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# Central limit theorems for multiple Skorohod integrals

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## Abstract

In this paper, we prove a central limit theorem for a sequence of multiple Skorohod integrals using the techniques of Malliavin calculus. The convergence is stable, and the limit is a conditionally Gaussian random variable. Some applications to sequences of multiple stochastic integrals, and renormalized weighted Hermite variations of the fractional Brownian motion are discussed.

**Key words:** central limit theorem, fractional Brownian motion, Malliavin calculus.

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# 1 Introduction

Consider a sequence of random variables  $\{F_n, n \geq 1\}$  defined on a complete probability space  $(\Omega, \mathcal{F}, P)$ . Suppose that the  $\sigma$ -field  $\mathcal{F}$  is generated by an isonormal Gaussian process  $X = \{X(h), h \in \mathfrak{H}\}$  on a real separable infinite-dimensional Hilbert space  $\mathfrak{H}$ . This just means that  $X$  is a centered Gaussian family of random variables indexed by the elements of  $\mathfrak{H}$ , and such that, for every  $h, g \in \mathfrak{H}$ ,

$$E[X(h)X(g)] = \langle h, g \rangle_{\mathfrak{H}}. \quad (1.1)$$

Suppose that the sequence  $\{F_n, n \geq 1\}$  is normalized, that is,  $E(F_n) = 0$  and  $\lim_{n \rightarrow \infty} E(F_n^2) = 1$ . A natural problem is to find suitable conditions ensuring that  $F_n$  converges in law towards a given distribution. When the random variables  $F_n$  belong to the  $q$ th Wiener chaos of  $X$  (for a fixed  $q \geq 2$ ), then it turns out that the following conditions are equivalent:

- (i)  $F_n$  converges in law to  $N(0, 1)$ ;
- (ii)  $\lim_{n \rightarrow \infty} E[F_n^4] = 3$ ;
- (iii)  $\lim_{n \rightarrow \infty} \|DF_n\|_{\mathfrak{H}}^2 = q$  in  $L^2(\Omega)$ .

Here,  $D$  stands for the derivative operator in the sense of Malliavin calculus (see Section 2 below for more details). More precisely, the following bound is in order, where  $N$  denotes a standard Gaussian random variable:

$$\sup_{z \in \mathbb{R}} |P(F_n \leq z) - P(N \leq z)| \leq \sqrt{E \left[ \left( 1 - \frac{1}{q} \|DF_n\|_{\mathfrak{H}}^2 \right)^2 \right]} \quad (1.2)$$

$$\leq \sqrt{\frac{q-1}{3q}} \sqrt{|E(F_n^4) - 3|}. \quad (1.3)$$

The equivalence between conditions (i) and (ii) was proved in Nualart and Peccati [22] by means of the Dambis, Dubins and Schwarz theorem. It implies that the convergence in distribution of a sequence of multiple stochastic integrals towards a Gaussian random variable is completely determined by the asymptotic behavior of their second and fourth moments, which represents a drastic simplification of the classical “method of moments and diagrams” (see, for instance, the survey by Peccati and Taqqu [26], as well as the references therein). The equivalence with condition (iii) was proved later by Nualart and Ortiz-Latorre [21] using tools of Malliavin

calculus. Finally, the Berry-Esseen's type bound (1.2) is taken from Nourdin and Peccati [16], while (1.3) was shown in Nourdin, Peccati and Reinert [17].

Peccati and Tudor [27] also obtained a multidimensional version of the equivalence between (i) and (ii). In particular, they proved that, given a sequence  $\{F_n, n \geq 1\}$  of  $d$ -dimensional random vectors such that  $F_n^i$  belongs to the  $q_i$ th Wiener chaos for  $i = 1, \dots, d$ , where  $1 \leq q_1 \leq \dots \leq q_d$ , then if the covariance matrix of  $F_n$  converges to the  $d \times d$  identity matrix  $I_d$ , the convergence in distribution to each component towards the law  $N(0, 1)$  implies the convergence in distribution of the whole sequence  $F_n$  towards the standard centered Gaussian law  $N(0, I_d)$ .

Recent examples of application of these results are, among others, the study of  $p$ -variations of fractional stochastic integrals (Corcuera *et al.* [4]), quadratic functionals of bivariate Gaussian processes (Deheuvels *et al.* [5]), self-intersection local times of fractional Brownian motion (Hu and Nualart [7]), approximation schemes for scalar fractional differential equations (Neuenkirch and Nourdin [12]), high-frequency CLTs for random fields on homogeneous spaces (Marinucci and Peccati [10, 11] and Peccati [23]), needlets analysis on the sphere (Baldi *et al.* [1]), estimation of self-similarity orders (Tudor and Viens [31]), weighted power variations of iterated Brownian motion (Nourdin and Peccati [15]) or bipower variations of Gaussian processes with stationary increments (Barndorff-Nielsen *et al.* [2]).

Since the works by Nualart and Peccati [22] and Peccati and Tudor [27], great efforts have been made to find similar statements in the case where the limit is not necessarily Gaussian. In the references [24] and [25], Peccati and Taqqu propose sufficient conditions ensuring that a given sequence of multiple Wiener-Itô integrals converges stably towards mixtures of Gaussian random variables. In another direction, Nourdin and Peccati [14] proved an extension of the above equivalence (i) – (iii) for a sequence of random variables  $\{F_n, n \geq 1\}$  in a fixed  $q$ th Wiener chaos,  $q \geq 2$ , where the limit law is  $2G_{\nu/2} - \nu$ ,  $G_{\nu/2}$  being the Gamma distribution with parameter  $\nu/2$ .

The purpose of the present paper is to study the convergence in distribution of a sequence of random variables of the form  $F_n = \delta^q(u_n)$ , where  $u_n$  are random variables with values in  $\mathfrak{H}^{\otimes q}$  (the  $q$ th tensor product of  $\mathfrak{H}$ ) and  $\delta^q$  denotes the multiple Skorohod integral (that is,  $\delta^2(u) = \delta(\delta(u))$ ,  $\delta^3(u) = \delta(\delta(\delta(u)))$ , and so on), towards a mixture of Gaussian random variables. Our main abstract result, Theorem 3.1, roughly says that under some technical conditions, if  $\langle u_n, D^q F_n \rangle_{\mathfrak{H}^{\otimes q}}$  converges in  $L^1(\Omega)$  to a nonnegative

random variable  $S^2$ , then the sequence  $F_n$  converges stably to a random variable  $F$  with conditional characteristic function  $E(e^{i\lambda F} | X) = E(e^{-\frac{\lambda^2}{2} S^2})$ . Notice that if  $u_n$  is deterministic, then  $F_n$  belongs to the  $q$ th Wiener chaos, and we have a sequence of the type considered above. In particular, if  $S^2$  is also deterministic, we recover the fact that condition (iii) above implies the convergence in distribution to the law  $N(0, 1)$ .

We develop some particular applications of Theorem 3.1 in the following directions. First, we consider a sequence of random variables in a fixed Wiener chaos and we derive new criteria for the convergence to a mixture of Gaussian laws. Second, we show the convergence in law of the sequence  $\delta^q(u_n)$ , where  $q \geq 2$  and  $u_n$  is a  $q$ -parameter process of the form

$$u_n = n^{qH - \frac{1}{2}} \sum_{k=0}^{n-1} f(B_{k/n}) \mathbf{1}_{(k/n, (k+1)/n]^q},$$

towards the random variable  $\sigma_{H,q} \int_0^1 f(B_s) dW_s$ , where  $B$  is a fractional Brownian motion with Hurst parameter  $H \in \left(\frac{1}{4q}, \frac{1}{2}\right)$ ,  $W$  is a standard Brownian motion independent of  $B$ , and  $\sigma_{H,q}$  denotes some positive constant. This convergence allows us to establish a new asymptotic result for the behavior of the weighted  $q$ th Hermite variation of the fractional Brownian motion with Hurst parameter  $H \in \left(\frac{1}{4q}, \frac{1}{2}\right)$ , which complements and provides a new perspective to the results proved by Nourdin [13], Nourdin, Nualart and Tudor [18], and Nourdin and Réveillac [19]. The reader is referred to Section 5 for a detailed description of these results.

The paper is organized as follows. In Section 2, we present some preliminary results about Malliavin calculus. Section 3 contains the statement and the proof of the main abstract result. In Section 4, we apply it to sequences of multiple stochastic integrals, while Section 5 focuses on the applications to the weighted Hermite variations of the fractional Brownian motion.

## 2 Preliminaries

Let  $\mathfrak{H}$  be a real separable infinite-dimensional Hilbert space. For any integer  $q \geq 1$ , let  $\mathfrak{H}^{\otimes q}$  be the  $q$ th tensor product of  $\mathfrak{H}$ . Also, we denote by  $\mathfrak{H}^{\odot q}$  the  $q$ th symmetric tensor product.

Suppose that  $X = \{X(h), h \in \mathfrak{H}\}$  is an isonormal Gaussian process on  $\mathfrak{H}$ , defined on some probability space  $(\Omega, \mathcal{F}, P)$ . Recall that this means that

the covariance of  $X$  is given in terms of the scalar product of  $\mathfrak{H}$  by (1.1). Assume from now on that  $\mathcal{F}$  is generated by  $X$ .

For every integer  $q \geq 1$ , let  $\mathcal{H}_q$  be the  $q$ th Wiener chaos of  $X$ , that is, the closed linear subspace of  $L^2(\Omega)$  generated by the random variables  $\{H_q(X(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$ , where  $H_q$  is the  $q$ th Hermite polynomial defined by

$$H_q(x) = \frac{(-1)^q}{q!} e^{x^2/2} \frac{d^q}{dx^q} (e^{-x^2/2}).$$

We denote by  $\mathcal{H}_0$  the space of constant random variables. For any  $q \geq 1$ , the mapping  $I_q(h^{\otimes q}) = q!H_q(X(h))$  provides a linear isometry between  $\mathfrak{H}^{\otimes q}$  (equipped with the modified norm  $\sqrt{q!} \|\cdot\|_{\mathfrak{H}^{\otimes q}}$ ) and  $\mathcal{H}_q$  (equipped with the  $L^2(\Omega)$  norm). For  $q = 0$ , by convention  $\mathcal{H}_0 = \mathbb{R}$ , and  $I_0$  is the identity map.

It is well-known (Wiener chaos expansion) that  $L^2(\Omega)$  can be decomposed into the infinite orthogonal sum of the spaces  $\mathcal{H}_q$ . That is, any square integrable random variable  $F \in L^2(\Omega)$  admits the following chaotic expansion:

$$F = \sum_{q=0}^{\infty} I_q(f_q), \quad (2.1)$$

where  $f_0 = E[F]$ , and the  $f_q \in \mathfrak{H}^{\otimes q}$ ,  $q \geq 1$ , are uniquely determined by  $F$ . For every  $q \geq 0$ , we denote by  $J_q$  the orthogonal projection operator on the  $q$ th Wiener chaos. In particular, if  $F \in L^2(\Omega)$  is as in (2.1), then  $J_q F = I_q(f_q)$  for every  $q \geq 0$ .

