}^{n,2}$ .

### 3 Main result

In the sequel, we will make use of the following hypothesis on  $f : \mathbb{R} \rightarrow \mathbb{R}$ :

**(H)**  $f$  belongs to  $\mathcal{C}^6$  and  $|f^{(6)}(x)| \leq e^{cx^2}$  for some constant  $c < \frac{1}{12}$ .

Notice that **(H)** implies

$$\sup_{t \in [0,1]} E(|f^{(i)}(B_t)|^p) < \infty,$$

for all  $1 \leq p \leq 6$ , and  $i = 1, \dots, 6$ . We also note  $G_n$  instead of  $G_n(f)$  for simplicity. Notice that  $n^{2H}(\Delta B_{k/n})^2 - 1 = n^{2H}I_2(\delta_{k/n}^{\otimes 2})$ , where  $\delta_{k/n} = \mathbf{1}_{[k/n, (k+1)/n]}$ , and  $I_2$  denotes the double stochastic integral. We will make use of the following technical lemma.

**Lemma 1** *Let  $s < t$  and  $r$  belong to  $[0, 1]$ . Then*

$$|E(B_r(B_t - B_s))| \leq (t - s)^{2H}.$$

**Proof.** We have

$$\begin{aligned} E(B_r(B_t - B_s)) &= \frac{1}{2} (r^{2H} + t^{2H} - |t - r|^{2H}) \\ &\quad - \frac{1}{2} (r^{2H} + s^{2H} - |s - r|^{2H}) \\ &= \frac{1}{2} (t^{2H} - s^{2H}) + \frac{1}{2} (|s - r|^{2H} - |t - r|^{2H}). \end{aligned}$$

We have  $|b^{2H} - a^{2H}| \leq |b - a|^{2H}$  for any  $a, b \in [0, 1]$ , because  $H < \frac{1}{2}$ . The desired result follows. ■

Our aim is to show the following result.

**Theorem 2** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfy **(H)**. We have

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(B_{k/n}) (n^{2H} (\Delta B_{k/n})^2 - 1) \implies \sigma_H \int_0^1 f(B_s) dW_s, \quad \text{as } n \rightarrow \infty, \quad (3.5)$$

where  $W$  is a Brownian motion independent of  $B$ . Moreover  $\sigma_{\frac{1}{2}} = \sqrt{2}$  while, for  $H \in (\frac{1}{4}, \frac{1}{2})$ :

$$\sigma_H = \sqrt{\frac{1}{2} \sum_{p=-\infty}^{\infty} (|p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H})^2}. \quad (3.6)$$

**Proof.** The proof will be done in several steps. We will denote by  $C$  a generic constant that can be different from line to line.

*Step 1.-* We will first show that  $\lim_{n \rightarrow \infty} E(G_n) = 0$ . In the case  $H = \frac{1}{2}$  we actually have  $E(G_n) = 0$ . Suppose  $H < \frac{1}{2}$ . By the duality formula between the iterated derivative and double stochastic integrals we have

$$\begin{aligned} E \left[ f(B_{k/n}) I_2(\delta_{k/n}^{\otimes 2}) \right] &= \left\langle E(D^2 f(B_{k/n})), \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} \\ &= E(f''(B_{k/n})) \left\langle \varepsilon_{k/n}^{\otimes 2}, \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}}, \end{aligned}$$

where  $\varepsilon_{k/n} = \mathbf{1}_{[0, k/n]}$ . Hence, by Lemma 1

$$\begin{aligned} |E(G_n)| &\leq n^{-\frac{1}{2}+2H} \sum_{k=0}^{n-1} |E(f''(B_{k/n}))| \left\langle \varepsilon_{k/n}, \delta_{k/n} \right\rangle_{\mathfrak{H}}^2 \\ &\leq C n^{\frac{1}{2}-2H}, \end{aligned}$$

which converges to zero as  $n$  tends to infinity because  $H > \frac{1}{4}$ .

