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ON THE BASE LOCUS OF THE LINEAR SYSTEM OF GENERALIZED THETA FUNCTIONS

CHRISTIAN PAULY

ABSTRACT. Let \mathcal{M}_r denote the moduli space of semi-stable rank- r vector bundles with trivial determinant over a smooth projective curve C of genus g . In this paper we study the base locus $\mathcal{B}_r \subset \mathcal{M}_r$ of the linear system of the determinant line bundle \mathcal{L} over \mathcal{M}_r , i.e., the set of semi-stable rank- r vector bundles without theta divisor. We construct base points in \mathcal{B}_{g+2} over any curve C , and base points in \mathcal{B}_4 over any hyperelliptic curve. We also show that \mathcal{B}_4 over a curve of genus 2 consists of 16 reduced points, which are in canonical bijection with the set of theta-characteristics of C .

1. INTRODUCTION

Let C be a complex smooth projective curve of genus g and let \mathcal{M}_r denote the coarse moduli space parametrizing semi-stable rank- r vector bundles with trivial determinant over the curve C . Let \mathcal{L} be the determinant line bundle over the moduli space \mathcal{M}_r and let $\Theta \subset \text{Pic}^{g-1}(C)$ be the Riemann theta divisor in the degree $g-1$ component of the Picard variety of C . By [BNR] there is a canonical isomorphism $|\mathcal{L}|^* \xrightarrow{\sim} |r\Theta|$, under which the natural rational map $\varphi_{\mathcal{L}} : \mathcal{M}_r \dashrightarrow |\mathcal{L}|^*$ is identified with the so-called theta map

$$\theta : \mathcal{M}_r \dashrightarrow |r\Theta|, \quad E \mapsto \theta(E) \subset \text{Pic}^{g-1}(C).$$

The underlying set of $\theta(E)$ consists of line bundles $L \in \text{Pic}^{g-1}(C)$ with $h^0(C, E \otimes L) > 0$. For a general semi-stable vector bundle E , $\theta(E)$ is a divisor. If $\theta(E) = \text{Pic}^{g-1}(C)$, we say that E has no theta divisor. We note that the indeterminacy locus of the theta map θ , i.e., the set of bundles E without theta divisor, coincides with the base locus $\mathcal{B}_r \subset \mathcal{M}_r$ of the linear system $|\mathcal{L}|$.

Over the past years many authors [A], [B2], [He], [Hi], [P], [R], [S] have studied the base locus \mathcal{B}_r of $|\mathcal{L}|$ and their analogues for the powers $|\mathcal{L}^k|$. For a recent survey of this subject we refer to [B1].

It is natural to introduce for a curve C the integer $r(C)$ defined as the minimal rank for which there exists a semi-stable rank- $r(C)$ vector bundle with trivial determinant over C without theta divisor (see also [B1] section 6). It is known [R] that $r(C) \geq 3$ for any curve C and that $r(C) \geq 4$ for a generic curve C . Our first result shows the existence of vector bundles of low ranks without theta divisor.

Theorem 1.1. *We assume that $g \geq 2$. Then we have the following bounds.*

- (1) $r(C) \leq g + 2$.
- (2) $r(C) \leq 4$, if C is hyperelliptic.

The first part of the theorem improves the upper bound $r(C) \leq \frac{(g+1)(g+2)}{2}$ given in [A]. The statements of the theorem are equivalent to the existence of a semi-stable rank- $(g+2)$ (resp. rank-4) vector bundle without theta divisor — see section 2.1 (resp. 2.2). The construction of these vector bundles uses ingredients which are already implicit in [Hi].

Theorem 1.1 seems to hint towards a dependence of the integer $r(C)$ on the curve C .

In the second part of this paper we specialize to the case $g = 2$. By a result of Raynaud [R] we have $r(C) = 4$ for any curve C . We give a full description of the base locus $\mathcal{B}_4 \subset \mathcal{M}_4$.

Theorem 1.2. *Let C be a curve of genus 2.*

- (1) *The base locus \mathcal{B}_4 is of dimension 0, reduced and of cardinality 16.*
- (2) *There exists a canonical bijection between \mathcal{B}_4 and the set of theta-characteristics of C . Let $E_\kappa \in \mathcal{B}_4$ denote the stable vector bundle associated with the theta-characteristic κ . Then*

$$\Lambda^2 E_\kappa = \bigoplus_{\alpha \in S(\kappa)} \alpha, \quad \text{Sym}^2 E_\kappa = \bigoplus_{\alpha \in J[2] \setminus S(\kappa)} \alpha,$$

where $S(\kappa)$ is the set of 2-torsion line bundles $\alpha \in J[2]$ such that $\kappa\alpha \in \Theta \subset \text{Pic}^1(C)$.