Let  $\{e_k, k \geq 1\}$  be a complete orthonormal system in  $\mathfrak{H}$ . Given  $f \in \mathfrak{H}^{\otimes p}$ ,  $g \in \mathfrak{H}^{\otimes q}$  and  $r \in \{0, \dots, p \wedge q\}$ , the  $r$ th contraction of  $f$  and  $g$  is the element of  $\mathfrak{H}^{\otimes(p+q-2r)}$  defined by

$$f \otimes_r g = \sum_{i_1, \dots, i_r=1}^{\infty} \langle f, e_{i_1} \otimes \dots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \dots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}}. \quad (2.2)$$

Notice that  $f \otimes_r g$  is not necessarily symmetric. We denote its symmetrization by  $f \widetilde{\otimes}_r g \in \mathfrak{H}^{\otimes(p+q-2r)}$ . Moreover,  $f \otimes_0 g = f \otimes g$  equals the tensor product of  $f$  and  $g$  while, for  $p = q$ ,  $f \otimes_q g = \langle f, g \rangle_{\mathfrak{H}^{\otimes q}}$ .

In the particular case  $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$ , where  $(A, \mathcal{A})$  is a measurable space and  $\mu$  is a  $\sigma$ -finite and non-atomic measure, one has that  $\mathfrak{H}^{\otimes q} = L^2_s(A^q, \mathcal{A}^{\otimes q}, \mu^{\otimes q})$  is the space of symmetric and square integrable functions on  $A^q$ . Moreover, for every  $f \in \mathfrak{H}^{\otimes q}$ ,  $I_q(f)$  coincides with the multiple Wiener-Itô integral of order  $q$  of  $f$  with respect to  $X$  (introduced by Itô in

[8]) and (2.2) can be written as

$$(f \otimes_r g)(t_1, \dots, t_{p+q-2r}) = \int_{A^r} f(t_1, \dots, t_{p-r}, s_1, \dots, s_r) \\ \times g(t_{p-r+1}, \dots, t_{p+q-2r}, s_1, \dots, s_r) d\mu(s_1) \dots d\mu(s_r).$$

Let us now introduce some basic elements of the Malliavin calculus with respect to the isonormal Gaussian process  $X$ . We refer the reader to Nualart [20] for a more detailed presentation of these notions. Let  $\mathcal{S}$  be the set of all smooth and cylindrical random variables of the form

$$F = g(X(\phi_1), \dots, X(\phi_n)), \quad (2.3)$$

where  $n \geq 1$ ,  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  is a infinitely differentiable function with compact support, and  $\phi_i \in \mathfrak{H}$ . The Malliavin derivative of  $F$  with respect to  $X$  is the element of  $L^2(\Omega, \mathfrak{H})$  defined as

$$DF = \sum_{i=1}^n \frac{\partial g}{\partial x_i}(X(\phi_1), \dots, X(\phi_n)) \phi_i.$$

By iteration, one can define the  $q$ th derivative  $D^q F$  for every  $q \geq 2$ , which is an element of  $L^2(\Omega, \mathfrak{H}^{\odot q})$ .

For  $q \geq 1$  and  $p \geq 1$ ,  $\mathbb{D}^{q,p}$  denotes the closure of  $\mathcal{S}$  with respect to the norm  $\|\cdot\|_{\mathbb{D}^{q,p}}$ , defined by the relation

$$\|F\|_{\mathbb{D}^{q,p}}^p = E[|F|^p] + \sum_{i=1}^q E\left(\|D^i F\|_{\mathfrak{H}^{\otimes i}}^p\right).$$

The Malliavin derivative  $D$  verifies the following chain rule. If  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuously differentiable with bounded partial derivatives and if  $F = (F_1, \dots, F_n)$  is a vector of elements of  $\mathbb{D}^{1,2}$ , then  $\varphi(F) \in \mathbb{D}^{1,2}$  and

$$D\varphi(F) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F) DF_i.$$

We denote by  $\delta$  the adjoint of the operator  $D$ , also called the divergence operator. The operator  $\delta$  is also called the Skorohod integral because in the case of the Brownian motion it coincides with the anticipating stochastic integral introduced by Skorohod in [30]. A random element  $u \in L^2(\Omega, \mathfrak{H})$  belongs to the domain of  $\delta$ , noted  $\text{Dom}\delta$ , if and only if it verifies

$$|E(\langle DF, u \rangle_{\mathfrak{H}})| \leq c_u \sqrt{E(F^2)}$$

for any  $F \in \mathbb{D}^{1,2}$ , where  $c_u$  is a constant depending only on  $u$ . If  $u \in \text{Dom}\delta$ , then the random variable  $\delta(u)$  is defined by the duality relationship (called ‘integration by parts formula’):

$$E(F\delta(u)) = E(\langle DF, u \rangle_{\mathfrak{H}}), \quad (2.4)$$

which holds for every  $F \in \mathbb{D}^{1,2}$ . The formula (2.4) extends to the multiple Skorohod integral  $\delta^q$ , and we have

$$E(F\delta^q(u)) = E(\langle D^q F, u \rangle_{\mathfrak{H}^{\otimes q}}) \quad (2.5)$$

for any element  $u$  in the domain of  $\delta^q$  and any random variable  $F \in \mathbb{D}^{q,2}$ . Moreover,  $\delta^q(h) = I_q(h)$  for any  $h \in \mathfrak{H}^{\odot q}$ .

The following property will be extensively used in the paper.

**Lemma 2.1** *Let  $q \geq 1$  be an integer. Suppose that  $F \in \mathbb{D}^{q,2}$ , and let  $u$  be a symmetric element in  $\text{Dom}\delta^q$ . Assume that, for any  $0 \leq r + j \leq q$ ,  $\langle D^r F, \delta^j(u) \rangle_{\mathfrak{H}^{\otimes r}} \in L^2(\Omega, \mathfrak{H}^{\otimes q-r-j})$ . Then, for any  $r = 0, \dots, q-1$ ,  $\langle D^r F, u \rangle_{\mathfrak{H}^{\otimes r}}$  belongs to the domain of  $\delta^{q-r}$  and we have*

$$F\delta^q(u) = \sum_{r=0}^q \binom{q}{r} \delta^{q-r}(\langle D^r F, u \rangle_{\mathfrak{H}^{\otimes r}}). \quad (2.6)$$

(We use the convention that  $\delta^0(v) = v$ ,  $v \in \mathbb{R}$ , and  $D^0 F = F$ ,  $F \in L^2(\Omega)$ .)

**Proof.** We prove this lemma by induction on  $q$ . For  $q = 1$  it reads  $F\delta(u) = \delta(Fu) + \langle DF, u \rangle_{\mathfrak{H}}$ , and this formula is well-known, see e.g. [20, Proposition 1.3.3]. Suppose the result is true for  $q$ . Then, if  $u$  belongs to the domain of  $\delta^{q+1}$ , by the induction hypothesis applied to  $\delta(u)$ ,

$$F\delta^{q+1}(u) = F\delta^q(\delta(u)) = \sum_{r=0}^q \binom{q}{r} \delta^{q-r}(\langle D^r F, \delta(u) \rangle_{\mathfrak{H}^{\otimes r}}). \quad (2.7)$$

On the other hand

$$\langle D^r F, \delta(u) \rangle_{\mathfrak{H}^{\otimes r}} = \delta(\langle D^r F, u \rangle_{\mathfrak{H}^{\otimes r}}) + \langle D^{r+1} F, u \rangle_{\mathfrak{H}^{\otimes r}}. \quad (2.8)$$

Finally, substituting (2.8) into (2.7) yields the desired result. ■

For any Hilbert space  $V$ , we denote by  $\mathbb{D}^{k,p}(V)$  the corresponding Sobolev space of  $V$ -valued random variables (see [20, page 31]). The operator  $\delta^q$



is continuous from  $\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})$  to  $\mathbb{D}^{k-q,p}$ , for any  $p > 1$  and any integers  $k \geq q \geq 1$ , that is, we have

$$\|\delta^q(u)\|_{\mathbb{D}^{k-q,p}} \leq c_{k,p} \|u\|_{\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})} \quad (2.9)$$

for all  $u \in \mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})$ , and some constant  $c_{k,p} > 0$ . These estimates are consequences of Meyer inequalities (see [20, Proposition 1.5.7]). In particular, these estimates imply that  $\mathbb{D}^{q,2}(\mathfrak{H}^{\otimes q}) \subset \text{Dom}\delta^q$  for any integer  $q \geq 1$ .

We will also use the following commutation relationship between the Malliavin derivative and the Skorohod integral (see [20, Proposition 1.3.2])

$$D\delta(u) = u + \delta(Du), \quad (2.10)$$

for any  $u \in \mathbb{D}^{2,2}(\mathfrak{H})$ . By induction we can show the following formula for any symmetric element  $u$  in  $\mathbb{D}^{j+k,2}(\mathfrak{H}^{\otimes j})$

$$D^k \delta^j(u) = \sum_{i=0}^{j \wedge k} \binom{k}{i} \binom{j}{i} i! \delta^{j-i}(D^{k-i}u). \quad (2.11)$$

We will make use of the following formula for the variance of a multiple Skorohod integral. Let  $u, v \in \mathbb{D}^{2q,2}(\mathfrak{H}^{\otimes q}) \subset \text{Dom}\delta^q$  be two symmetric functions. Then

$$\begin{aligned} E(\delta^q(u)\delta^q(v)) &= E(\langle u, D^q(\delta^q(v)) \rangle_{\mathfrak{H}^{\otimes q}}) \\ &= \sum_{i=0}^q \binom{q}{i}^2 i! E(\langle u, \delta^{q-i}(D^{q-i}v) \rangle_{\mathfrak{H}^{\otimes q}}) \\ &= \sum_{i=0}^q \binom{q}{i}^2 i! E(\langle D^{q-i}u, D^{q-i}v \rangle_{\mathfrak{H}^{\otimes(2q-i)}}). \end{aligned} \quad (2.12)$$

The operator  $L$  is defined on the Wiener chaos expansion as

$$L = \sum_{q=0}^{\infty} -qJ_q,$$

and is called the infinitesimal generator of the Ornstein-Uhlenbeck semigroup. The domain of this operator in  $L^2(\Omega)$  is the set

$$\text{Dom}L = \{F \in L^2(\Omega) : \sum_{q=1}^{\infty} q^2 \|J_q F\|_{L^2(\Omega)}^2 < \infty\} = \mathbb{D}^{2,2}.$$

There is an important relation between the operators  $D$ ,  $\delta$  and  $L$  (see [20, Proposition 1.4.3]). A random variable  $F$  belongs to the domain of  $L$  if and only if  $F \in \text{Dom}(\delta D)$  (i.e.  $F \in \mathbb{D}^{1,2}$  and  $DF \in \text{Dom}\delta$ ), and in this case

$$\delta DF = -LF. \quad (2.13)$$

Note also that a random variable  $F$  as in (2.1) is in  $\mathbb{D}^{1,2}$  if and only if

$$\sum_{q=1}^{\infty} qq! \|f_q\|_{\mathfrak{H}^{\otimes q}}^2 < \infty,$$

and, in this case,  $E(\|DF\|_{\mathfrak{H}}^2) = \sum_{q \geq 1} qq! \|f_q\|_{\mathfrak{H}^{\otimes q}}^2$ . If  $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$  (with  $\mu$  non-atomic), then the derivative of a random variable  $F$  as in (2.1) can be identified with the element of  $L^2(A \times \Omega)$  given by

$$D_a F = \sum_{q=1}^{\infty} q I_{q-1}(f_q(\cdot, a)), \quad a \in A. \quad (2.14)$$

Finally, we need the definition of stable convergence (see, for instance, the original paper [29], or the book [9] for an exhaustive discussion of stable convergence).