*Step 2.-* We claim that  $\lim_{n \rightarrow \infty} E(G_n^2) = \sigma_H^2 \int_0^1 E(f(B_s)^2) ds$ . We have

$$E(G_n^2) = n^{4H-1} \sum_{j,k=0}^{n-1} E \left( f(B_{j/n}) I_2(\delta_{j/n}^{\otimes 2}) f(B_{k/n}) I_2(\delta_{k/n}^{\otimes 2}) \right).$$

The product formula for multiple stochastic integrals yields

$$\begin{aligned} I_2(\delta_{j/n}^{\otimes 2}) I_2(\delta_{k/n}^{\otimes 2}) &= \sum_{r=0}^2 r! \binom{2}{r}^2 I_{4-2r}(\delta_{j/n}^{\otimes 2} \otimes_r \delta_{k/n}^{\otimes 2}) \\ &= I_4(\delta_{j/n}^{\otimes 2} \widetilde{\otimes} \delta_{k/n}^{\otimes 2}) + 4 I_2(\delta_{j/n} \widetilde{\otimes} \delta_{k/n}) \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\ &\quad + 2 \left\langle \delta_{j/n}^{\otimes 2}, \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}}, \end{aligned}$$

and, as a consequence,

$$\begin{aligned}
E(G_n^2) &= n^{4H-1} \sum_{j,k=0}^{n-1} E \left[ f(B_{j/n}) f(B_{k/n}) I_4(\delta_{j/n}^{\otimes 2} \tilde{\otimes} \delta_{k/n}^{\otimes 2}) \right] \\
&\quad + 4n^{4H-1} \sum_{j,k=0}^{n-1} E \left[ f(B_{j/n}) f(B_{k/n}) I_2(\delta_{j/n} \tilde{\otimes} \delta_{k/n}) \right] \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\
&\quad + 2n^{4H-1} \sum_{j,k=0}^{n-1} E \left[ f(B_{j/n}) f(B_{k/n}) \right] \langle \delta_{j/n}^{\otimes 2}, \delta_{k/n}^{\otimes 2} \rangle_{\mathfrak{H}^{\otimes 2}} \\
&= A_{1,n} + B_{1,n} + C_{1,n}.
\end{aligned}$$

The term  $A_{1,n}$  makes no contribution in the limit as  $n$  tends to infinity because by the duality formula (2.4) we have

$$\begin{aligned}
A_{1,n} &= n^{4H-1} \sum_{j,k=0}^{n-1} \left\langle E \left[ D^4 \left( f(B_{j/n}) f(B_{k/n}) \right) \right], \delta_{j/n}^{\otimes 2} \tilde{\otimes} \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 4}} \\
&= n^{4H-1} \sum_{j,k=0}^{n-1} \sum_{a+b=4} E \left( f^{(a)}(B_{j/n}) f^{(b)}(B_{k/n}) \right) \left\langle \varepsilon_{j/n}^{\otimes a} \otimes \varepsilon_{k/n}^{\otimes b}, \delta_{j/n}^{\otimes 2} \tilde{\otimes} \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 4}},
\end{aligned}$$

and by Lemma 1 we obtain

$$|A_{1,n}| \leq Cn^{1-4H}.$$

In the same way,

$$\begin{aligned}
B_{1,n} &= 4n^{4H-1} \sum_{j,k=0}^{n-1} E \left[ \left\langle D^2 \left( f(B_{j/n}) f(B_{k/n}) \right), \delta_{j/n} \tilde{\otimes} \delta_{k/n} \right\rangle_{\mathfrak{H}^{\otimes 2}} \right] \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\
&= 4n^{4H-1} \sum_{j,k=0}^{n-1} \sum_{a+b=2} E \left( f^{(a)}(B_{j/n}) f^{(b)}(B_{k/n}) \right) \left\langle \varepsilon_{j/n}^{\otimes a} \otimes \varepsilon_{k/n}^{\otimes b}, \delta_{j/n} \tilde{\otimes} \delta_{k/n} \right\rangle_{\mathfrak{H}^{\otimes 2}} \\
&\quad \times \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}},
\end{aligned}$$

and by Lemma 1 we obtain

$$|B_{1,n}| \leq Cn^{-1} \sum_{j,k=0}^{n-1} \left| \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right|,$$

which tends to zero as  $n$  tends to infinity. In fact, this is clear for  $H = \frac{1}{2}$ , and for  $H < \frac{1}{2}$  we have

$$\begin{aligned}
n^{-1} \sum_{j,k=0}^{n-1} \left| \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right| &= \frac{n^{-1-2H}}{2} \sum_{j,k=0}^{n-1} \left| |k-j+1|^{2H} + |k-j-1|^{2H} - 2|k-j|^{2H} \right| \\
&\leq Cn^{-2H} \sum_{p=-\infty}^{\infty} \left| |p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H} \right|,
\end{aligned}$$