- (3) *If κ is odd, then E_κ is a symplectic bundle. If κ is even, then E_κ is an orthogonal bundle with non-trivial Stiefel-Whitney class.*
- (4) *The 16 vector bundles E_κ are invariant under the tensor product with the group $J[2]$.*

The 16 base points E_κ already appeared in Raynaud's paper [R] as Fourier-Mukai transforms and were further studied in [Hi] and [He], where they are called Raynaud bundles — see section 3.2. We note that for $g = 2$ the two constructions of Theorem 1.1 coincide and give the 16 Raynaud bundles. We also remark that the restriction of \mathcal{B}_4 to *symplectic* rank-4 bundles has been worked out in [Hi]. The method of this paper is different.

For $g = 2$ the theta map θ is a generically finite, dominant map [B2]. The degree of this map is known only for rank-3 bundles (see e.g. [O]). An alternate interpretation of this degree is given in [BV]. Theorem 1.2 can be applied to compute the degree of θ for rank-4 bundles.

Corollary 1.3. *The degree of the rational theta map $\theta : \mathcal{M}_4 \dashrightarrow |4\Theta|$ equals 30.*

Notations: If E is a vector bundle over C , we will write $H^i(E)$ for $H^i(C, E)$ and $h^i(E)$ for $\dim H^i(C, E)$. We denote the slope of E by $\mu(E) := \frac{\deg E}{\text{rk} E}$, the canonical bundle over C by K and the degree d component of the Picard variety of C by $\text{Pic}^d(C)$. We denote by $J := \text{Pic}^0(C)$ the Jacobian of C and by $J[n]$ its group of n -torsion points. The divisor $\Theta_\kappa \subset J$ is the translate of the Riemann theta divisor $\Theta \subset \text{Pic}^{g-1}(C)$ by a theta-characteristic κ . The line bundle $\mathcal{O}_J(2\Theta_\kappa)$ does not depend on κ and will be denoted by $\mathcal{O}_J(2\Theta)$.

2. PROOF OF THEOREM 1.1

2.1. Semi-stable rank- $(g+2)$ vector bundles without theta divisor. We consider a line bundle $L \in \text{Pic}^{2g+1}(C)$. Then L is globally generated, $h^0(L) = g+2$ and the evaluation bundle E_L , which is defined by the exact sequence

$$(1) \quad 0 \longrightarrow E_L^* \longrightarrow H^0(L) \otimes \mathcal{O}_C \xrightarrow{ev} L \longrightarrow 0,$$

is stable (see e.g. [Bu]), with $\deg E_L = 2g + 1$, $\text{rk} E_L = g + 1$ and $\mu(E_L) = 2 - \frac{1}{g+1}$.

A cohomology class $e \in \text{Ext}^1(LK^{-1}, E_L) = H^1(E_L \otimes KL^{-1})$ determines a rank- $(g + 2)$ vector bundle \mathcal{E}_e given as an extension

$$(2) \quad 0 \longrightarrow E_L \longrightarrow \mathcal{E}_e \longrightarrow LK^{-1} \longrightarrow 0.$$

Proposition 2.1. *For any non-zero class e , the rank- $(g + 2)$ vector bundle \mathcal{E}_e is semi-stable.*

Proof. Consider a proper subbundle $A \subset \mathcal{E}_e$ and let $S = A \cap E_L \subset E_L$. Then there is an exact sequence

$$0 \longrightarrow S \longrightarrow A \longrightarrow LK^{-1}(-D) \longrightarrow 0,$$

where D is an effective divisor. If $\text{rk} S = g + 1$ or $S = 0$, we easily conclude that $\mu(A) < \mu(\mathcal{E}_e) = 2$. If $\text{rk} S < g + 1$ and $S \neq 0$, then stability of E_L gives the inequality $\mu(S) < \mu(E_L) = 2 - \frac{1}{g+1}$. We introduce the integer $\delta = 2\text{rk} S - \deg S$. Then the previous inequality is equivalent to $\delta \geq 1$. Now we compute

$$\mu(A) = \frac{\deg S + \deg LK^{-1}(-D)}{\text{rk} S + 1} \leq \frac{2\text{rk} S - \delta + 3}{\text{rk} S + 1} = 2 + \frac{1 - \delta}{\text{rk} S + 1} \leq 2 = \mu(\mathcal{E}_e),$$

which shows the semi-stability of \mathcal{E}_e . \square

We tensorize the exact sequence (1) with L and take the cohomology

$$(3) \quad 0 \longrightarrow H^0(E_L^* \otimes L) \longrightarrow H^0(L) \otimes H^0(L) \xrightarrow{\mu} H^0(L^2) \longrightarrow 0.$$

Note that $h^1(E_L^* \otimes L) = h^0(E_L \otimes KL^{-1}) = 0$ by stability of E_L . The second map μ is the multiplication map and factorizes through $\text{Sym}^2 H^0(L)$, i.e.,

$$\Lambda^2 H^0(L) \subset H^0(E_L^* \otimes L) = \ker \mu.$$

By Serre duality a cohomology class $e \in \text{Ext}^1(LK^{-1}, E_L) = H^1(E_L \otimes KL^{-1}) = H^0(E_L^* \otimes L)^*$ can be viewed as a hyperplane $H_e \subset H^0(E_L^* \otimes L)$. Then we have the following