**Definition 2.2** *Let  $F_n$  be a sequence of random variables defined on the probability space  $(\Omega, \mathcal{F}, P)$ , and suppose that  $F$  is a random variable defined on an enlarged probability space  $(\Omega, \mathcal{G}, P)$ , with  $\mathcal{F} \subseteq \mathcal{G}$ . We say that  $F_n$  converges  $\mathcal{G}$ -stably to  $F$  (or only stably when the context is clear) if, for any continuous and bounded function  $f : \mathbb{R} \rightarrow \mathbb{R}$  and any bounded  $\mathcal{F}$ -measurable random variable  $Z$ , we have  $E[f(F_n)Z] \rightarrow E[f(F)Z]$  as  $n$  tends to infinity.*

### 3 Convergence in law of multiple Skorohod integrals

As in the previous section,  $X = \{X(h), h \in \mathfrak{H}\}$  is an isonormal Gaussian process associated with a real separable infinite-dimensional Hilbert space  $\mathfrak{H}$ . The next theorem is the main abstract result of the present paper.

**Theorem 3.1** *Fix an integer  $q \geq 1$ , and suppose that  $F_n$  is a sequence of random variables of the form  $F_n = \delta^q(u_n)$ , for some symmetric functions  $u_n$  in  $\mathbb{D}^{2q, 2q}(\mathfrak{H}^{\otimes q})$ . Suppose moreover that the sequence  $F_n$  is bounded in  $L^1(\Omega)$ , and that:*

(i)  $\langle u_n, (DF_n)^{\otimes k_1} \otimes \dots \otimes (D^{q-1}F_n)^{\otimes k_{q-1}} \otimes h \rangle_{\mathfrak{H}^{\otimes q}}$  converges in  $L^1(\Omega)$  to zero, for all integers  $r, k_1, \dots, k_{q-1} \geq 0$  such that

$$k_1 + 2k_2 + \dots + (q-1)k_{q-1} + r = q,$$

and all  $h \in \mathfrak{H}^{\otimes r}$ ;

(ii)  $\langle u_n, D^q F_n \rangle_{\mathfrak{H}^{\otimes q}}$  converges in  $L^1(\Omega)$  to a nonnegative random variable  $S^2$ .

Then,  $F_n$  converges stably to a random variable with conditional Gaussian law  $N(0, S^2)$  given  $X$ .

**Remark 3.2** When  $q = 1$ , condition (i) of the theorem is that  $\langle u_n, h \rangle_{\mathfrak{H}}$  converges to zero in  $L^1(\Omega)$ , for each  $h \in \mathfrak{H}$ . When  $q = 2$ , condition (i) means that  $\langle u_n, h \otimes g \rangle_{\mathfrak{H}^{\otimes 2}}$ ,  $\langle u_n, DF_n \otimes h \rangle_{\mathfrak{H}^{\otimes 2}}$  and  $\langle u_n, DF_n \otimes DF_n \rangle_{\mathfrak{H}^{\otimes 2}}$  converge to zero in  $L^1(\Omega)$ , for each  $h, g \in \mathfrak{H}$ . And so on.

**Proof of Theorem 3.1.** Taking into account Definition 2.2, it suffices to show that for any  $h_1, \dots, h_m \in \mathfrak{H}$ , the sequence

$$\xi_n = (F_n, X(h_1), \dots, X(h_m))$$

converges in distribution to a vector  $(F_\infty, X(h_1), \dots, X(h_m))$ , where  $F_\infty$  satisfies, for any  $\lambda \in \mathbb{R}$ ,

$$E(e^{i\lambda F_\infty} | X(h_1), \dots, X(h_m)) = e^{-\frac{\lambda^2}{2} S^2}. \quad (3.1)$$

Since the sequence  $F_n$  is bounded in  $L^1(\Omega)$ , the sequence  $\xi_n$  is tight. Assume that  $(F_\infty, X(h_1), \dots, X(h_m))$  denotes the limit in law of a certain subsequence of  $\xi_n$ , denoted again by  $\xi_n$ .

Let  $Y = \phi(X(h_1), \dots, X(h_m))$ , with  $\phi \in \mathcal{C}_b^\infty(\mathbb{R}^m)$  ( $\phi$  is infinitely differentiable, bounded, with bounded partial derivatives of all orders), and consider  $\phi_n(\lambda) = E(e^{i\lambda F_n} Y)$  for  $\lambda \in \mathbb{R}$ . The convergence in law of  $\xi_n$ , together with the fact that  $F_n$  is bounded in  $L^1(\Omega)$ , imply that

$$\lim_{n \rightarrow \infty} \phi_n'(\lambda) = \lim_{n \rightarrow \infty} iE(F_n e^{i\lambda F_n} Y) = iE(F_\infty e^{i\lambda F_\infty} Y). \quad (3.2)$$

On the other hand, by (2.5) and the Leibnitz rule for  $D^q$ , we obtain

$$\begin{aligned}
\phi'_n(\lambda) &= iE(F_n e^{i\lambda F_n} Y) = iE\left(\delta^q(u_n) e^{i\lambda F_n} Y\right) \\
&= iE\left(\left\langle u_n, D^q\left(e^{i\lambda F_n} Y\right)\right\rangle_{\mathfrak{H}^{\otimes q}}\right) \\
&= i\sum_{a=0}^q \binom{q}{a} E\left(\left\langle u_n, D^a\left(e^{i\lambda F_n}\right) \tilde{\otimes} D^{q-a} Y\right\rangle_{\mathfrak{H}^{\otimes q}}\right) \\
&= i\sum_{a=0}^q \binom{q}{a} \sum \frac{a!}{k_1! \dots k_a!} (i\lambda)^{k_1 + \dots + k_a} \\
&\quad \times E\left(e^{i\lambda F_n} \left\langle u_n, (DF_n)^{\otimes k_1} \tilde{\otimes} \dots \tilde{\otimes} (D^a F_n)^{\otimes k_a} \tilde{\otimes} D^{q-a} Y\right\rangle_{\mathfrak{H}^{\otimes q}}\right) \\
&= i\sum_{a=0}^q \binom{q}{a} \sum \frac{a!}{k_1! \dots k_a!} (i\lambda)^{k_1 + \dots + k_a} \\
&\quad \times E\left(e^{i\lambda F_n} \left\langle u_n, (DF_n)^{\otimes k_1} \otimes \dots \otimes (D^a F_n)^{\otimes k_a} \otimes D^{q-a} Y\right\rangle_{\mathfrak{H}^{\otimes q}}\right),
\end{aligned}$$

where the second sum in the two last equalities runs over all sequences of integers  $(k_1, \dots, k_a)$  such that  $k_1 + 2k_2 + \dots + ak_a = a$ , due to the Faá di Bruno's formula. By condition (i), this yields that

$$\phi'_n(\lambda) = -\lambda E\left(e^{i\lambda F_n} \langle u_n, D^q F_n \rangle_{\mathfrak{H}^{\otimes q}} Y\right) + R_n,$$

with  $R_n$  converging to zero as  $n \rightarrow \infty$ . Using condition (ii) and (3.2), we obtain that

$$iE(F_\infty e^{i\lambda F_\infty} Y) = -\lambda E\left(e^{i\lambda F_\infty} S^2 Y\right).$$

Since  $S^2$  is defined through condition (ii), it is in particular measurable with respect to  $X$ . Thus, the following linear differential equation verified by the conditional characteristic function of  $F_\infty$  holds:

$$\frac{\partial}{\partial \lambda} E(e^{i\lambda F_\infty} | X(h_1), \dots, X(h_m)) = -\lambda S^2 E(e^{i\lambda F_\infty} | X(h_1), \dots, X(h_m)).$$

By solving it, we obtain (3.1), which yields the desired conclusion. ■

The next corollary provides stronger but easier conditions for the stable convergence.

**Corollary 3.3** *For a fixed  $q \geq 1$ , suppose that  $F_n$  is a sequence of random variables of the form  $F_n = \delta^q(u_n)$ , for some symmetric functions  $u_n$  in  $\mathbb{D}^{2q, 2q}(\mathfrak{H}^{\otimes q})$ . Suppose moreover that the sequence  $F_n$  is bounded in  $\mathbb{D}^{q, p}$  for all  $p \geq 2$ , and that:*

- (i')  $\langle u_n, h \rangle_{\mathfrak{H}^{\otimes q}}$  converges to zero in  $L^1(\Omega)$  for all  $h \in \mathfrak{H}^{\otimes q}$ ; and  $u_n \otimes_l D^l F_n$  converges to zero in  $L^2(\Omega; \mathfrak{H}^{\otimes(q-l)})$  for all  $l = 1, \dots, q-1$ ;
- (ii)  $\langle u_n, D^q F_n \rangle_{\mathfrak{H}^{\otimes q}}$  converges in  $L^1(\Omega)$  to a nonnegative random variable  $S^2$ .

Then,  $F_n$  converges stably to a random variable with conditional Gaussian law  $N(0, S^2)$  given  $X$ .