and this series is convergent. Finally,

$$\begin{aligned}
C_{1,n} &= 2n^{4H-1} \sum_{j,k=0}^{n-1} E [f(B_{j/n})f(B_{k/n})] \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \\
&= \frac{1}{2n} \sum_{j,k=0}^{n-1} E [f(B_{j/n})f(B_{k/n})] (|k-j+1|^{2H} + |k-j-1|^{2H} - 2|k-j|^{2H})^2 \\
&= \frac{1}{2n} \sum_{p=-\infty}^{\infty} \sum_{j=0 \vee -p}^{(n-1) \wedge (n-1-p)} E [f(B_{j/n})f(B_{(j+p)/n})] \\
&\quad \times (|p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H})^2.
\end{aligned}$$

This converges to

$$\frac{1}{2} \left( \sum_{p=-\infty}^{\infty} (|p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H})^2 \right) \int_0^1 E [f(B_s)^2] ds.$$

This gives us the value (3.6).

*Step 3.-* By step 2, the sequence  $(G_n, (B_t)_{t \in [0,1]})$  is tight in  $\mathbb{R} \times \mathcal{C}([0,1])$ . Assume that  $(G_\infty, (B_t)_{t \in [0,1]})$  denotes the limit in law of a certain subsequence of  $(G_n, (B_t)_{t \in [0,1]})$ . In order to show that (3.5) holds, we have to prove that

$$G_\infty \stackrel{\mathcal{L}}{=} \sigma_H \int_0^1 f(B_s) dW_s$$

for a standard Brownian motion  $W$  independent of  $B$ . Let  $H = \phi(B_{t_1}, \dots, B_{t_m})$ , with  $\phi \in \mathcal{C}_b^\infty(\mathbb{R}^m)$  and  $0 \leq t_1 < \dots < t_m \leq 1$ , be a smooth random variable and consider  $\phi_n(\lambda) \triangleq E(e^{i\lambda G_n} H)$  for  $\lambda \in \mathbb{R}$ . We have

$$\phi'_n(\lambda) = iE(G_n e^{i\lambda G_n} H) = in^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} E\left(f(B_{k/n}) I_2(\delta_{k/n}^{\otimes 2}) e^{i\lambda G_n} H\right). \quad (3.7)$$

Our next step is to analyze the asymptotic behavior of  $\phi'_n(\lambda)$ , as  $n$  tends to infinity. To do this we will compute the right-hand side of Equation (3.7) using the duality relationship (2.4).

*Step 4.-* We claim that

$$\begin{aligned}
\phi'_n(\lambda) &= -2\lambda n^{4H-1} \sum_{k,\ell=0}^{n-1} E(f(B_{k/n})f(B_{\ell/n})e^{i\lambda G_n} H) \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \\
&\quad + in^{2H-\frac{1}{2}} \sum_{k=0}^{n-1} r_{k,n},
\end{aligned} \quad (3.8)$$



where

$$\sup_k |r_{k,n}| \leq Cn^{-4H}. \quad (3.9)$$

The proof of the equality (3.8) and the estimate (3.9) is rather involved. By the duality between the derivative and divergence operators we have

$$E \left( f(B_{k/n}) I_2(\delta_{k/n}^{\otimes 2}) e^{i\lambda G_n} H \right) = \left\langle E \left( D^2 \left( f(B_{k/n}) e^{i\lambda G_n} H \right) \right), \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}}. \quad (3.10)$$

The first and second derivatives of  $f(B_{k/n}) e^{i\lambda G_n} H$  are given by

$$\begin{aligned} D \left( f(B_{k/n}) e^{i\lambda G_n} H \right) &= f'(B_{k/n}) e^{i\lambda G_n} H \varepsilon_{k/n} \\ &\quad + i\lambda f(B_{k/n}) e^{i\lambda G_n} H D G_n + f(B_{k/n}) e^{i\lambda G_n} D H \end{aligned}$$

and

$$\begin{aligned} D^2 \left( f(B_{k/n}) e^{i\lambda G_n} H \right) &= f''(B_{k/n}) e^{i\lambda G_n} H (\varepsilon_{k/n} \otimes \varepsilon_{k/n}) \\ &\quad + 2i\lambda f'(B_{k/n}) e^{i\lambda G_n} H (\varepsilon_{k/n} \tilde{\otimes} D G_n) \\ &\quad + 2f'(B_{k/n}) e^{i\lambda G_n} (\varepsilon_{k/n} \tilde{\otimes} D H) \\ &\quad - \lambda^2 f(B_{k/n}) e^{i\lambda G_n} H (D G_n \otimes D G_n) \\ &\quad + 2i\lambda f(B_{k/n}) e^{i\lambda G_n} (D G_n \tilde{\otimes} D H) \\ &\quad + i\lambda f(B_{k/n}) e^{i\lambda G_n} H D^2 G_n + f(B_{k/n}) e^{i\lambda G_n} D^2 H. \end{aligned}$$