Proposition 2.2. *If $\Lambda^2 H^0(L) \subset H_e$, then the vector bundle \mathcal{E}_e satisfies*

$$h^0(\mathcal{E}_e \otimes \lambda) > 0, \quad \forall \lambda \in \text{Pic}^{g-3}(C).$$

Proof. We tensorize the exact sequence (2) with $\lambda \in \text{Pic}^{g-3}(C)$ and take the cohomology

$$0 \longrightarrow H^0(E_L \otimes \lambda) \longrightarrow H^0(\mathcal{E}_e \otimes \lambda) \longrightarrow H^0(LK^{-1}\lambda) \xrightarrow{\cup e} H^1(E_L \otimes \lambda) \longrightarrow \dots$$

Since $\deg LK^{-1}\lambda = g$, we can write $LK^{-1}\lambda = \mathcal{O}_C(D)$ for some effective divisor D . It is enough to show that $h^0(\mathcal{E}_e \otimes \lambda) > 0$ holds for λ general. Hence we can assume that $h^0(LK^{-1}\lambda) = h^0(\mathcal{O}_C(D)) = 1$.

If $h^0(E_L \otimes \lambda) > 0$, we are done. So we assume $h^0(E_L \otimes \lambda) = 0$, which implies $h^1(E_L \otimes \lambda) = 1$ by Riemann-Roch. Hence we obtain that $h^0(\mathcal{E}_e \otimes \lambda) > 0$ if and only if the cup product map

$$\cup e : H^0(\mathcal{O}_X(D)) \longrightarrow H^1(E_L \otimes \lambda) = H^0(E_L^* \otimes L(-D))^*$$

is zero. Furthermore $\cup e$ is zero if and only if $H^0(E_L^* \otimes L(-D)) \subset H_e$. The proposition then will follow from the inclusion

$$H^0(E_L^* \otimes L(-D)) \subset \Lambda^2 H^0(L),$$

which is seen as follows. We tensorize the exact sequence (1) with $L(-D)$ and take cohomology

$$0 \longrightarrow H^0(E_L^* \otimes L(-D)) \longrightarrow H^0(L) \otimes H^0(L(-D)) \xrightarrow{\mu} H^0(L^2(-D)) \longrightarrow \dots$$

Since $h^0(E_L^* \otimes L(-D)) = 1$, we conclude that $h^0(L(-D)) = 2$ and $H^0(E_L^* \otimes L(-D)) = \Lambda^2 H^0(L(-D)) \subset \Lambda^2 H^0(L)$. \square

We introduce the linear subspace $\Gamma \subset \text{Ext}^1(LK^{-1}, E_L)$ defined by

$$\Gamma := \ker \left(\text{Ext}^1(LK^{-1}, E_L) = H^0(E_L^* \otimes L)^* \longrightarrow \Lambda^2 H^0(L)^* \right),$$

which has dimension $\frac{g(g-1)}{2} > 0$. Then for any non-zero cohomology class $e \in \Gamma$ and any $\gamma \in \text{Pic}^2(C)$ satisfying $\gamma^{g+2} = L^2 K^{-1} = \det \mathcal{E}_e$, the rank- $(g+2)$ vector bundle

$$\mathcal{E}_e \otimes \gamma^{-1}$$

has trivial determinant, is semi-stable by Proposition 2.1 and has no theta divisor by Proposition 2.2.

2.2. Hyperelliptic curves. In this subsection we assume that C is hyperelliptic and we denote by σ the hyperelliptic involution. The construction of a semi-stable rank-4 vector bundle without theta divisor has been given in [Hi] section 6 in the case $g = 2$, but it can be carried out for any $g \geq 2$ without major modification. For the convenience of the reader, we recall the construction and refer to [Hi] for the details and the proofs.

Let $w \in C$ be a Weierstrass point. Any non-trivial extension

$$0 \longrightarrow \mathcal{O}_C(-w) \longrightarrow G \longrightarrow \mathcal{O}_C \longrightarrow 0$$

is a stable, σ -invariant, rank-2 vector bundle with $\deg G = -1$. By [Hi] Theorem 4 a cohomology class $e \in H^1(\text{Sym}^2 G)$ determines a symplectic rank-4 bundle

$$0 \longrightarrow G \longrightarrow \mathcal{E}_e \longrightarrow G^* \longrightarrow 0.$$