**Proof.** It suffices to show that condition (i') implies condition (i) in Theorem 3.1. When  $k_a \neq 0$  for  $1 \leq a \leq q-1$ , we have, for all  $h \in \mathfrak{H}^{\otimes r}$  (with  $r = q - k_1 - 2k_2 - \dots - ak_a$ ),

$$\begin{aligned}
& \left| \left\langle u_n, (DF_n)^{\otimes k_1} \otimes \dots \otimes (D^a F_n)^{\otimes k_a} \otimes h \right\rangle_{\mathfrak{H}^{\otimes q}} \right| \\
&= \left| \left\langle u_n \otimes_a D^a F_n, \right. \right. \\
&\quad \left. \left. (DF_n)^{\otimes k_1} \otimes \dots \otimes (D^{a-1} F_n)^{\otimes k_{a-1}} \otimes (D^a F_n)^{\otimes(k_a-1)} \otimes h \right\rangle_{\mathfrak{H}^{\otimes(q-a)}} \right| \\
&\leq \|u_n \otimes_a D^a F_n\|_{\mathfrak{H}^{\otimes(q-a)}} \\
&\quad \times \left\| (DF_n)^{\otimes k_1} \otimes \dots \otimes (D^{a-1} F_n)^{\otimes k_{a-1}} \otimes (D^a F_n)^{\otimes(k_a-1)} \otimes h \right\|_{\mathfrak{H}^{\otimes(q-a)}}.
\end{aligned}$$

The second factor is bounded in  $L^2(\Omega)$ , and the first factor converges to zero in  $L^2(\Omega)$ , for all  $a = 1, \dots, q-1$ . In the case  $a = 0$  we have that  $\langle u_n, h \rangle_{\mathfrak{H}^{\otimes q}}$  converges to zero in  $L^1(\Omega)$ , for all  $h \in \mathfrak{H}^{\otimes q}$ , by condition (i'). This completes the proof. ■

## 4 Multiple stochastic integrals

Suppose that  $\mathfrak{H}$  is a Hilbert space  $L^2(A, \mathcal{A}, \mu)$ , where  $(A, \mathcal{A})$  is a measurable space and  $\mu$  is a  $\sigma$ -finite and non-atomic measure.

Fix an integer  $m \geq 2$ , and consider a sequence of multiple stochastic integrals  $\{F_n = I_m(g_n), n \geq 1\}$  with  $g_n \in \mathfrak{H}^{\otimes m}$ . We would like to apply Theorem 3.1 with  $q = 1$  to the sequence  $F_n$ . To do this, we represent each  $F_n$  as

$$F_n = \delta(u_n), \quad \text{with } u_n = I_{m-1}(\widehat{g}_n),$$

for  $\widehat{g}_n \in \mathfrak{H}^{\otimes m}$  some function which is symmetric in the first  $m-1$  variables.

Notice that, from (2.14), we have  $DF_n = mI_{m-1}(g_n)$ . Hence, since  $F_n = -\frac{1}{m}LF_n = \frac{1}{m}\delta(DF_n)$  by (2.13),  $g_n$  is always a possible choice for  $\widehat{g}_n$ . (In this case,  $\widehat{g}_n$  is symmetric in all the variables.) However, as observed,

for instance, in Example 4.2 below, the choice  $\widehat{g}_n = g_n$  does not allow to conclude in general.

**Proposition 4.1** *For a fixed integer  $m \geq 2$ , let  $F_n$  be a sequence of random variables of the form  $F_n = I_m(g_n)$ , with  $g_n \in \mathfrak{H}^{\odot m}$ . Suppose moreover that  $F_n$  is bounded in  $L^2(\Omega)$  and that  $F_n = \delta(u_n)$ , where  $u_n = I_{m-1}(\widehat{g}_n)$ , for  $\widehat{g}_n \in \mathfrak{H}^{\otimes m}$  some function which is symmetric in the first  $m - 1$  variables. Finally, assume that:*

- (a)  $\langle \widehat{g}_n \otimes_{m-1} \widehat{g}_n, h^{\otimes 2} \rangle_{\mathfrak{H}^{\otimes 2}}$  converges to zero for all  $h \in \mathfrak{H}$ ;
- (b)  $\langle u_n, DF_n \rangle_{\mathfrak{H}}$  converges in  $L^1(\Omega)$  to a non negative random variable  $S^2$ .

Then,  $F_n$  converges stably to a random variable with conditional Gaussian law  $N(0, S^2)$  given  $X$ .

**Proof.** It suffices to apply Theorem 3.1 to  $u_n = I_{m-1}(\widehat{g}_n)$  and  $q = 1$ . Indeed, we have

$$\begin{aligned} E(\langle u_n, h \rangle_{\mathfrak{H}}^2) &= E\left(\langle I_{m-1}(\widehat{g}_n), h \rangle_{\mathfrak{H}}^2\right) = E\left(I_{m-1}(\widehat{g}_n \otimes_1 h)^2\right) \\ &= (m-1)! \|\widehat{g}_n \otimes_1 h\|_{\mathfrak{H}^{\otimes(m-1)}}^2 \\ &= (m-1)! \langle \widehat{g}_n \otimes_{m-1} \widehat{g}_n, h^{\otimes 2} \rangle_{\mathfrak{H}^{\otimes 2}} \rightarrow 0, \end{aligned}$$

which implies condition (i) in Theorem 3.1, see also Remark 3.2. Condition (ii) in Theorem 3.1 follows from (b). ■

**Example 4.2** (see also [28, Proposition 2.1] or [24, Proposition 18] for two different proofs using other techniques). Suppose that  $\{W_t, t \in [0, 1]\}$  is a standard Brownian motion. (This corresponds to  $A = [0, 1]$  and  $\mu$  the Lebesgue measure.) Assume that  $m = 2$  and take  $g_n(s, t) = \frac{1}{2}\sqrt{n}(s \vee t)^n$ . Then

$$F_n = I_2(g_n) = \sqrt{n} \int_0^1 t^n W_t dW_t,$$

and

$$D_s F_n = \sqrt{n} s^n W_s + \sqrt{n} \int_s^1 t^n W_t dW_t.$$

We can take  $u_n(t) = \sqrt{n} t^n W_t$ , that is,  $\widehat{g}_n(s, t) = \sqrt{n} t^n \mathbf{1}_{[0, t]}(s)$ . In this case,

$$(\widehat{g}_n \otimes_1 \widehat{g}_n)(s, t) = n s^n t^n (s \wedge t),$$

which converges to zero weakly in  $L^2(\Omega)$ , and

$$\langle u_n, DF_n \rangle_{\mathfrak{H}} = \int_0^1 nt^{2n}W_t^2 dt + n \int_0^1 t^n W_t \left( \int_0^t s^n W_s dW_s \right) dt,$$

which converges in  $L^2(\Omega)$  to  $\frac{1}{2}W_1^2$ . Therefore, conditions (a) and (b) of Proposition 4.1 are satisfied with  $S^2 = \frac{1}{2}W_1^2$ , and  $F_n$  converges in distribution to  $\frac{1}{\sqrt{2}}W_1 \times N$ , with  $N \sim N(0, 1)$ . One easily see on this particular example that the choice  $\widehat{g}_n = g_n$  does not allows us to conclude in general (except when  $S^2$  is deterministic); indeed, one can check here that  $\langle u_n, DF_n \rangle_{\mathfrak{H}} = \frac{1}{m} \|DF_n\|_{\mathfrak{H}}^2$  does not converge in  $L^1(\Omega)$ .

If we take  $\widehat{g}_n = g_n$  and  $S^2 = 1$ , then condition (b) coincides with condition (iii) in the introduction. In this case, Nualart and Peccati criterion combined with Lemma 6 in [21] tells us that, if the sequence of variances converges to one, then condition (a) is automatically satisfied.

On the other hand, we can also apply Theorem 3.1 with  $u_n = g_n$ . In this way, applying Corollary 3.3, we obtain that the following conditions imply that  $F_n$  converges to a normal random variable  $N(0, 1)$  independent of  $X$ :

- ( $\alpha$ )  $g_n$  converges weakly to zero;
- ( $\beta$ )  $\|g_n \otimes_l g_n\|_{\mathfrak{H}^{\otimes 2(q-l)}}$  converges to zero for all  $l = 1, \dots, q-1$ ;
- ( $\gamma$ )  $q! \|g_n\|_{\mathfrak{H}^{\otimes q}}^2$  converges to 1.

Indeed, notice first that if  $g_n$  is bounded in  $\mathfrak{H}^{\otimes q}$ , then  $F_n$  is bounded in all the Sobolev spaces  $\mathbb{D}^{q,p}$ ,  $p \geq 2$ . Then, condition (ii) in Corollary 3.3 follows from ( $\gamma$ ) and the equality  $D^q(I_q(g_n)) = q!g_n$ . On the other hand, condition (i') in Corollary 3.3 follows from (ii) and

$$\begin{aligned} E \left[ \left\| g_n \otimes_l D^l F_n \right\|_{\mathfrak{H}^{\otimes (q-l)}}^2 \right] &= \frac{q!^2}{(q-l)!^2} E \left[ \left\| g_n \otimes_l I_{q-l}(g_n) \right\|_{\mathfrak{H}^{\otimes (q-l)}}^2 \right] \\ &= \frac{q!^2}{(q-l)!^2} E \left[ \left\| I_{q-l}(g_n \otimes_l g_n) \right\|_{\mathfrak{H}^{\otimes (q-l)}}^2 \right] \\ &= \frac{q!^2}{(q-l)!} \left\| g_n \widetilde{\otimes}_l g_n \right\|_{\mathfrak{H}^{\otimes 2(q-l)}}^2 \\ &\leq \frac{q!^2}{(q-l)!} \left\| g_n \otimes_l g_n \right\|_{\mathfrak{H}^{\otimes 2(q-l)}}^2. \end{aligned}$$

In this way we recover the fact that condition (iii) in the introduction implies the normal convergence.

## 5 Weighted Hermite variations of the fractional Brownian motion

### 5.1 Description of the results

The fractional Brownian motion (fBm) with Hurst parameter  $H \in (0, 1)$  is a centered Gaussian process  $B = \{B_t, t \geq 0\}$  with the covariance function

$$E(B_s B_t) = R_H(s, t) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}). \quad (5.1)$$

From (5.1), it follows that  $E|B_t - B_s|^2 = (t - s)^{2H}$  for all  $0 \leq s < t$  and that, for each  $a > 0$ , the process  $\{a^{-H} B_{at}, t \geq 0\}$  is also a fBm with Hurst parameter  $H$  (self-similarity property). As a consequence, the sequence  $\{B_j - B_{j-1}, j = 1, 2, \dots\}$  is stationary, Gaussian and ergodic, with correlation given by

$$\rho_H(n) = \frac{1}{2} [|n + 1|^{2H} - 2|n|^{2H} + |n - 1|^{2H}], \quad (5.2)$$

which behaves as  $H(2H - 1)|n|^{2H-2}$  as  $n$  tends to infinity.