Hence, taking expectation and multiplying by  $\delta_{k/n}^{\otimes 2}$  yields

$$\begin{aligned} &\left\langle E \left( D^2 \left( f(B_{k/n}) e^{i\lambda G_n} H \right) \right), \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} \\ &= E \left( f''(B_{k/n}) e^{i\lambda G_n} H \right) \left\langle \varepsilon_{k/n}, \delta_{k/n} \right\rangle_{\mathfrak{H}}^2 \\ &\quad + 2i\lambda E \left( f'(B_{k/n}) e^{i\lambda G_n} H \left\langle D G_n, \delta_{k/n} \right\rangle_{\mathfrak{H}} \right) \left\langle \varepsilon_{k/n}, \delta_{k/n} \right\rangle_{\mathfrak{H}} \\ &\quad + 2E \left( f'(B_{k/n}) e^{i\lambda G_n} \left\langle D H, \delta_{k/n} \right\rangle_{\mathfrak{H}} \right) \left\langle \varepsilon_{k/n}, \delta_{k/n} \right\rangle_{\mathfrak{H}} \\ &\quad - \lambda^2 E \left( f(B_{k/n}) e^{i\lambda G_n} H \left\langle D G_n, \delta_{k/n} \right\rangle_{\mathfrak{H}}^2 \right) \\ &\quad + 2i\lambda E \left( f(B_{k/n}) e^{i\lambda G_n} \left\langle D H, \delta_{k/n} \right\rangle_{\mathfrak{H}} \left\langle D G_n, \delta_{k/n} \right\rangle_{\mathfrak{H}} \right) \\ &\quad + i\lambda E \left( f(B_{k/n}) e^{i\lambda G_n} H \left\langle D^2 G_n, \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} \right) \\ &\quad + E \left( f(B_{k/n}) e^{i\lambda G_n} \left\langle D^2 H, \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} \right). \end{aligned} \quad (3.11)$$

We also need explicit expressions for  $\left\langle D G_n, \delta_{k/n} \right\rangle_{\mathfrak{H}}$  and for  $\left\langle D^2 G_n, \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}}$ . Differentiating  $G_n$  we obtain

$$D G_n = n^{2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} \left[ f'(B_{\ell/n}) I_2(\delta_{\ell/n}^{\otimes 2}) \varepsilon_{\ell/n} + 2f(B_{\ell/n}) \Delta B_{\ell/n} \delta_{\ell/n} \right]$$

and, as a consequence,

$$\begin{aligned}
\langle DG_n, \delta_{k/n} \rangle_{\mathfrak{H}} &= n^{2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} f'(B_{\ell/n}) I_2(\delta_{\ell/n}^{\otimes 2}) \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\
&\quad + 2n^{2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} f(B_{\ell/n}) \Delta B_{\ell/n} \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}.
\end{aligned} \tag{3.12}$$

Also

$$\begin{aligned}
D^2 G_n &= n^{2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} \left[ f''(B_{\ell/n}) I_2(\delta_{\ell/n}^{\otimes 2}) \varepsilon_{\ell/n}^{\otimes 2} \right. \\
&\quad \left. + 4f'(B_{\ell/n}) \Delta B_{\ell/n} (\varepsilon_{\ell/n} \widetilde{\otimes} \delta_{\ell/n}) + 2f(B_{\ell/n}) \delta_{\ell/n}^{\otimes 2} \right],
\end{aligned}$$

and, as a consequence,

$$\begin{aligned}
\langle D^2 G_n, \delta_{k/n}^{\otimes 2} \rangle_{\mathfrak{H}^{\otimes 2}} &= n^{2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} \left[ f''(B_{\ell/n}) I_2(\delta_{\ell/n}^{\otimes 2}) \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \right. \\
&\quad + 4f'(B_{\ell/n}) \Delta B_{\ell/n} \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\
&\quad \left. + 2f(B_{\ell/n}) \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \right].
\end{aligned} \tag{3.13}$$