Moreover it is easily seen that, for any non-zero class e , the vector bundle \mathcal{E}_e is semi-stable. By [Hi] Lemma 16 the composite map

$$D_G : \mathbb{P}H^1(\text{Sym}^2 G) \longrightarrow \mathcal{M}_4 \xrightarrow{\theta} |4\Theta|, \quad e \mapsto \theta(\mathcal{E}_e)$$

is the projectivization of a linear map

$$\widetilde{D}_G : H^1(\text{Sym}^2 G) \longrightarrow H^0(\text{Pic}^{g-1}(C), 4\Theta).$$

The involution $i(L) = KL^{-1}$ on $\text{Pic}^{g-1}(C)$ induces a linear involution on $|4\Theta|$ with eigenspaces $|4\Theta|_{\pm}$. Note that $4\Theta \in |4\Theta|_+$. We now observe that $\theta(\mathcal{E}) \in |4\Theta|_+$ for any symplectic rank-4 vector bundle \mathcal{E} — see e.g. [B2]. Moreover we have the equality $\theta(\sigma^* \mathcal{E}) = i^* \theta(\mathcal{E})$ for any vector bundle \mathcal{E} . These two observations imply that the linear map \widetilde{D}_G is equivariant with respect to the induced involutions σ and i . Since $\text{im } \widetilde{D}_G \subset H^0(\text{Pic}^{g-1}(C), 4\Theta)_+$, we obtain that one of the two eigenspaces $H^1(\text{Sym}^2 G)_{\pm}$ is contained in the kernel $\ker \widetilde{D}_G$, hence give base points for the theta map. We now compute as in [Hi] using the Atiyah-Bott-fixed-point formula

$$h^1(\text{Sym}^2 G)_+ = g - 1, \quad h^1(\text{Sym}^2 G)_- = 2g + 1.$$

One can work out that $H^1(\text{Sym}^2 G)_+ \subset \ker \widetilde{D}_G$. Hence any \mathcal{E}_e with non-zero $e \in H^1(\text{Sym}^2 G)_+$ is a semi-stable rank-4 vector bundle without theta divisor.

3. PROOF OF THEOREM 1.2

3.1. The 16 base points E_κ . We first show that the set-theoretical support of \mathcal{B}_4 consists of 16 stable vector bundles E_κ , which are canonically labelled by the theta-characteristics of C .

We note that $\mathcal{B}_4 \neq \emptyset$ by Theorem 1.1. We consider a vector bundle $\mathcal{E} \in \mathcal{B}_4$. Since $r(C) = 4$, we deduce that \mathcal{E} is stable. We introduce $\mathcal{E}' = \mathcal{E}^* \otimes K$. Then $\mu(\mathcal{E}') = 2$ and since $\mathcal{E} \in \mathcal{B}_4$, we obtain that $h^0(\mathcal{E}' \otimes \lambda^{-1}) = h^1(\mathcal{E} \otimes \lambda) = h^0(\mathcal{E} \otimes \lambda) > 0$ for any $\lambda \in \text{Pic}^1(C)$. In particular for any $x \in C$ we have $h^0(\mathcal{E}' \otimes \mathcal{O}_C(-x)) > 0$. On the other hand stability of \mathcal{E} implies that $h^0(\mathcal{E}) = h^1(\mathcal{E}') = 0$. Hence $h^0(\mathcal{E}') = 4$ by Riemann-Roch. Thus we obtain that the evaluation map of global sections

$$\mathcal{O}_C \otimes H^0(\mathcal{E}') \xrightarrow{ev} \mathcal{E}'$$

is not of maximal rank. Let us denote by $I := \text{im } ev$ the subsheaf of \mathcal{E}' given by the image of ev . Then clearly $h^0(I) = 4$. The cases $\text{rk } I \leq 2$ are easily ruled out using stability of \mathcal{E}' . Hence we conclude that $\text{rk } I = 3$. We then consider the natural exact sequence

$$(4) \quad 0 \longrightarrow L^{-1} \longrightarrow \mathcal{O}_C \otimes H^0(\mathcal{E}') \xrightarrow{ev} I \longrightarrow 0,$$

where L is the line bundle such that $L^{-1} := \ker ev$.

Proposition 3.1. *We have $h^0(I^*) = 0$.*

Proof. Suppose on the contrary that there exists a non-zero map $I \rightarrow \mathcal{O}_C$. Its kernel $S \subset I$ is a rank-2 subsheaf of \mathcal{E}' and by stability of \mathcal{E}' we obtain $\mu(S) < \mu(\mathcal{E}') = 2$, hence $\deg S \leq 3$. Moreover $h^0(S) \geq h^0(I) - 1 = 3$.

Assume that $\deg S = 3$. Then S is stable and S can be written as an extension

$$0 \longrightarrow \mu \longrightarrow S \longrightarrow \nu \longrightarrow 0,$$

with $\deg \mu = 1$ and $\deg \nu = 2$. The condition $h^0(S) \geq 3$ then implies that $\mu = \mathcal{O}_C(x)$ for some $x \in C$, $\nu = K$ and that the extension has to be split, i.e., $S = K \oplus \mathcal{O}_C(x)$. This contradicts stability of S .