Set  $\Delta B_{k/n} = B_{(k+1)/n} - B_{k/n}$ , where  $k = 0, 1, \dots, n$ , and  $n \geq 1$ . The ergodic theorem combined with the self-similarity property implies that the sequence  $n^{2H-1} \sum_{k=0}^{n-1} (\Delta B_{k/n})^2$  converges, almost surely and in  $L^1(\Omega)$ , to  $E(B_1^2) = 1$ . Moreover, it is well-known (see, e.g., [3]) that, provided  $H \in (0, \frac{3}{4})$ , a central limit theorem holds: the sequence

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left( n^{2H} (\Delta B_{k/n})^2 - 1 \right) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} H_2(n^H \Delta B_{k/n}) \quad (5.3)$$

converges in law to  $N(0, \sigma_H^2)$  as  $n \rightarrow \infty$ , for some constant  $\sigma_H > 0$ . (Notice also that, by normalizing with  $\sqrt{n \log n}$  instead of  $\sqrt{n}$ , the central limit theorem continues to hold in the critical case  $H = \frac{3}{4}$ .) When  $H > \frac{3}{4}$ , the situation is very different. Indeed, we have in contrast that

$$n^{1-2H} \sum_{k=0}^{n-1} \left( n^{2H} (\Delta B_{k/n})^2 - 1 \right) = n^{1-2H} \sum_{k=0}^{n-1} H_2(n^H \Delta B_{k/n})$$

converges in  $L^2(\Omega)$ . More generally, consider an integer  $q \geq 2$ . If  $H < 1 - \frac{1}{2q}$ , then the sequence

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} H_q(n^H \Delta B_{k/n}) \quad (5.4)$$



converges in law to  $N(0, \sigma_{q,H}^2)$  (for some constant  $\sigma_{q,H} > 0$ ), whereas, if  $H > 1 - \frac{1}{2q}$ , then the sequence

$$n^{q-qH-1} \sum_{k=0}^{n-1} H_q(n^H \Delta B_{k/n})$$

converges in  $L^2(\Omega)$ .

Some unexpected results happen when we introduce a weight of the form  $f(B_{k/n})$  in (5.4). In fact, a new critical value ( $H = \frac{1}{2q}$ ) plays an important role. More precisely, consider the following sequence of random variables:

$$G_n = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(B_{k/n}) H_q(n^H \Delta B_{k/n}). \quad (5.5)$$

Here, the integer  $q \geq 2$  is fixed and the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is supposed to satisfy some suitable regularity and growth conditions. In [13, 18], the following convergences as  $n \rightarrow \infty$  are shown:

- If  $H < \frac{1}{2q}$ , then

$$n^{qH-\frac{1}{2}} G_n \xrightarrow{L^2(\Omega)} \frac{(-1)^q}{2^q q!} \int_0^1 f^{(q)}(B_s) ds. \quad (5.6)$$

- If  $\frac{1}{2q} < H < 1 - \frac{1}{2q}$ , then

$$G_n \xrightarrow{\text{stably}} \sigma_{H,q} \int_0^1 f(B_s) dW_s, \quad (5.7)$$

where  $W$  is a Brownian motion independent of  $B$ , and

$$\sigma_{H,q}^2 = q! \sum_{r \in \mathbb{Z}} \rho_H(r)^q < \infty. \quad (5.8)$$

- If  $H = 1 - \frac{1}{2q}$ , then

$$\frac{G_n}{\sqrt{\log n}} \xrightarrow{\text{stably}} \sqrt{\frac{2}{q!}} \left(1 - \frac{1}{2q}\right)^{q/2} \left(1 - \frac{1}{q}\right)^{q/2} \int_0^1 f(B_s) dW_s,$$

where  $W$  is a Brownian motion independent of  $B$ .

- If  $H > 1 - \frac{1}{2q}$ , then

$$n^{q(1-H)-\frac{1}{2}} G_n \xrightarrow{L^2(\Omega)} \int_0^1 f(B_s) dZ_s^{(q)},$$

where  $Z^{(q)}$  denotes the Hermite process of order  $q$  canonically constructed from  $B$  (see [18] for the details).

In addition, when  $q = 2$  and  $H = \frac{1}{4}$ , it was shown in [19] that  $G_n$  converges stably to a linear combination of the limits in (5.7) and (5.6). (The proof of this last result follows an approach similar to the proof of our Theorem 3.1, and allows to derive a change of variable formula for the fBm of Hurst index  $\frac{1}{4}$ , with a correction term that is an ordinary Itô integral with respect to a Brownian motion that is independent of  $B$ .) But the convergence of  $G_n$  in the critical case  $H = \frac{1}{2q}$ ,  $q \geq 3$ , was open till now.

In the present paper, we are going to show that Theorem 3.1 provides a proof of the following new result, valid for any integer  $q \geq 2$  and any index  $H \in \left(\frac{1}{4q}, \frac{1}{2}\right)$ :

$$G_n - n^{-\frac{1}{2}-qH} \frac{(-1)^q}{2^q q!} \sum_{k=0}^{n-1} f^{(q)}(B_{k/n}) \xrightarrow{\text{stably}} \sigma_{H,q} \int_0^1 f(B_s) dW_s. \quad (5.9)$$

(See Theorem 5.3 below for a precise statement.) Notice that (5.9) provides a new proof of (5.7) in the case  $H \in \left(\frac{1}{2q}, \frac{1}{2}\right)$  (without considering two different levels of discretization  $n \leq m$ , as in [18]). More importantly, in the critical case  $H = \frac{1}{2q}$ , convergence (5.9) yields:

$$G_n \xrightarrow{\text{stably}} \frac{(-1)^q}{2^q q!} \int_0^1 f^{(q)}(B_s) ds + \sigma_{1/(2q),q} \int_0^1 f(B_s) dW_s.$$

Hence, the understanding of the asymptotic behavior of the weighted Hermite variations of the fBm is now complete (indeed, the case  $H = \frac{1}{2q}$ ,  $q \geq 3$ , was the only remaining case, as mentioned in the discussion above).

The main idea of the proof of (5.9) is a decomposition of the random variable  $G_n$  using equation (2.6). The term with  $r = 0$  is a multiple Skorohod integral of order  $q$  and, by Theorem 5.2 below, it converges in law for any  $H \in \left(\frac{1}{4q}, \frac{1}{2}\right)$ . The term with  $r = q$  behaves as  $-n^{-\frac{1}{2}-qH} \frac{(-1)^q}{2^q q!} \sum_{k=0}^n f^{(q)}(B_{k/n})$ . The remaining terms ( $1 \leq r \leq q-1$ ) converge to zero in  $L^2(\Omega)$ .

## 5.2 Some preliminaries on the fractional Brownian motion

Before proving (5.9), we need some preliminaries on the Malliavin calculus associated with the fBm and some technical results (see [20, Chapter 5]).

In the following we assume  $H \in (0, \frac{1}{2})$ . We denote by  $\mathcal{E}$  the set of step functions on  $[0, 1]$ . Let  $\mathfrak{H}$  be the Hilbert space defined as the closure of  $\mathcal{E}$  with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathfrak{H}} = R_H(t, s) = \frac{1}{2}(s^H + t^H - |t - s|^H).$$

The mapping  $\mathbf{1}_{[0,t]} \rightarrow B_t$  can be extended to a linear isometry between the Hilbert space  $\mathfrak{H}$  and the Gaussian space spanned by  $B$ . We denote this isometry by  $\phi \rightarrow B(\phi)$ . In this way  $\{B(\phi), \phi \in \mathfrak{H}\}$  is an isonormal Gaussian space. (In fact, we know that the space  $\mathfrak{H}$  coincides with  $I_{0+}^{H-\frac{1}{2}}(L^2[0, 1])$ , where

$$I_{0+}^{H-\frac{1}{2}} f(x) = \frac{1}{\Gamma(H-\frac{1}{2})} \int_0^x (x-y)^{H-\frac{3}{2}} f(y) dy$$

is the left-sided Liouville fractional integral of order  $H - \frac{1}{2}$ , see [6].)

From now on, we will make use of the notation

$$\begin{aligned} \varepsilon_t &= \mathbf{1}_{[0,t]}, \\ \partial_{k/n} &= \varepsilon_{(k+1)/n} - \varepsilon_{k/n} = \mathbf{1}_{(k/n, (k+1)/n]}, \end{aligned}$$

for  $t \in [0, 1]$ ,  $n \geq 1$ , and  $k = 0, \dots, n-1$ . Notice that  $H_q(n^H \Delta B_{k/n}) = n^{qH} I_q(\partial_{k/n}^{\otimes q})$ .

We need the following technical lemma.

**Lemma 5.1** *Recall that  $H < \frac{1}{2}$ . Let  $n \geq 1$  and  $k = 0, \dots, n-1$ . We have*

$$(a) \quad |E(B_r(B_t - B_s))| \leq (t-s)^{2H} \text{ for any } r \in [0, 1] \text{ and } 0 \leq s < t \leq 1.$$

$$(b) \quad \left| \langle \varepsilon_t, \partial_{k/n} \rangle_{\mathfrak{H}} \right| \leq n^{-2H} \text{ for any } t \in [0, 1].$$

$$(c) \quad \sup_{t \in [0, 1]} \sum_{k=0}^{n-1} \left| \langle \varepsilon_t, \partial_{k/n} \rangle_{\mathfrak{H}} \right| = O(1) \text{ as } n \text{ tends to infinity.}$$

(d) For any integer  $q \geq 2$ ,

$$\sum_{k=0}^{n-1} \left| \langle \varepsilon_{k/n}, \partial_{k/n} \rangle_{\mathfrak{H}}^q - \frac{(-1)^q}{2^q n^{2qH}} \right| = O(n^{-2H(q-1)}) \text{ as } n \text{ tends to infinity.} \quad (5.10)$$

(e) Recall the definition (5.2) of  $\rho_H$ . We have

$$\langle \partial_{j/n}, \partial_{k/n} \rangle_{\mathfrak{H}} = n^{-2H} \rho_H(k - j).$$

Consequently, for any integer  $q \geq 1$ , we can write

$$\sum_{k,j=0}^{n-1} \left| \langle \partial_{j/n}, \partial_{k/n} \rangle_{\mathfrak{H}} \right|^q = O(n^{1-2qH}) \quad \text{as } n \text{ tends to infinity.} \quad (5.11)$$

**Proof.** We have

$$\begin{aligned} E(B_r(B_t - B_s)) &= \frac{1}{2} (r^{2H} + t^{2H} - |t - r|^{2H}) - \frac{1}{2} (r^{2H} + s^{2H} - |s - r|^{2H}) \\ &= \frac{1}{2} (t^{2H} - s^{2H}) + \frac{1}{2} (|s - r|^{2H} - |t - r|^{2H}). \end{aligned}$$

Using the inequality  $|b^{2H} - a^{2H}| \leq |b - a|^{2H}$  for any  $a, b \in [0, 1]$ , we deduce (a). Property (b) is an immediate consequence of (a). To show property (c) we use

$$\langle \varepsilon_t, \partial_{k/n} \rangle_{\mathfrak{H}} = \frac{1}{2n^{2H}} [(k+1)^{2H} - k^{2H} - |k+1-nt|^{2H} + |k-nt|^{2H}].$$