Substituting (3.13) into (3.11) yields

$$\begin{aligned}
&\langle E(D^2(f(B_{k/n})e^{i\lambda G_n}H)), \delta_{k/n}^{\otimes 2} \rangle_{\mathfrak{H}^{\otimes 2}} \\
&= 2i\lambda n^{2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} E(f(B_{k/n})f(B_{\ell/n})e^{i\lambda G_n}H) \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 + r_{k,n},
\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
r_{k,n} &= E \left( f''(B_{k/n}) e^{i\lambda G_n} H \right) \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \\
&\quad + 2i\lambda E \left( f'(B_{k/n}) e^{i\lambda G_n} H \langle DG_n, \delta_{k/n} \rangle_{\mathfrak{H}} \right) \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\
&\quad + 2E \left( f'(B_{k/n}) e^{i\lambda G_n} \langle DH, \delta_{k/n} \rangle_{\mathfrak{H}} \right) \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\
&\quad - \lambda^2 E \left( f(B_{k/n}) e^{i\lambda G_n} H \langle DG_n, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \right) \\
&\quad + 2i\lambda E \left( f(B_{k/n}) e^{i\lambda G_n} \langle DH, \delta_{k/n} \rangle_{\mathfrak{H}} \langle DG_n, \delta_{k/n} \rangle_{\mathfrak{H}} \right) \\
&\quad + i\lambda n^{2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} E \left( f(B_{k/n}) e^{i\lambda G_n} H f''(B_{\ell/n}) I_2(\delta_{\ell/n}^{\otimes 2}) \right) \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \\
&\quad + 4i\lambda n^{2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} E \left( f(B_{k/n}) e^{i\lambda G_n} H f'(B_{\ell/n}) \Delta B_{\ell/n} \right) \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\
&\quad + E \left( f(B_{k/n}) e^{i\lambda G_n} \langle D^2 H, \delta_{k/n}^{\otimes 2} \rangle_{\mathfrak{H}^{\otimes 2}} \right) \\
&= \sum_{j=1}^8 R_{k,n}^j.
\end{aligned}$$

Then, substituting (3.14) into (3.10) and using (3.7) yields (3.8). To complete the proof of this step we need to show the estimate (3.9). For this we will get estimates of the form  $Cn^{-4H}$  for each of the eight terms in the above expression. Clearly, by Lemma 1 we obtain  $|R_{k,n}^1| \leq Cn^{-4H}$ . On the other hand, since  $H$  is a random variable of the form  $\phi(B_{t_1}, \dots, B_{t_m})$  with  $\phi \in \mathcal{C}_b^\infty(\mathbb{R}^m)$ , we also have  $|R_{k,n}^8| \leq Cn^{-4H}$ , and  $|R_{k,n}^3| \leq Cn^{-4H}$ . For the other terms we have

$$\begin{aligned}
|R_{k,n}^2| + |R_{k,n}^5| &\leq Cn^{-2H} \left\| \langle DG_n, \delta_{k/n} \rangle_{\mathfrak{H}} \right\|_2, \\
|R_{k,n}^4| &\leq CE \left( (1 + f^2(B_{k/n})) \langle DG_n, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \right), \\
|R_{k,n}^6| &\leq Cn^{-2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} \left| E \left( f(B_{k/n}) e^{i\lambda G_n} H f''(B_{\ell/n}) I_2(\delta_{\ell/n}^{\otimes 2}) \right) \right|, \\
|R_{k,n}^7| &\leq Cn^{-\frac{1}{2}} \sum_{\ell=0}^{n-1} \left| E \left( f(B_{k/n}) e^{i\lambda G_n} H f'(B_{\ell/n}) \Delta B_{\ell/n} \right) \right| \left| \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right|.
\end{aligned} \tag{3.15}$$

*Step 5.-* Set  $H_{k/n} = 1 + f^2(B_{k/n})$ . We claim that

$$E \left( H_{k/n} \langle DG_n, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \right) \leq Cn^{-4H},$$

and substituting this estimate in (3.15) we will get the desired estimate for  $R_{k,n}^2$ ,  $R_{k,n}^4$  and  $R_{k,n}^5$ . We have

$$E \left( H_{k/n} \langle DG_n, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \right) \leq M_{k,n} + N_{k,n},$$

where

$$M_{k,n} = 2n^{4H-1} \sum_{\ell,j=0}^{n-1} E(H_{k/n} f'(B_{\ell/n}) I_2(\delta_{\ell/n}^{\otimes 2}) f'(B_{j/n}) I_2(\delta_{j/n}^{\otimes 2}) \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}})$$

and

$$N_{k,n} = 8n^{4H-1} \sum_{\ell,j=0}^{n-1} E(H_{k/n} f(B_{\ell/n}) \Delta B_{\ell/n} f(B_{j/n}) \Delta B_{j/n}) \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}}.$$