The assumption $\deg S \leq 2$ similarly leads to a contradiction. We leave the details to the reader. \square

Now we take the cohomology of the dual of the exact sequence (4) and we obtain — using $h^0(I^*) = 0$ — an inclusion $H^0(\mathcal{E}')^* \subset H^0(L)$. Hence $h^0(L) \geq 4$, which implies $\deg L \geq 5$. On the other hand $\deg L = \deg I$ and by stability of \mathcal{E}' , we have $\mu(I) < 2$, i.e., $\deg L \leq 5$. So we can conclude that $\deg L = 5$, that $H^0(\mathcal{E}')^* = H^0(L)$ and that $I = E_L$, where E_L is the *evaluation bundle* associated to L defined in (1).

Moreover the subsheaf $E_L \subset \mathcal{E}'$ is of maximal degree, hence E_L is a subbundle of \mathcal{E}' and we have an exact sequence

$$0 \longrightarrow E_L \longrightarrow \mathcal{E}' \longrightarrow K^4 L^{-1} \longrightarrow 0,$$

with extension class $e \in \text{Ext}^1(K^4 L^{-1}, E_L) = H^1(E_L \otimes K^{-4} L) = H^0(E_L^* \otimes K^5 L^{-1})^*$. Using Riemann-Roch and stability of E_L one shows that

$$h^0(E_L^* \otimes K^5 L^{-1}) = 7, \quad h^0(E_L^* \otimes K^5 L^{-1}(-x)) = 4, \quad h^0(E_L^* \otimes K^5 L^{-1}(-x-y)) = 1$$

for general points $x, y \in C$. In that case we denote by $\mu_{x,y} \in \mathbb{P}H^0(E_L^* \otimes K^5L^{-1})$ the point determined by the 1-dimensional subspace $H^0(E_L^* \otimes K^5L^{-1}(-x-y))$. We also denote by

$$\mathbb{S} \subset \mathbb{P}H^0(E_L^* \otimes K^5L^{-1})$$

the linear span of the points $\mu_{x,y}$ when x and y vary in C and by $H_e \subset \mathbb{P}H^0(E_L^* \otimes K^5L^{-1})$ the hyperplane determined by the non-zero class e .

By the same argument as in the proof of Theorem 1.1 one shows that $\mu_{x,y} \in H_e$ if and only if $h^0(\mathcal{E}' \otimes K^{-4}L(x+y)) > 0$. Since we assume $\mathcal{E} \in \mathcal{B}_4$, we obtain

$$\mathbb{S} \subset H_e.$$

We consider a *general* point $x \in C$ such that $h^0(E_L^* \otimes K^5L^{-1}(-x)) = 4$ and denote for simplicity

$$A := E_L^* \otimes K^5L^{-1}(-x).$$

Then A is stable with $\mu(A) = \frac{7}{3}$. We consider the evaluation map of global sections

$$ev_A : \mathcal{O}_C \otimes H^0(A) \longrightarrow A$$

and consider the set S_A of points $p \in C$ for which $(ev_A)_p$ is not surjective, i.e.

$$S_A = \{p \in C \mid h^0(A(-p)) \geq 2\}.$$

Then we have the following

Lemma 3.2. *We assume that x is general.*

- (1) *If $L^2 \neq K^5$, then the set S_A consists of the 2 distinct points p_1, p_2 determined by $\mathcal{O}_C(p_1 + p_2) = K^4L^{-1}(-x)$.*
- (2) *If $L^2 = K^5$, then the set S_A consists of the 2 distinct points p_1, p_2 introduced in (1) and the conjugate $\sigma(x)$ of x under the hyperelliptic involution σ .*

Proof. Given a point $p \in C$, we tensorize the exact sequence (1) with $K^5L^{-1}(-x-p)$ and take cohomology:

$$0 \longrightarrow H^0(A(-p)) \longrightarrow H^0(L) \otimes H^0(K^5L^{-1}(-x-p)) \longrightarrow H^0(K^5(-x-p)) \longrightarrow \cdots$$

We note that $h^0(K^5L^{-1}(-x-p)) = 2$. We distinguish two cases.

(a) The pencil $|K^5L^{-1}(-x-p)|$ has a base-point, i.e. there exists a point $q \in C$ such that $K^5L^{-1}(-x-p) = K(q)$, or equivalently $K^4L^{-1}(-x) = \mathcal{O}_C(p+q)$. Since x is general, we have $h^0(K^4L^{-1}(-x)) = 1$, which determines p and q , i.e., $\{p, q\} = \{p_1, p_2\}$. In this case $|K^5L^{-1}(-x-p)| = |K(q)| = |K|$ and $h^0(A(-p)) = h^0(K^{-1}L) = 2$. This shows that $p_1, p_2 \in S_A$.