Property (d) follows from

$$\langle \varepsilon_{k/n}, \partial_{k/n} \rangle_{\mathfrak{H}} = \frac{1}{2n^{2H}} [(k+1)^{2H} - k^{2H} - 1],$$

and

$$\begin{aligned} \left| \langle \varepsilon_{k/n}, \partial_{k/n} \rangle_{\mathfrak{H}}^q - \frac{(-1)^q}{2^q n^{2qH}} \right| &= \frac{1}{2^q n^{2qH}} \left| [(k+1)^{2H} - k^{2H} - 1]^q - (-1)^q \right| \\ &= \frac{1}{2^q n^{2qH}} \sum_{i=1}^q \binom{q}{i} [(k+1)^{2H} - k^{2H}]^i \\ &\leq \frac{1}{2^q n^{2qH}} [(k+1)^{2H} - k^{2H}] \sum_{i=1}^q \binom{q}{i}. \end{aligned}$$

Finally, property (e) follows from

$$\sum_{k,j=0}^{n-1} \left| \langle \partial_{j/n}, \partial_{k/n} \rangle_{\mathfrak{H}} \right|^q \leq n^{-2qH} \sum_{k,j=0}^{n-1} |\rho_H(j-k)|^q \leq n^{1-2qH} \sum_{r \in \mathbb{Z}} |\rho_H(r)|^q.$$

■

### 5.3 An auxiliary convergence result

From now on, we fix  $q \geq 2$  and we make use of the following hypothesis on  $f : \mathbb{R} \rightarrow \mathbb{R}$ :

(H)  $f$  belongs to  $\mathcal{C}^{2q}$  and, for any  $p \geq 2$  and  $i = 0, \dots, 2q$ ,

$$E\left(\sup_{t \in [0,1]} |f^{(i)}(B_t)|^p\right) < \infty. \quad (5.12)$$

Notice that a sufficient condition for (5.12) to hold is that  $f$  satisfies an exponential growth condition of the form  $|f^{(2q)}(x)| \leq ke^{c|x|^p}$  for some constants  $c, k > 0$  and  $0 < p < 2$ .

The aim of this section is to prove the following auxiliary convergence result.

**Theorem 5.2** *Suppose  $H \in \left(\frac{1}{4q}, \frac{1}{2}\right)$ , and let  $f$  be a function satisfying Hypothesis (H). Consider the sequence of  $q$ -parameter step processes defined by*

$$u_n = n^{qH - \frac{1}{2}} \sum_{k=0}^{n-1} f(B_{k/n}) \partial_{k/n}^{\otimes q}. \quad (5.13)$$

*Then  $u_n \in \text{Dom} \delta^q$ , and  $\delta^q(u_n)$  converges stably to  $\sigma_{H,q} \int_0^1 f(B_s) dW_s$ , where  $W$  is a Brownian motion independent of  $B$ , and  $\sigma_{H,q} > 0$  is defined in (5.8).*

**Proof.** The fact that  $u_n$  belongs to  $\text{Dom} \delta^q$  is a consequence of the inclusion  $\mathbb{D}^{q,2}(\mathfrak{H}^{\otimes q}) \subset \text{Dom} \delta^q$  and hypothesis (H). We are now going to show that the sequence  $F_n = \delta^q(u_n)$  satisfies the conditions of Theorem 3.1. We make use of the notation

$$\alpha_{k,j} = \langle \varepsilon_{k/n}, \partial_{j/n} \rangle_{\mathfrak{H}}, \quad \beta_{k,j} = \langle \partial_{k/n}, \partial_{j/n} \rangle_{\mathfrak{H}}, \quad (5.14)$$

for  $k, j = 0, \dots, n-1$  and  $n \geq 1$ . Also  $C$  will denote a generic constant.

*Step 1.* Let us show first that  $F_n$  is bounded in  $L^2(\Omega)$ . Taking into account the continuity of the Skorohod integral from the space  $\mathbb{D}^{q,2}(\mathfrak{H}^{\otimes q})$  into  $L^2(\Omega)$  (see (2.9)), it suffices to show that  $u_n$  is bounded in  $\mathbb{D}^{q,2}(\mathfrak{H}^{\otimes q})$ . Actually we are going to show that  $u_n$  is bounded in  $\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes k})$  for any integer  $k \leq 2q$  and any real number  $p \geq 2$ . Using the estimate (5.11) we obtain

$$\|u_n\|_{\mathfrak{H}^{\otimes q}}^2 = n^{2qH-1} \sum_{k,j=0}^{n-1} f(B_{k/n}) f(B_{j/n}) \beta_{k,j}^q \leq C \sup_{0 \leq t \leq 1} |f(B_t)|^2.$$

Moreover for any integer  $k \geq 1$ ,

$$D^k u_n = n^{qH - \frac{1}{2}} \sum_{j=0}^{n-1} f^{(k)}(B_{j/n}) \varepsilon_{j/n}^{\otimes k} \otimes \partial_{j/n}^{\otimes q},$$

and we obtain in the same way

$$\begin{aligned} \left\| D^k u_n \right\|_{\mathfrak{H}^{\otimes(q+k)}}^2 &= n^{2qH-1} \sum_{l,j=0}^{n-1} f^{(k)}(B_{l/n}) f^{(k)}(B_{j/n}) \langle \varepsilon_{l/n}, \varepsilon_{j/n} \rangle^k \beta_{l,j}^q \\ &\leq C \sup_{0 \leq t \leq 1} \left| f^{(k)}(B_t) \right|^2. \end{aligned}$$

Then the result follows from hypothesis **(H)**.

*Step 2.* Let us show condition (i) of Theorem 3.1. Fix some integers  $r, k_1, \dots, k_{q-1} \geq 0$  such that  $k_1 + 2k_2 + \dots + (q-1)k_{q-1} + r = q$ . Let  $h \in \mathfrak{H}^{\otimes r}$ . We claim that  $\langle u_n, (DF_n)^{\otimes k_1} \otimes \dots \otimes (D^{q-1}F_n)^{\otimes k_{q-1}} \otimes h \rangle_{\mathfrak{H}^{\otimes q}}$  converges to zero in  $L^1(\Omega)$ . Suppose first that  $r \geq 1$ . Without loss of generality, we can assume that  $h$  has the form  $g \otimes \varepsilon_t$ , with  $g \in \mathfrak{H}^{\otimes(r-1)}$ . Set  $\Phi_n = (DF_n)^{\otimes k_1} \otimes \dots \otimes (D^{q-1}F_n)^{\otimes k_{q-1}} \otimes g$ . Then we can write

$$\langle u_n, \Phi_n \otimes \varepsilon_t \rangle_{\mathfrak{H}^{\otimes q}} = n^{qH - \frac{1}{2}} \sum_{k=0}^{n-1} f(B_{k/n}) \left\langle \partial_{k/n}^{\otimes(q-1)}, \Phi_n \right\rangle_{\mathfrak{H}^{\otimes(q-1)}} \langle \partial_{k/n}, \varepsilon_t \rangle_{\mathfrak{H}}.$$

As a consequence,

$$\begin{aligned} E \left( \left| \langle u_n, \Phi_n \otimes \varepsilon_t \rangle_{\mathfrak{H}^{\otimes q}} \right| \right) &\leq n^{qH - \frac{1}{2}} \sum_{k=0}^{n-1} E \left( \left| f(B_{k/n}) \left\langle \partial_{k/n}^{\otimes(q-1)}, \Phi_n \right\rangle_{\mathfrak{H}^{\otimes(q-1)}} \right| \right) \\ &\quad \times \left| \langle \partial_{k/n}, \varepsilon_t \rangle_{\mathfrak{H}} \right|. \end{aligned}$$

Condition (c) of Lemma 5.1 implies

$$\sum_{k=0}^{n-1} \left| \langle \partial_{k/n}, \varepsilon_t \rangle_{\mathfrak{H}} \right| \leq C.$$

Hence,

$$E \left( \left| \langle u_n, \Phi_n \otimes \varepsilon_t \rangle_{\mathfrak{H}^{\otimes q}} \right| \right) \leq C n^{H - \frac{1}{2}} \left( E \left( \|\Phi_n\|_{\mathfrak{H}^{\otimes(q-1)}}^2 \right) \right)^{\frac{1}{2}}.$$

On the other hand

$$\|\Phi_n\|_{\mathfrak{H}^{\otimes(q-1)}}^2 = \|g\|_{\mathfrak{H}^{\otimes(r-1)}}^2 \prod_{m=1}^{q-1} \|D^m F_n\|_{\mathfrak{H}^{\otimes m}}^{2k_m},$$

and applying the generalized Hölder's inequality

$$\begin{aligned} E\left(\|\Phi_n\|_{\mathfrak{H}^{\otimes(q-1)}}^2\right) &\leq C \prod_{m=1}^{q-1} \left(E\left(\|D^m F_n\|_{\mathfrak{H}^{\otimes m}}^{2k_m(q-1)}\right)\right)^{\frac{1}{q-1}} \\ &= C \prod_{m=1}^{q-1} \|D^m F_n\|_{L^{2k_m(q-1)}(\Omega; \mathfrak{H}^{\otimes m})}^{2k_m}. \end{aligned}$$

By Meyer's inequalities (2.9), for any  $1 \leq m \leq q-1$  and any  $p \geq 2$ , we obtain, using Step 1, that

$$\begin{aligned} \|D^m F_n\|_{L^p(\Omega; \mathfrak{H}^{\otimes m})} &= \|D^m \delta^q(u_n)\|_{L^p(\Omega; \mathfrak{H}^{\otimes m})} \\ &\leq \|\delta^q(u_n)\|_{\mathbb{D}^{m,p}} \leq C \|u_n\|_{\mathbb{D}^{m+q,p}(\mathfrak{H}^{\otimes q})} \leq C. \end{aligned}$$

Therefore,

$$E\left(|\langle u_n, \Phi_n \otimes \varepsilon_t \rangle_{\mathfrak{H}^{\otimes q}}|\right) \leq C n^{H-\frac{1}{2}},$$

which converges to zero as  $n$  tends to infinity because  $H < \frac{1}{2}$ .

