



HAL
open science

On the base locus of the linear system of generalized theta functions

Christian Pauly

► **To cite this version:**

Christian Pauly. On the base locus of the linear system of generalized theta functions. 2008. hal-00164700v2

HAL Id: hal-00164700

<https://hal.science/hal-00164700v2>

Preprint submitted on 14 Apr 2008

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON THE BASE LOCUS OF THE LINEAR SYSTEM OF GENERALIZED THETA FUNCTIONS

CHRISTIAN PAULY

ABSTRACT. Let \mathcal{M}_r denote the moduli space of semi-stable rank- r vector bundles with trivial determinant over a smooth projective curve C of genus g . In this paper we study the base locus $\mathcal{B}_r \subset \mathcal{M}_r$ of the linear system of the determinant line bundle \mathcal{L} over \mathcal{M}_r , i.e., the set of semi-stable rank- r vector bundles without theta divisor. We construct base points in \mathcal{B}_{g+2} over any curve C , and base points in \mathcal{B}_4 over any hyperelliptic curve.

1. INTRODUCTION

Let C be a complex smooth projective curve of genus g and let \mathcal{M}_r denote the coarse moduli space parametrizing semi-stable rank- r vector bundles with trivial determinant over the curve C . Let \mathcal{L} be the determinant line bundle over the moduli space \mathcal{M}_r and let $\Theta \subset \text{Pic}^{g-1}(C)$ be the Riemann theta divisor in the degree $g-1$ component of the Picard variety of C . By [BNR] there is a canonical isomorphism $|\mathcal{L}|^* \xrightarrow{\sim} |r\Theta|$, under which the natural rational map $\varphi_{\mathcal{L}} : \mathcal{M}_r \dashrightarrow |\mathcal{L}|^*$ is identified with the so-called theta map

$$\theta : \mathcal{M}_r \dashrightarrow |r\Theta|, \quad E \mapsto \theta(E) \subset \text{Pic}^{g-1}(C).$$

The underlying set of $\theta(E)$ consists of line bundles $L \in \text{Pic}^{g-1}(C)$ with $h^0(C, E \otimes L) > 0$. For a general semi-stable vector bundle E , $\theta(E)$ is a divisor. If $\theta(E) = \text{Pic}^{g-1}(C)$, we say that E has no theta divisor. We note that the indeterminacy locus of the theta map θ , i.e., the set of bundles E without theta divisor, coincides with the base locus $\mathcal{B}_r \subset \mathcal{M}_r$ of the linear system $|\mathcal{L}|$.

Over the past years many authors [A], [B2], [He], [Hi], [P], [R], [S] have studied the base locus \mathcal{B}_r of $|\mathcal{L}|$ and their analogues for the powers $|\mathcal{L}^k|$. For a recent survey of this subject we refer to [B1].

It is natural to introduce for a curve C the integer $r(C)$ defined as the minimal rank for which there exists a semi-stable rank- $r(C)$ vector bundle with trivial determinant over C without theta divisor (see also [B1] section 6). It is known [R] that $r(C) \geq 3$ for any curve C and that $r(C) \geq 4$ for a generic curve C . Our main result shows the existence of vector bundles of low ranks without theta divisor.

Theorem 1.1. *We assume that $g \geq 2$. Then we have the following bounds.*

- (1) $r(C) \leq g + 2$.
- (2) $r(C) \leq 4$, if C is hyperelliptic.

The first part of the theorem improves the upper bound $r(C) \leq \frac{(g+1)(g+2)}{2}$ given in [A]. The statements of the theorem are equivalent to the existence of a semi-stable rank- $(g+2)$ (resp.

rank-4) vector bundle without theta divisor — see section 2.1 (resp. 2.2). The construction of these vector bundles uses ingredients which are already implicit in [Hi].

Theorem 1.1 seems to hint towards a dependence of the integer $r(C)$ on the curve C .

Notations: If E is a vector bundle over C , we will write $H^i(E)$ for $H^i(C, E)$ and $h^i(E)$ for $\dim H^i(C, E)$. We denote the slope of E by $\mu(E) := \frac{\deg E}{\operatorname{rk} E}$, the canonical bundle over C by K and the degree d component of the Picard variety of C by $\operatorname{Pic}^d(C)$.