(b) The pencil $|K^5L^{-1}(-x-p)|$ is base-point-free. By the base-point-free-pencil-trick, we have $H^0(A(-p)) \cong H^0(L^2K^{-5}(x+p))$. Since $\deg L^2K^{-5}(x+p) = 2$, we have $h^0(L^2K^{-5}(x+p)) = 2$ if and only if $L^2K^{-5}(x+p) = K$, or equivalently $\mathcal{O}_C(p) = K^6L^{-2}(-x)$. If $K^6L^{-2} \neq K$, then for general $x \in C$ the line bundle $K^6L^{-2}(-x)$ is not of the form $\mathcal{O}_C(p)$. If $K^6L^{-2} = K$, then for any $x \in C$, $K^6L^{-2}(-x) = \mathcal{O}_C(\sigma(x))$, which implies that $\sigma(x) \in S_A$.

This shows the lemma. □

Proposition 3.3. *If $L^2 \neq K^5$, then $\mathbb{S} = \mathbb{P}H^0(E_L^* \otimes K^5L^{-1})$.*

Proof. We consider a general point $x \in C$ and the rank-3 bundle A . Let $B \subset A$ denote the subsheaf given by the image of ev_A . By Lemma 3.2 (1) we have $\deg B = \deg A - 2 = 5$. Moreover $H^0(B) = H^0(A)$ and there is an exact sequence

$$(5) \quad 0 \longrightarrow M^{-1} \longrightarrow \mathcal{O}_C \otimes H^0(B) \xrightarrow{ev_A} B \longrightarrow 0,$$

with $M \in \text{Pic}^5(C)$. It follows that the rational map

$$\phi_x : C \dashrightarrow \mathbb{P}H^0(B) = \mathbb{P}H^0(A) = \mathbb{P}^3, \quad y \mapsto \mu_{x,y}$$

factorizes through

$$C \xrightarrow{\varphi_M} |M|^* \longrightarrow \mathbb{P}H^0(B),$$

where φ_M is the morphism given by the linear system $|M|$ and the second map is linear and identifies with the projectivization of the dual of δ , which is given by the long exact sequence obtained from (5) by dualizing and taking cohomology:

$$0 \longrightarrow H^0(B^*) \longrightarrow H^0(B)^* \xrightarrow{\delta} H^0(M) \longrightarrow H^1(B^*) \longrightarrow \dots$$

We obtain that the linear span of $\text{im } \phi_x$ is non-degenerate if and only if $h^0(B^*) = 0$.

We now show that $h^0(B^*) = 0$. Suppose on the contrary that there exists a non-zero map $B \rightarrow \mathcal{O}_C$. Its kernel $S \subset B$ is a rank-2 subsheaf of A with $\deg S \geq \deg B = 5$, hence $\mu(S) \geq \frac{5}{2}$, which contradicts stability of A — recall that $\mu(A) = \frac{7}{3}$.

This shows that $\text{im } \phi_x$ spans $\mathbb{P}H^0(A) \subset \mathbb{P}H^0(E_L^* \otimes K^5 L^{-1})$ for general $x \in C$. We now take 2 general points $x, x' \in C$ and deduce from $\dim H^0(A) \cap H^0(A') = \dim H^0(E_L^* \otimes K^5 L^{-1}(-x - x')) = 1$ that the linear span of the union $\mathbb{P}H^0(A) \cup \mathbb{P}H^0(A')$ equals the full space $\mathbb{P}H^0(E_L^* \otimes K^5 L^{-1})$. This shows the proposition. \square

We deduce from the proposition that the line bundle L satisfies the relation $L^2 = K^5$, i.e.

$$L = K^2 \kappa$$

for some theta-characteristic κ of C . In that case the extension space $H^0(E_L \otimes K^5 L^{-1})^*$ equals $H^0(E_L^* \otimes L)^*$ and we have the exact sequence (3). As already noticed in section 2.1 we have a natural inclusion $\Lambda^2 H^0(L) \subset H^0(E_L^* \otimes L)$. More precisely we can show

Proposition 3.4. *The linear span \mathbb{S} equals*

$$\mathbb{S} = \mathbb{P}\Lambda^2 H^0(L) \subset \mathbb{P}H^0(E_L^* \otimes L).$$

Proof. Using the standard exact sequences and the base-point-free-pencil-trick, one easily works out that for general points $x, y \in C$

$$\mu_{x,y} = \mathbb{P}\Lambda^2 H^0(L(-x - y)) \subset \mathbb{P}\Lambda^2 H^0(L) \subset \mathbb{P}H^0(E_L^* \otimes L).$$