Notice that  $M_{k,n}$  has an expression similar to that of  $E(G_n^2)$ , but replacing  $f$  by  $f'$ , and adding the factors  $H_{k/n}$  and  $\langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}}$ , and we have

$$\left| \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right| \leq n^{-4H}.$$

As a consequence, estimations similar to those of Step 2 lead to

$$|M_{k,n}| \leq Cn^{-4H}.$$

The term  $N_{k,n}$  is treated in a similar way. In fact, by the duality relation

$$\begin{aligned} & \left| E(H_{k/n} f(B_{\ell/n}) f(B_{j/n}) \Delta B_{\ell/n} \Delta B_{j/n}) \right| \\ & \leq \left| E(H_{k/n} f''(B_{\ell/n}) f(B_{j/n})) \langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{\ell/n}, \delta_{j/n} \rangle_{\mathfrak{H}} \right| \\ & \quad + 2 \left| E(H_{k/n} f'(B_{\ell/n}) f'(B_{j/n})) \langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{j/n}, \delta_{j/n} \rangle_{\mathfrak{H}} \right| \\ & \quad + \left| E(H_{k/n} f(B_{\ell/n}) f''(B_{j/n})) \langle \varepsilon_{j/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{j/n}, \delta_{j/n} \rangle_{\mathfrak{H}} \right| \\ & \quad + \left| E(H_{k/n} f(B_{\ell/n}) f(B_{j/n})) \langle \delta_{\ell/n}, \delta_{j/n} \rangle_{\mathfrak{H}} \right| \\ & \quad + 2E(\langle D^2 H_{k/n}, \delta_{\ell/n} \otimes \delta_{j/n} \rangle_{\mathfrak{H}^{\otimes 2}} f(B_{\ell/n}) f(B_{j/n})) \\ & \leq Cn^{-2H} \end{aligned} \tag{3.16}$$

Thus

$$\begin{aligned} N_{k,n} & \leq Cn^{-1+2H} \left( \sum_{\ell=0}^{n-1} |\langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}| \right)^2 \\ & \leq Cn^{-1-2H} \left( \sum_{p=-\infty}^{\infty} \left| |p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H} \right| \right)^2 = Cn^{-1-2H} \leq Cn^{-4H}, \end{aligned} \tag{3.17}$$

since  $H < \frac{1}{2}$ , and we get the desired estimate.

Step 6.- Estimation of the term  $|R_{k,n}^6|$ . Applying the duality formula yields

$$\begin{aligned}
& E \left( f(B_{k/n}) e^{i\lambda G_n} H f''(B_{\ell/n}) I_2(\delta_{\ell/n}^{\otimes 2}) \right) \\
&= \left\langle E \left( D^2 \left( f(B_{k/n}) e^{i\lambda G_n} H f''(B_{\ell/n}) \right) \right), \delta_{\ell/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} \\
&= E \left( f''(B_{k/n}) e^{i\lambda G_n} H f''(B_{\ell/n}) \right) \langle \varepsilon_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}}^2 \\
&\quad + 2i\lambda E \left( f'(B_{k/n}) e^{i\lambda G_n} H f''(B_{\ell/n}) \langle DG_n, \delta_{\ell/n} \rangle_{\mathfrak{H}} \right) \langle \varepsilon_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \\
&\quad + 2E \left( f'(B_{k/n}) e^{i\lambda G_n} f''(B_{\ell/n}) \langle DH, \delta_{\ell/n} \rangle_{\mathfrak{H}} \right) \langle \varepsilon_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \\
&\quad + 2E \left( f'(B_{k/n}) e^{i\lambda G_n} H f'''(B_{\ell/n}) \right) \langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \\
&\quad - \lambda^2 E \left( f(B_{k/n}) e^{i\lambda G_n} H f''(B_{\ell/n}) \langle DG_n, \delta_{\ell/n} \rangle_{\mathfrak{H}}^2 \right) \\
&\quad + 2i\lambda E \left( f(B_{k/n}) e^{i\lambda G_n} f''(B_{\ell/n}) \langle DH, \delta_{\ell/n} \rangle_{\mathfrak{H}} \langle DG_n, \delta_{\ell/n} \rangle_{\mathfrak{H}} \right) \\
&\quad + 2i\lambda E \left( f(B_{k/n}) e^{i\lambda G_n} f'''(B_{\ell/n}) \langle DG_n, \delta_{\ell/n} \rangle_{\mathfrak{H}} \right) \langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \\
&\quad + i\lambda E \left( f(B_{k/n}) e^{i\lambda G_n} H f''(B_{\ell/n}) \left\langle D^2 G_n, \delta_{\ell/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} \right) \\
&\quad + 2\lambda E \left( f(B_{k/n}) e^{i\lambda G_n} f'''(B_{\ell/n}) \langle DH, \delta_{\ell/n} \rangle_{\mathfrak{H}} \right) \langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \\
&\quad + E \left( f(B_{k/n}) e^{i\lambda G_n} f''(B_{\ell/n}) \left\langle D^2 H, \delta_{\ell/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} \right) \\
&\quad + E \left( f(B_{k/n}) e^{i\lambda G_n} H f^{(4)}(B_{\ell/n}) \right) \langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}}^2.
\end{aligned}$$