2. PROOF OF THEOREM 1.1

2.1. Semi-stable rank- $(g+2)$ vector bundles without theta divisor. We consider a line bundle $L \in \operatorname{Pic}^{2g+1}(C)$. Then L is globally generated, $h^0(L) = g+2$ and the evaluation bundle E_L , which is defined by the exact sequence

$$(1) \quad 0 \longrightarrow E_L^* \longrightarrow H^0(L) \otimes \mathcal{O}_C \xrightarrow{ev} L \longrightarrow 0,$$

is stable (see e.g. [Bu]), with $\deg E_L = 2g+1$, $\operatorname{rk} E_L = g+1$ and $\mu(E_L) = 2 - \frac{1}{g+1}$.

A cohomology class $e \in \operatorname{Ext}^1(LK^{-1}, E_L) = H^1(E_L \otimes KL^{-1})$ determines a rank- $(g+2)$ vector bundle \mathcal{E}_e given as an extension

$$(2) \quad 0 \longrightarrow E_L \longrightarrow \mathcal{E}_e \longrightarrow LK^{-1} \longrightarrow 0.$$

Proposition 2.1. *For any non-zero class e , the rank- $(g+2)$ vector bundle \mathcal{E}_e is semi-stable.*

Proof. Consider a proper subbundle $A \subset \mathcal{E}_e$. If $A \subset E_L$, then $\mu(A) \leq \mu(E_L) = 2 - \frac{1}{g+1}$ by stability of E_L , so the subbundles of E_L cannot destabilize \mathcal{E}_e . If $A \not\subset E_L$, we introduce $S = A \cap E_L \subset E_L$ and consider the exact sequence

$$0 \longrightarrow S \longrightarrow A \longrightarrow LK^{-1}(-D) \longrightarrow 0,$$

where D is an effective divisor. If $\operatorname{rk} S = g+1$ or $S = 0$, we easily conclude that $\mu(A) < \mu(\mathcal{E}_e) = 2$. If $\operatorname{rk} S < g+1$ and $S \neq 0$, then stability of E_L gives the inequality $\mu(S) < \mu(E_L) = 2 - \frac{1}{g+1}$. We introduce the integer $\delta = 2\operatorname{rk} S - \deg S$. Then the previous inequality is equivalent to $\delta \geq 1$. Now we compute

$$\mu(A) = \frac{\deg S + \deg LK^{-1}(-D)}{\operatorname{rk} S + 1} \leq \frac{2\operatorname{rk} S - \delta + 3}{\operatorname{rk} S + 1} = 2 + \frac{1 - \delta}{\operatorname{rk} S + 1} \leq 2 = \mu(\mathcal{E}_e),$$

which shows the semi-stability of \mathcal{E}_e . □

We tensorize the exact sequence (1) with L and take the cohomology

$$(3) \quad 0 \longrightarrow H^0(E_L^* \otimes L) \longrightarrow H^0(L) \otimes H^0(L) \xrightarrow{\mu} H^0(L^2) \longrightarrow 0.$$

Note that $h^1(E_L^* \otimes L) = h^0(E_L \otimes KL^{-1}) = 0$ by stability of E_L . The second map μ is the multiplication map and factorizes through $\operatorname{Sym}^2 H^0(L)$, i.e.,

$$\Lambda^2 H^0(L) \subset H^0(E_L^* \otimes L) = \ker \mu.$$

By Serre duality a cohomology class $e \in \operatorname{Ext}^1(LK^{-1}, E_L) = H^1(E_L \otimes KL^{-1}) = H^0(E_L^* \otimes L)^*$ can be viewed as a hyperplane $H_e \subset H^0(E_L^* \otimes L)$. Then we have the following

Proposition 2.2. *If $\Lambda^2 H^0(L) \subset H_e$, then the vector bundle \mathcal{E}_e satisfies*

$$h^0(\mathcal{E}_e \otimes \lambda) > 0, \quad \forall \lambda \in \text{Pic}^{g-3}(C).$$