This implies that $\mathbb{S} \subset \mathbb{P}\Lambda^2 H^0(L)$. In order to show equality one chooses 4 general points $x_i \in C$ such that their images $C \rightarrow |L|^* = \mathbb{P}^3$ linearly span the \mathbb{P}^3 . We denote by $s_i \in H^0(L)$ the global section vanishing on the points x_j for $j \neq i$ and not vanishing on x_i . Then one checks that for any choice of the indices i, j, k, l such that $\{i, j, k, l\} = \{1, 2, 3, 4\}$ one has $s_i \wedge s_j = \mu_{x_k, x_l}$. Since the 6 tensors $s_i \wedge s_j$ are a basis of $\Lambda^2 H^0(L)$, we obtain equality. \square

The hyperplane $\mathbb{S} = \mathbb{P}\Lambda^2 H^0(L) \subset \mathbb{P}H^0(E_L^* \otimes L)$ determines a unique (up to a scalar) non-zero extension class $e \in H^0(E_L^* \otimes L)^*$ by $\mathbb{S} = H_e$, which in turn determines a unique stable vector bundle $\mathcal{E} \in \mathcal{B}_4$, which we will denote by E_κ .

This shows that \mathcal{B}_4 is of dimension 0 and of cardinality 16.

3.2. The Raynaud bundles. In this subsection we recall the construction of the Raynaud bundles introduced in [R] as Fourier-Mukai transforms. We refer to [Hi] section 9.2 for the details and the proofs.

The rank-4 vector bundle $\mathcal{O}_J(2\Theta) \otimes H^0(J, \mathcal{O}_J(2\Theta))^*$ over J admits a canonical $J[2]$ -linearization and descends therefore under the duplication map $[2] : J \rightarrow J$, i.e., there exists a rank-4 vector bundle M over J such that

$$[2]^*M \cong \mathcal{O}_J(2\Theta) \otimes H^0(J, \mathcal{O}_J(2\Theta))^*.$$

Proposition 3.5. *For any theta-characteristic κ of C there exists an isomorphism*

$$\xi_\kappa : M \xrightarrow{\sim} M^* \otimes \mathcal{O}_J(\Theta_\kappa).$$

Moreover if κ is even (resp. odd), then ξ_κ is symmetric (resp. skew-symmetric).

Let $\gamma_\kappa : C \rightarrow J$ be the Abel-Jacobi map defined by $\gamma_\kappa(p) = \kappa^{-1}(p)$. We define the Raynaud bundle

$$R_\kappa := \gamma_\kappa^*M \otimes \kappa^{-1}.$$

Then by [R] the bundle $R_\kappa \in \mathcal{B}_4$. Since $\gamma_\kappa^*\mathcal{O}_J(\Theta_\kappa) = K$ we see that the isomorphism ξ_κ induces an orthogonal (resp. symplectic) structure on the bundle R_κ , if κ is even (resp. odd). In particular the bundle R_κ is self-dual, i.e., $R_\kappa = R_\kappa^*$. The pull-back $\gamma_{\kappa'}^*(\xi_{\kappa'})$ for a theta-characteristic $\kappa' = \kappa\alpha$ with $\alpha \in J[2]$ gives an isomorphism

$$R_\kappa \xrightarrow{\sim} R_\kappa^* \otimes \alpha,$$

hence a non-zero section in $H^0(\Lambda^2 R_\kappa \otimes \alpha)$ (resp. $H^0(\text{Sym}^2 R_\kappa \otimes \alpha)$) if $h^0(\kappa\alpha) = 1$ (resp. $h^0(\kappa\alpha) = 0$). We deduce that there are isomorphisms

$$(6) \quad \Lambda^2 R_\kappa = \bigoplus_{\alpha \in S(\kappa)} \alpha, \quad \text{Sym}^2 R_\kappa = \bigoplus_{\alpha \in J[2] \setminus S(\kappa)} \alpha.$$

In particular the 16 bundles R_κ are non-isomorphic. Each R_κ is invariant under tensor product with $J[2]$. The isomorphisms (6) can be used to prove the relation

$$(7) \quad R_\kappa \otimes \beta = R_{\kappa\beta^2}, \quad \forall \beta \in J[4].$$

3.3. Symplectic and orthogonal bundles. In this subsection we give a third construction of the bundles in \mathcal{B}_4 as symplectic and orthogonal extension bundles. Let κ be a theta-characteristic.

If κ is odd, then $\kappa = \mathcal{O}_C(w)$ for some Weierstrass point $w \in C$. The construction outlined in section 2.2 gives a unique symplectic bundle $\mathcal{E}_e \in \mathcal{B}_4$ with $e \in H^1(\text{Sym}^2 G)_+$. We denote this bundle by V_κ .