By Lemma 1 we have that  $|\langle \varepsilon_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}}|$ ,  $|\langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}}|$ , and  $\|\langle DH, \delta_{\ell/n} \rangle_{\mathfrak{H}}\|_2$  are bounded by  $Cn^{-2H}$  while  $\|\langle D^2 H, \delta_{\ell/n}^{\otimes 2} \rangle_{\mathfrak{H}^{\otimes 2}}\|_2$  is bounded by  $Cn^{-4H}$ . On the other hand, as in Step 5, and taking into account hypothesis **(H)**, we can show that

$$E \left( \widehat{H}_{k/n} \langle DG_n, \delta_{k/n} \rangle_{\mathfrak{H}} \right) \leq Cn^{-4H},$$

where  $\widehat{H}_{k/n} = 1 + (f(B_{k/n}) f''(B_{\ell/n}))^2$ . As a consequence,

$$|R_{k,n}^6| \leq Cn^{-2H-\frac{1}{2}} \left[ n^{1-4H} + \sum_{\ell=0}^{n-1} \left\| \left\langle D^2 G_n, \delta_{\ell/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} \right\|_2 \right].$$

Then, since  $H > \frac{1}{4}$ , it suffices to show that

$$\left\| \left\langle D^2 G_n, \delta_{\ell/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} \right\|_2 \leq Cn^{-2H-\frac{1}{2}}. \quad (3.18)$$

We have

$$\begin{aligned}
\left\langle D^2 G_n, \delta_{k/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} &= n^{2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} \left[ f''(B_{\ell/n}) I_2(\delta_{\ell/n}^{\otimes 2}) \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \right. \\
&\quad + 4f'(B_{\ell/n}) \Delta B_{\ell/n} \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\
&\quad \left. + 2f(B_{\ell/n}) \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \right]. \tag{3.19}
\end{aligned}$$

$$\begin{aligned}
&E \left( \left\langle D^2 G_n, \delta_{\ell/n}^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}}^2 \right) \\
&\leq 3n^{4H-1} \sum_{\ell,j=0}^{n-1} E \left( f''(B_{\ell/n}) I_2(\delta_{\ell/n}^{\otimes 2}) f''(B_{j/n}) I_2(\delta_{j/n}^{\otimes 2}) \right) \\
&\quad \times \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \langle \varepsilon_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \\
&\quad + 48n^{4H-1} \sum_{\ell,j=0}^{n-1} E \left( f'(B_{\ell/n}) \Delta B_{\ell/n} f'(B_{j/n}) \Delta B_{j/n} \right) \\
&\quad \times \langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \varepsilon_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \\
&\quad + 12n^{4H-1} \sum_{\ell,j=0}^{n-1} E \left( f(B_{\ell/n}) f(B_{j/n}) \right) \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \\
&= R_{1,n} + S_{1,n} + T_{1,n}.
\end{aligned}$$