Proof. We tensorize the exact sequence (2) with $\lambda \in \text{Pic}^{g-3}(C)$ and take the cohomology

$$0 \longrightarrow H^0(E_L \otimes \lambda) \longrightarrow H^0(\mathcal{E}_e \otimes \lambda) \longrightarrow H^0(LK^{-1}\lambda) \xrightarrow{\cup_e} H^1(E_L \otimes \lambda) \longrightarrow \dots$$

Since $\deg LK^{-1}\lambda = g$, we can write $LK^{-1}\lambda = \mathcal{O}_C(D)$ for some effective divisor D . It is enough to show that $h^0(\mathcal{E}_e \otimes \lambda) > 0$ holds for λ general. Hence we can assume that $h^0(LK^{-1}\lambda) = h^0(\mathcal{O}_C(D)) = 1$.

If $h^0(E_L \otimes \lambda) > 0$, we are done. So we assume $h^0(E_L \otimes \lambda) = 0$, which implies $h^1(E_L \otimes \lambda) = 1$ by Riemann-Roch. Hence we obtain that $h^0(\mathcal{E}_e \otimes \lambda) > 0$ if and only if the cup product map

$$\cup_e : H^0(\mathcal{O}_X(D)) \longrightarrow H^1(E_L \otimes \lambda) = H^0(E_L^* \otimes L(-D))^*$$

is zero. Furthermore \cup_e is zero if and only if $H^0(E_L^* \otimes L(-D)) \subset H_e$. Now we will show the inclusion

$$(4) \quad H^0(E_L^* \otimes L(-D)) \subset \Lambda^2 H^0(L).$$

We tensorize the exact sequence (1) with $L(-D)$ and take cohomology

$$0 \longrightarrow H^0(E_L^* \otimes L(-D)) \longrightarrow H^0(L) \otimes H^0(L(-D)) \xrightarrow{\mu} H^0(L^2(-D)) \longrightarrow \dots$$

Since $h^0(E_L^* \otimes L(-D)) = 1$, we conclude that $h^0(L(-D)) = 2$ and $H^0(E_L^* \otimes L(-D)) = \Lambda^2 H^0(L(-D)) \subset \Lambda^2 H^0(L)$.

Finally the proposition follows: if $\Lambda^2 H^0(L) \subset H_e$, then by (4) $H^0(E_L^* \otimes L(-D)) \subset H_e$ for general D , or equivalently $h^0(\mathcal{E}_e \otimes \lambda) > 0$ for general $\lambda \in \text{Pic}^{g-3}(C)$. \square

We introduce the linear subspace $\Gamma \subset \text{Ext}^1(LK^{-1}, E_L)$ defined by

$$\Gamma := \ker \left(\text{Ext}^1(LK^{-1}, E_L) = H^0(E_L^* \otimes L)^* \longrightarrow \Lambda^2 H^0(L)^* \right),$$

which has dimension $\frac{g(g-1)}{2} > 0$. Then for any non-zero cohomology class $e \in \Gamma$ and any $\gamma \in \text{Pic}^2(C)$ satisfying $\gamma^{g+2} = L^2 K^{-1} = \det \mathcal{E}_e$, the rank- $(g+2)$ vector bundle

$$\mathcal{E}_e \otimes \gamma^{-1}$$

has trivial determinant, is semi-stable by Proposition 2.1 and has no theta divisor by Proposition 2.2.

2.2. Hyperelliptic curves. In this subsection we assume that C is hyperelliptic and we denote by σ the hyperelliptic involution. The construction of a semi-stable rank-4 vector bundle without theta divisor has been given in [Hi] section 6 in the case $g = 2$, but it can be carried out for any $g \geq 2$ without major modification. For the convenience of the reader, we recall the construction and refer to [Hi] for the details and the proofs.