Suppose now that  $r = 0$ . In this case, we have  $\Phi_n = (DF_n)^{\otimes k_1} \otimes \dots \otimes (D^{q-1}F_n)^{\otimes k_{q-1}}$ . Then

$$\left\langle \partial_{j/n}^{\otimes q}, \Phi_n \right\rangle_{\mathfrak{H}^{\otimes q}} = \left\langle \partial_{j/n}, DF_n \right\rangle_{\mathfrak{H}}^{k_1} \dots \left\langle \partial_{j/n}^{\otimes(q-1)}, D^{q-1}F_n \right\rangle_{\mathfrak{H}^{\otimes(q-1)}}^{k_{q-1}}. \quad (5.15)$$

From (5.15) and (5.13) we obtain

$$\langle u_n, \Phi_n \rangle_{\mathfrak{H}^{\otimes q}} = n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} f(B_{k/n}) \prod_{m=1}^{q-1} \left\langle \partial_{j/n}^{\otimes m}, D^m F_n \right\rangle_{\mathfrak{H}^{\otimes m}}^{k_m}. \quad (5.16)$$

Notice that for any  $m = 1, \dots, q-1$ , the term  $\left\langle \partial_{j/n}^{\otimes m}, D^m F_n \right\rangle_{\mathfrak{H}^{\otimes m}}$  can be estimated by  $n^{-mH} \|D^m F_n\|_{\mathfrak{H}^{\otimes m}}$ . Then, taking into account that

$$\sup_n E\left(\|D^m F_n\|_{\mathfrak{H}^{\otimes m}}^p\right) < \infty$$

for any  $p \geq 2$ , and that  $\sum_{m=1}^{q-1} m k_m = q$ , we obtain for  $E\left(|\langle u_n, \Phi_n \rangle_{\mathfrak{H}^{\otimes q}}|\right)$  an estimate of the form  $C\sqrt{n}$ , which is unfortunately not satisfactory. For this reason, a finer analysis of the terms  $\left\langle \partial_{j/n}^{\otimes m}, D^m F_n \right\rangle_{\mathfrak{H}^{\otimes m}}$  is required.

First we are going to apply formula (2.11) to compute the derivative  $D^m F_n$ ,  $m = 1, \dots, q-1$ :

$$\begin{aligned}
D^m F_n &= \sum_{i=0}^m \binom{m}{i} \binom{q}{i} i! \delta^{q-i} (D^{m-i} u_n) \\
&= n^{qH-\frac{1}{2}} \sum_{i=0}^m \binom{m}{i} \binom{q}{i} i! \sum_{l=0}^{n-1} \left( \varepsilon_{l/n}^{\otimes(m-i)} \otimes \partial_{l/n}^{\otimes i} \right) \\
&\quad \times \delta^{q-i} \left( f^{(m-i)}(B_{l/n}) \partial_{l/n}^{\otimes(q-i)} \right). \tag{5.17}
\end{aligned}$$

Set  $\Psi_n^{m,j} = \left\langle \partial_{j/n}^{\otimes m}, D^m F_n \right\rangle_{\mathfrak{H}^{\otimes m}}$ , and recall the definition of  $\alpha_{k,j}$  and  $\beta_{k,j}$  from (5.14). From (5.17) we obtain

$$\begin{aligned}
\Psi_n^{m,j} &= n^{qH-\frac{1}{2}} \sum_{i=0}^m \binom{m}{i} \binom{q}{i} i! \sum_{l=0}^{n-1} \alpha_{l,j}^{m-i} \beta_{l,j}^i \delta^{q-i} \left( f^{(m-i)}(B_{l/n}) \partial_{l/n}^{\otimes(q-i)} \right) \\
&= \sum_{i=0}^m \Phi_n^{i,m,j}, \tag{5.18}
\end{aligned}$$

with

$$\Phi_n^{i,m,j} = n^{qH-\frac{1}{2}} \binom{m}{i} \binom{q}{i} i! \sum_{l=0}^{n-1} \alpha_{l,j}^{m-i} \beta_{l,j}^i \delta^{q-i} \left( f^{(m-i)}(B_{l/n}) \partial_{l/n}^{\otimes(q-i)} \right).$$

By Meyer inequalities (2.9) we obtain, using also assumption **(H)**, that, for any  $p \geq 2$ ,

$$\begin{aligned}
\left\| \delta^{q-i} \left( f^{(m-i)}(B_{l/n}) \partial_{l/n}^{\otimes(q-i)} \right) \right\|_{L^p} &\leq C \left\| f^{(m-i)}(B_{l/n}) \partial_{l/n}^{\otimes(q-i)} \right\|_{\mathbb{D}^{q-i,p}(\mathfrak{H}^{\otimes q-i})} \\
&\leq C n^{-(q-i)H}. \tag{5.19}
\end{aligned}$$

Using Lemma 5.1 (b) and (e) we have  $\left| \alpha_{l,j}^{m-i} \right| \leq C n^{-(m-i)2H}$  and  $\sum_{l=0}^{n-1} \left| \beta_{l,j}^i \right| \leq C n^{-2iH}$ . Therefore, for any  $i \geq 1$ , we have

$$\left\| \Phi_n^{i,m,j} \right\|_{L^p} \leq C n^{iH-\frac{1}{2}} \sum_{l=0}^{n-1} \left| \alpha_{l,j}^{m-i} \beta_{l,j}^i \right| \leq C n^{-\frac{1}{2}-2mH+iH}. \tag{5.20}$$

On the other hand, if  $i = 0$ , Lemma 5.1 (c) and (5.19) yield

$$\left\| \Phi_n^{0,m,j} \right\|_{L^p} \leq C n^{-\frac{1}{2}-2mH+2H}. \tag{5.21}$$



Notice that the estimate for the  $L^p(\Omega)$ -norm of  $\Phi_n^{0,m,j}$  in the case  $i = 0$  is worst than for  $i \geq 1$ . We will see later that, for  $p = 2$ , we can get a better estimate for  $\Phi_n^{0,m,j}$ .

Because  $\sum_{m=1}^{q-1} k_m \geq 2$ , the number of factors in  $\prod_{m=1}^{q-1} \langle \partial_{j/n}, D^m F_n \rangle_{\mathfrak{H}^{\otimes m}}^{k_m}$  is at least two. As a consequence, we can write

$$\langle u_n, \Phi_n \rangle_{\mathfrak{H}^{\otimes q}} = n^{qH - \frac{1}{2}} \sum_{j=0}^{n-1} f(B_{j/n}) \Psi_n^{\mu,j} \Psi_n^{\nu,j} \Theta_n^j,$$

for some  $\mu, \nu$  (not necessarily distinct), where

$$\Theta_n^j = (\Psi_n^{\mu,j})^{k_\mu - 1} (\Psi_n^{\nu,j})^{k_\nu - 1} \prod_{\substack{m=1 \\ m \neq \mu, \nu}}^{q-1} (\Psi_n^{m,j})^{k_m}. \quad (5.22)$$

Consider the decomposition

$$\langle u_n, \Phi_n \rangle_{\mathfrak{H}^{\otimes q}} = A_n + B_n,$$

where

$$\begin{aligned} A_n &= n^{qH - \frac{1}{2}} \sum_{j=0}^{n-1} f(B_{j/n}) \left( \sum_{i=0}^{\mu} \sum_{k=0}^{\nu} \mathbf{1}_{i+k \geq 1} \Phi_n^{i,\mu,j} \Phi_n^{k,\nu,j} \right) \Theta_n^j, \\ B_n &= n^{qH - \frac{1}{2}} \sum_{j=0}^{n-1} f(B_{j/n}) \Phi_n^{0,\mu,j} \Phi_n^{0,\nu,j} \Theta_n^j. \end{aligned}$$

From (5.22) and the estimate  $\|\Psi_n^{m,j}\|_{L^p} \leq Cn^{-mH}$ , for all  $p \geq 2$  and  $1 \leq m \leq q$ , we obtain

$$\|\Theta_n^j\|_{L^p} \leq Cn^{-H(q-\mu-\nu)}. \quad (5.23)$$

Then, from (5.20), (5.21) and (5.23) we obtain

$$\begin{aligned} E(|A_n|) &\leq Cn^{qH + \frac{1}{2}} n^{-H(q-\mu-\nu)} \left( \sum_{i=1}^{\mu} \sum_{k=1}^{\nu} n^{-1-2(\mu+\nu)H+(i+k)H} \right. \\ &\quad \left. + \sum_{i=1}^{\mu} n^{-1-2(\mu+\nu)H+iH+2H} + \sum_{k=1}^{\nu} n^{-1-2(\mu+\nu)H+kH+2H} \right) \\ &= Cn^{-\frac{1}{2}} + n^{-\frac{1}{2}+2H-\mu H} + n^{-\frac{1}{2}+2H-\nu H}, \end{aligned}$$

which converges to zero as  $n$  tends to infinity, because  $\mu, \nu \geq 1$  and  $H < \frac{1}{2}$ .

For the term  $B_n$  using again the estimates (5.21) and (5.23) we get

$$\begin{aligned} E(|B_n|) &\leq Cn^{qH+\frac{1}{2}-H(q-\mu-\nu)-1-2H(\mu+\nu)+4H} = Cn^{-\frac{1}{2}-H(\mu+\nu)+4H} \\ &\leq Cn^{-\frac{1}{2}+2H}, \end{aligned}$$

which converges to zero as  $n$  tends to infinity if  $H < \frac{1}{4}$ . To handle the case  $H \in [\frac{1}{4}, \frac{1}{2})$  we need more precise estimates for the  $L^2(\Omega)$ -norm of  $\Phi_n^{0,\nu,j}$ . We have, using formula (2.12)

$$\begin{aligned} E\left[(\Phi_n^{0,\nu,j})^2\right] &= \binom{q}{i}^2 \binom{m}{i}^2 i!^2 E\left(\left|n^{qH-\frac{1}{2}} \sum_{l=0}^{n-1} \alpha_{l,j}^\nu \delta^q\left(f^{(\nu)}(B_{l/n}) \partial_{l/n}^{\otimes q}\right)\right|^2\right) \\ &= n^{2qH-1} \binom{q}{i}^2 \binom{m}{i}^2 i!^2 \sum_{l,l'=0}^{n-1} \alpha_{l,j}^\nu \alpha_{l',j}^\nu \\ &\quad \times E\left(\delta^q\left(f^{(\nu)}(B_{l/n}) \partial_{l/n}^{\otimes q}\right) \delta^q\left(f^{(\nu)}(B_{l'/n}) \partial_{l'/n}^{\otimes q}\right)\right) \\ &= n^{2qH-1} \binom{q}{i}^2 \binom{m}{i}^2 i!^2 \sum_{l,l'=0}^{n-1} \alpha_{l,j}^\nu \alpha_{l',j}^\nu \sum_{i=0}^q \binom{q}{i}^2 i! \alpha_{l,l'}^{q-i} \alpha_{l,l'}^{q-i} \beta_{l,l'}^{2i} \\ &\quad \times E\left(f^{(\nu+q-i)}(B_{l/n}) f^{(\nu+q-i)}(B_{l'/n})\right) \\ &= \sum_{i=0}^q R_n^i. \end{aligned}$$

If  $i \geq 1$ , then  $\sum_{l,l'=0}^{n-1} \beta_{l,l'}^{2i} \leq Cn^{1-4iH}$ , and we obtain an estimate of the form  $\|R_n^i\|_{L^2} \leq Cn^\gamma$ , where

$$\gamma = \frac{1}{2}(2qH - 1 - 4\nu H - 4(q-i)H + 1 - 4iH) = -qH - 2\nu H.$$

For  $i = 0$ , then  $\sup_n \sum_{l,l'=0}^{n-1} |\alpha_{l,l'} \alpha_{l',l}| < \infty$ , and we get

$$\gamma = \frac{1}{2}(2qH - 1 - 2H(2\nu + 2q - 2)) = -qH - 2\nu H - \frac{1}{2} + 2H.$$