The term  $R_{1,n}$  has an expression similar to that of  $E(G_n^2)$ , but replacing  $f$  by  $f''$ , and adding the factor  $\langle \varepsilon_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \langle \varepsilon_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2$ . As a consequence, we obtain

$$R_{1,n} \leq Cn^{-8H}.$$

For the term  $S_{1,n}$ , using (3.16) yields

$$\begin{aligned}
S_{1,n} &\leq Cn^{-2H-1} \left( \sum_{\ell=0}^{n-1} \left| \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right| \right)^2 \\
&\leq Cn^{-6H-1} \left( \sum_{p=-\infty}^{\infty} \left| |p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H} \right| \right)^2.
\end{aligned}$$

Finally,

$$\begin{aligned}
T_{1,n} &\leq Cn^{4H-1} \left( \sum_{\ell=0}^{n-1} \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \right)^2 \\
&\leq Cn^{-4H-1} \left( \sum_{p=-\infty}^{\infty} \left| |p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H} \right|^2 \right)^2.
\end{aligned}$$

Step 7.- Estimation of the term  $|R_{k,n}^7|$ . We have

$$\begin{aligned}
& E \left( f(B_{k/n}) e^{i\lambda G_n} H f'(B_{\ell/n}) \Delta B_{\ell/n} \right) \\
&= E \left( f'(B_{k/n}) e^{i\lambda G_n} H f'(B_{\ell/n}) \right) \langle \varepsilon_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}} \\
&\quad + i\lambda E \left( f(B_{k/n}) e^{i\lambda G_n} H f'(B_{\ell/n}) \langle DG_n, \delta_{\ell/n} \rangle_{\mathfrak{H}} \right) \\
&\quad + E \left( f(B_{k/n}) e^{i\lambda G_n} f'(B_{\ell/n}) \langle DH, \delta_{\ell/n} \rangle_{\mathfrak{H}} \right) \\
&\quad + E \left( f(B_{k/n}) e^{i\lambda G_n} H f''(B_{\ell/n}) \right) \langle \varepsilon_{\ell/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}}.
\end{aligned}$$

We deduce, using also Step 5 and (3.17):

$$|R_{k,n}^7| \leq C n^{2H-\frac{1}{2}} \sum_{\ell=0}^{n-1} \left| \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \right| \leq C n^{-4H-\frac{1}{2}}.$$

Step 8.- From (3.8), taking the limit as  $n \rightarrow \infty$ , we obtain

$$\begin{aligned}
\lim_{n \rightarrow \infty} \phi'_n(\lambda) &= -2\lambda \lim_{n \rightarrow \infty} n^{4H-1} \sum_{k,\ell=0}^{n-1} E \left( f(B_{k/n}) f(B_{\ell/n}) e^{i\lambda G_n} H \right) \langle \delta_{\ell/n}, \delta_{k/n} \rangle_{\mathfrak{H}}^2 \\
&= -\lambda 2\sigma_H^2 \int_0^1 E \left( f^2(B_s) H e^{i\lambda G_\infty} \right) ds.
\end{aligned}$$

This limit requires some justification. A way to prove it consists to use the Skorohod almost sure representation theorem (see, *e.g.*, [5, p. 281]). Indeed, by using the fact that, on an auxiliary space, there exists a fractional Brownian motion  $\overline{B}$  such that, with obvious notations,  $\overline{G}_n \xrightarrow{\text{a.s.}} \overline{G}_\infty$ , we easily obtain the desired convergence by bounded convergence.

On the other hand, due to the definition of  $\phi_n$ , we also have:

$$\phi'_n(\lambda) \longrightarrow iE \left( G_\infty H e^{i\lambda G_\infty} \right) = \frac{\partial}{\partial \lambda} E \left( H e^{i\lambda G_\infty} \right).$$

Since this result is valid for any smooth (with respect to the  $\sigma$ -field generated by  $(B_t)_{t \in [0,1]}$ ) random variable  $H$ , we have proved:

$$\frac{\partial}{\partial \lambda} E \left( e^{i\lambda G_\infty} | (B_t)_{t \in [0,1]} \right) = -\frac{1}{2} \sigma_H^2 \lambda \int_0^1 f^2(B_s) ds E \left( e^{i\lambda G_\infty} | (B_t)_{t \in [0,1]} \right).$$

In other words,  $\lambda \mapsto E \left( e^{i\lambda G_\infty} | (B_t)_{t \in [0,1]} \right)$  solves a linear differential equation, and we deduce:

$$E \left( e^{i\lambda G_\infty} | (B_t)_{t \in [0,1]} \right) = e^{-\frac{\sigma_H^2 \lambda^2}{2} \int_0^1 f^2(B_s) ds},$$

which proves the desired result.

■

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