Let $w \in C$ be a Weierstrass point. Any non-trivial extension

$$0 \longrightarrow \mathcal{O}_C(-w) \longrightarrow G \longrightarrow \mathcal{O}_C \longrightarrow 0$$

is a stable, σ -invariant, rank-2 vector bundle with $\deg G = -1$. By [Hi] Theorem 4 a cohomology class $e \in H^1(\text{Sym}^2 G)$ determines a symplectic rank-4 bundle

$$0 \longrightarrow G \longrightarrow \mathcal{E}_e \longrightarrow G^* \longrightarrow 0.$$

Moreover it is easily seen that, for any non-zero class e , the vector bundle \mathcal{E}_e is semi-stable. By [Hi] Lemma 16 the composite map

$$D_G : \mathbb{P}H^1(\mathrm{Sym}^2 G) \longrightarrow \mathcal{M}_4 \xrightarrow{\theta} |4\Theta|, \quad e \mapsto \theta(\mathcal{E}_e)$$

is the projectivization of a linear map

$$\widetilde{D}_G : H^1(\mathrm{Sym}^2 G) \longrightarrow H^0(\mathrm{Pic}^{g-1}(C), 4\Theta).$$

The involution $i(L) = KL^{-1}$ on $\mathrm{Pic}^{g-1}(C)$ induces a linear involution on $|4\Theta|$ with eigenspaces $|4\Theta|_{\pm}$. Note that $4\Theta \in |4\Theta|_+$. We now observe that $\theta(\mathcal{E}) \in |4\Theta|_+$ for any symplectic rank-4 vector bundle \mathcal{E} — see e.g. [B2]. Moreover we have the equality $\theta(\sigma^*\mathcal{E}) = i^*\theta(\mathcal{E})$ for any vector bundle \mathcal{E} . These two observations imply that the linear map \widetilde{D}_G is equivariant with respect to the induced involutions σ and i . Since $\mathrm{im} \widetilde{D}_G \subset H^0(\mathrm{Pic}^{g-1}(C), 4\Theta)_+$, we obtain that one of the two eigenspaces $H^1(\mathrm{Sym}^2 G)_{\pm}$ is contained in the kernel $\ker \widetilde{D}_G$, hence give base points for the theta map. We now compute as in [Hi] using the Atiyah-Bott-fixed-point formula

$$h^1(\mathrm{Sym}^2 G)_+ = g - 1, \quad h^1(\mathrm{Sym}^2 G)_- = 2g + 1.$$

One can work out that $H^1(\mathrm{Sym}^2 G)_+ \subset \ker \widetilde{D}_G$. Hence any \mathcal{E}_e with non-zero $e \in H^1(\mathrm{Sym}^2 G)_+$ is a semi-stable rank-4 vector bundle without theta divisor.

REFERENCES

- [A] D. Arcara: A lower bound for the dimension of the base locus of the generalized theta divisor, C. R. Acad. Sci. Paris Math. 340 (2005), 131-134
- [B1] A. Beauville: Vector bundles on curves and theta functions, Moduli Spaces and Arithmetic Geometry (Kyoto 2004), Advanced Studies in Pure Mathematics 45 (2006), 145-156
- [B2] A. Beauville: Vector bundles and theta functions on curves of genus 2 and 3, Amer. J. of Math. 128 (2006), 607-618
- [BNR] A. Beauville, M.S. Narasimhan, S. Ramanan: Spectral covers and the generalised theta divisor, J. Reine Angew. Math. 398 (1989), 169-179
- [Bu] D. Butler: Normal generation of vector bundles over a curve, J. Differential Geom. 39 (1994), 1-34
- [He] G. Hein: Raynaud's vector bundles and base points of the generalized Theta divisor, math.AG/0607338
- [Hi] G. Hitching: Rank four symplectic bundles without theta divisor over a curve of genus two, math.AG/0604637, to appear in International J. of Math.
- [P] M. Popa: On the base locus of the generalized theta divisor, C.R. Acad. Sci. Paris Math. 329 (1999), 507-512
- [R] M. Raynaud: Sections de fibrés vectoriels sur une courbe, Bull. Soc. Math. France, 110 (1982), 103-125
- [S] O. Schneider: Sur la dimension de l'ensemble des points de base du fibré déterminant sur $\mathcal{S}U_C(r)$, Ann. Inst. Fourier 57 (2007), 481-490

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE MONTPELLIER II - CASE COURRIER 051, PLACE EUGÈNE BATAILLON, 34095 MONTPELLIER CEDEX 5, FRANCE

E-mail address: pauly@math.univ-montp2.fr