We have obtained the estimate

$$\|\Phi_n^{0,\nu,j}\|_{L^2} \leq Cn^{-qH-2\nu H+2H-\frac{1}{2}}. \quad (5.24)$$

Fix  $\frac{1}{4qH} < \alpha < 1$ . This choice is possible because  $\frac{1}{4qH} < 1$ . We have, by Hölder's inequality,

$$E(|B_n|) \leq Cn^{qH-\frac{1}{2}} \sum_{j=0}^{n-1} \|\Phi_n^{0,\mu,j}\|_{L^2}^\alpha \|\Phi_n^{0,\nu,j}\|_{L^2}^\alpha \left\| |\Phi_n^{0,\mu,j} \Phi_n^{0,\nu,j}|^{1-\alpha} \Theta_n^j \right\|_{L^{\frac{1}{1-\alpha}}}.$$

Using (5.24), (5.21) and (5.23) we obtain

$$E(|B_n|) \leq Cn^\gamma, \quad (5.25)$$

where

$$\begin{aligned} \gamma &= qH + \frac{1}{2} + [-2qH - 2(\mu + \nu)H + 4H - 1] \alpha \\ &\quad - H(q - \mu - \nu) + (1 - \alpha)(-1 - 2H(\mu + \nu) + 4H) \\ &= -\frac{1}{2} + 4H - H(\mu + \nu) - 2\alpha qH \\ &\leq -\frac{1}{2} + 2H - 2\alpha qH \leq \frac{1}{2} - 2\alpha qH < 0, \end{aligned}$$

because  $H < \frac{1}{2}$ . Therefore  $E(|B_n|)$  converges to zero as  $n$  tends to infinity.

*Step 3.* Let us show condition (ii). We have

$$\langle u_n, D^q F_n \rangle_{\mathfrak{H}^{\otimes q}} = n^{qH - \frac{1}{2}} \sum_{j=0}^{n-1} f(B_{j/n}) \langle \partial_{j/n}^{\otimes q}, D^q F_n \rangle_{\mathfrak{H}^{\otimes q}}.$$

From (5.18) we get

$$\langle \partial_{j/n}^{\otimes q}, D^q F_n \rangle_{\mathfrak{H}^{\otimes q}} = n^{qH - \frac{1}{2}} \sum_{i=0}^q \binom{q}{i}^2 i! \sum_{l=0}^{n-1} \alpha_{l,j}^{q-i} \beta_{l,j}^i \delta^{q-i} \left( f^{(q-i)}(B_{l/n}) \partial_{l/n}^{\otimes (q-i)} \right).$$

Therefore, we can make the decomposition

$$\langle u_n, D^q F_n \rangle_{\mathfrak{H}^{\otimes q}} = A_n + B_n + C_n,$$

where

$$\begin{aligned} A_n &= n^{2qH-1} q! \sum_{l,j=0}^{n-1} \beta_{l,j}^q f(B_{l/n}) f(B_{j/n}), \\ B_n &= n^{2qH-1} \sum_{i=1}^{q-1} \binom{q}{i}^2 i! \sum_{l,j=0}^{n-1} \alpha_{l,j}^{q-i} \beta_{l,j}^i f(B_{j/n}) \delta^{q-i} \left( f^{(q-i)}(B_{l/n}) \partial_{l/n}^{\otimes (q-i)} \right), \\ C_n &= n^{2qH-1} \sum_{l,j=0}^{n-1} \alpha_{l,j}^q f(B_{j/n}) \delta^q \left( f^{(q)}(B_{l/n}) \partial_{l/n}^{\otimes (q)} \right). \end{aligned}$$

The term  $A_n$  converges to a nonnegative square integrable random variable. Indeed,

$$\begin{aligned} A_n &= \frac{q!}{2^{qn}} \sum_{k,j=0}^{n-1} f(B_{k/n})f(B_{j/n}) (|k-j+1|^{2H} + |k-j-1|^{2H} - 2|k-j|^{2H})^q \\ &= \frac{q!}{2^{qn}} \sum_{p=-\infty}^{\infty} \sum_{j=0 \vee -p}^{(n-1) \wedge (n-1-p)} f(B_{j/n})f(B_{(j+p)/n}) (|p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H})^q, \end{aligned}$$

which converges in  $L^1(\Omega)$  to

$$q! \left( \sum_{k \in \mathbb{Z}} \rho_H(k)^q \right) \int_0^1 f(B_s)^2 ds.$$

Then, it suffices to show that the terms  $B_n$  and  $C_n$  converge to zero in  $L^2(\Omega)$ . For the term  $B_n$  we can write, using the fact that  $\sum_{l,j=0}^{n-1} |\alpha_{l,j}^{q-i} \beta_{l,j}^i| \leq Cn^{-2qH+1}$

$$\begin{aligned} E(|B_n|) &\leq Cn^{2qH-1} \sum_{i=1}^{q-1} \sum_{l,j=0}^{n-1} |\alpha_{l,j}^{q-i} \beta_{l,j}^i| \left\| \delta^{q-i} \left( f^{(q-i)}(B_{l/n}) \partial_{l/n}^{\otimes(q-i)} \right) \right\|_{L^2} \\ &\leq C \sum_{i=1}^{q-1} n^{-H(q-i)}, \end{aligned}$$

which converges to zero as  $n$  tends to infinity. Finally, for the term  $C_n$  we can write

$$E(|C_n|) \leq Cn^{qH+\frac{1}{2}} \sup_j \|\Phi_n^{0,q;j}\|_{L^2} \leq Cn^{\frac{1}{2}-2qH+(2H-\frac{1}{2}) \vee 0},$$

and  $\frac{1}{2} - 2qH + (2H - \frac{1}{2}) \vee 0 < 0$ , because if  $2H - \frac{1}{2} \leq 0$  this is true due to  $\frac{1}{2} - 2qH < 0$ , and if  $2H - \frac{1}{2} \geq 0$ , then we get  $2H(1-q) < 0$ . This completes the proof of Theorem 5.2. ■

## 5.4 Proof of the stable convergence (5.9)

As a consequence of Theorem 5.2, we can derive the following result, which is nothing but (5.9):

**Theorem 5.3** Suppose that  $f$  is a function satisfying Hypothesis **(H)**. Let  $G_n$  be the sequence of random variables defined in (5.5). Then, provided  $H \in (\frac{1}{4q}, \frac{1}{2})$ , we have

$$G_n - n^{-\frac{1}{2}-qH} \frac{(-1)^q}{2^q q!} \sum_{k=0}^{n-1} f^{(q)}(B_{k/n}) \xrightarrow{\text{stably}} \sigma_{H,q} \int_0^1 f(B_s) dW_s,$$

where  $W$  is a Brownian motion independent of  $B$  and  $\sigma_{H,q} > 0$  is defined by (5.8).

**Proof.** We recall first that  $H_q(n^H(\Delta B_{k/n})) = \frac{1}{q!} n^{qH} \delta^q(\partial_{k/n}^{\otimes q})$ . Then, using

(2.6) yields

$$f(B_{k/n}) \delta^q(\partial_{k/n}^{\otimes q}) = \sum_{r=0}^q \binom{q}{r} \alpha_{k,k}^r \delta^{q-r}(f^{(r)}(B_{k/n}) \partial_{k/n}^{\otimes(q-r)}),$$

where  $\alpha_{k,k}$  is defined in (5.14). As a consequence,

$$\begin{aligned} G_n &= \frac{1}{q!} n^{qH-\frac{1}{2}} \sum_{r=0}^q \sum_{k=0}^{n-1} \binom{q}{r} \alpha_{k,k}^r \delta^{q-r}(f^{(r)}(B_{k/n}) \partial_{k/n}^{\otimes(q-r)}) \\ &= \frac{1}{q!} \delta^q(u_n) + \sum_{r=1}^{q-1} \delta^{q-r}(v_n^{(r)}) + R_n, \end{aligned}$$

where  $u_n$  is defined in (5.13),

$$v_n^{(r)} = \frac{1}{q!} \binom{q}{r} n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} \alpha_{k,k}^r f^{(r)}(B_{k/n}) \partial_{k/n}^{\otimes(q-r)},$$

and

$$R_n = \frac{1}{q!} n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} \alpha_{k,k}^q f^{(q)}(B_{k/n}).$$

The proof will be done in two steps.

*Step 1* We first show that if  $H \in (0, \frac{1}{2})$ , and  $r = 1, \dots, q-1$ ,  $\delta^{q-r}(v_n^{(r)})$  converges to zero in  $L^2(\Omega)$  as  $n$  tends to infinity. It suffices to show that  $v_n^{(r)}$

converges to zero in the norm of the space  $\mathbb{D}^{q-r,2}(\mathfrak{H}^{\otimes(q-r)})$ . For  $0 \leq m \leq q-r$ , we can write, using the notation  $\beta_{k,l}$  defined by (5.14),

$$\begin{aligned} E \left( \left\| D^m v_n^{(r)} \right\|_{\mathfrak{H}^{\otimes(q-r+m)}}^2 \right) &= \left( \frac{1}{q!} \binom{q}{r} \right)^2 n^{2qH-1} \\ &\quad \times \sum_{k,l=0}^{n-1} E \left( f^{(r+m)}(B_{k/n}) f^{(r+m)}(B_{l/n}) \right) \\ &\quad \times \alpha_{k,k}^r \alpha_{l,l}^r \alpha_{k,l}^m \beta_{k,l}^{q-r} \\ &\leq C n^{2qH-1} n^{-2H(2r-2+m+q-r)} \\ &= C n^{2H-1-2Hm}, \end{aligned}$$

which converges to zero as  $n$  tends to infinity.

*Step 2* To complete the proof it suffices to check that

$$R_n - n^{-\frac{1}{2}-qH} \frac{(-1)^q}{2^q q!} \sum_{k=0}^{n-1} f^{(q)}(B_{k/n})$$

converges to zero in  $L^2(\Omega)$  as  $n$  tends to infinity. This follows from (5.10) and the estimates

$$\begin{aligned} &\left\| \frac{1}{q!} n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} \alpha_{k,k}^q f^{(q)}(B_{k/n}) - \frac{(-1)^q}{2^q q!} n^{-\frac{1}{2}-qH} \sum_{k=0}^{n-1} f^{(q)}(B_{k/n}) \right\|_{L^2} \\ &\leq C n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} \left| \alpha_{k,k}^q - \frac{1}{2^q n^{2qH}} \right| \leq C n^{-qH+2H-\frac{1}{2}}. \end{aligned}$$

Notice that  $-qH + 2H - \frac{1}{2} < 0$ . The proof is now complete. ■

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