If κ is even, there is an analogue construction, which we briefly outline for the convenience of the reader. The proofs are similar to those given in [Hi]. Using the Atiyah-Bott-fixed-point formula one observes that among all non-trivial extensions

$$0 \longrightarrow \kappa^{-1} \longrightarrow G \longrightarrow \mathcal{O}_C \longrightarrow 0,$$

there are 2 extensions (up to scalar), which are σ -invariant. We take one of them. Then any non-zero class $e \in H^1(\Lambda^2 G) = H^1(\kappa^{-1})$ determines an orthogonal bundle \mathcal{E}_e , which fits in the exact sequence

$$(8) \quad 0 \longrightarrow G \longrightarrow \mathcal{E}_e \longrightarrow G^* \longrightarrow 0.$$

The composite map

$$D_G : \mathbb{P}H^1(\Lambda^2 G) \longrightarrow \mathcal{M}_4 \xrightarrow{\theta} |4\Theta|, \quad e \mapsto \theta(\mathcal{E}_e)$$

is the projectivization of a linear map

$$\widetilde{D}_G : H^1(\Lambda^2 G) \longrightarrow H^0(\text{Pic}^1(C), 4\Theta).$$

Moreover $\text{im } \widetilde{D}_G \subset H^0(\text{Pic}^1(C), 4\Theta)_-$, which can be seen as follows. By [Se] Thm 2 the second Stiefel-Whitney class $w_2(\mathcal{E}_e)$ of an orthogonal bundle \mathcal{E}_e is given by the parity of $h^0(\mathcal{E}_e \otimes \kappa')$ for any theta-characteristic κ' . This parity can be computed by taking the cohomology of the exact sequence (8) tensorized with κ' and taking into account that the coboundary map is skew-symmetric. One obtains that $w_2(\mathcal{E}_e) \neq 0$ and one can conclude the above-mentioned inclusion by [B3] Lemma 1.4.

We now observe that by the Atiyah-Bott-fixed-point-formula $h^1(\Lambda^2 G)_+ = h^1(\Lambda^2 G)_- = 1$. By the argument given in section 2.2 we conclude that one of the two eigenspaces $H^1(\Lambda^2 G)_\pm$ is contained in the kernel $\ker \widetilde{D}_G$. We denote the corresponding bundle \mathcal{E}_e by $V_\kappa \in \mathcal{B}_4$.

3.4. Three descriptions of the same bundle.

Proposition 3.6. *For any theta-characteristic κ the three bundles E_κ , R_κ and V_κ coincide.*

Proof. If κ is odd, this was worked out in detail in [Hi] section 8 and Theorem 29. If κ is even, the proofs are similar. \square

This proposition shows all assertions of Theorem 1.2 except reducedness of \mathcal{B}_4 .

3.5. Reducedness of \mathcal{B}_4 . We start with a description of the space of global sections $H^0(\mathcal{M}_4, \mathcal{L})$.

Proposition 3.7. *For any theta-characteristic κ there is a section $s_\kappa \in H^0(\mathcal{M}_4, \mathcal{L})$ with zero divisor*

$$\Delta_\kappa := \text{Zero}(s_\kappa) = \{E \in \mathcal{M}_4 \mid h^0(\Lambda^2 E \otimes \kappa) > 0\}.$$

The 16 sections s_κ form a basis of $H^0(\mathcal{M}_4, \mathcal{L})$.

Proof. The Dynkin index of the second fundamental representation $\rho : \mathfrak{sl}_4(\mathbb{C}) \rightarrow \text{End}(\Lambda^2 \mathbb{C}^4)$ equals 2 (see e.g. [LS] Proposition 2.6). Moreover the bundle $\Lambda^2 E \otimes \kappa$ admits a K -valued non-degenerate quadratic form, which allows to construct the Pfaffian divisor s_κ , which is a section of \mathcal{L} (see [LS]). The space $H^0(\mathcal{M}_4, \mathcal{L})$ is a representation of level 2 of the Heisenberg group $\text{Heis}(2)$, which is a central extension of $J[2]$ by \mathbb{C}^* . One can work out that the sections s_κ generate the 16 one-dimensional character spaces for the $\text{Heis}(2)$ -action on $H^0(\mathcal{M}_4, \mathcal{L})$. This shows that the sections s_κ are linearly independent. \square

Since $E_\kappa \in \mathcal{B}_4$, we have $E_\kappa \in \Delta_{\kappa'}$ for any theta-characteristic κ' . By the deformation theory of determinant and Pfaffian divisors (see e.g. [L], [LS]) the point $E_\kappa \in \mathcal{M}_4$ is a smooth point of the divisor $\Delta_{\kappa'} \subset \mathcal{M}_4$ if and only if the following two conditions hold

- (1) $h^0(\Lambda^2 E_\kappa \otimes \kappa') = 2$,
- (2) the natural linear form

$$\Phi_{\kappa'} : T_{E_\kappa} \mathcal{M}_4 = H^1(\text{End}_0(E_\kappa)) \longrightarrow \Lambda^2 H^0(\Lambda^2 E_\kappa \otimes \kappa')^*$$

is non-zero.

Moreover if these two conditions holds, then $T_{E_\kappa} \Delta_{\kappa'} = \ker \Phi_{\kappa'}$. The map $\Phi_{\kappa'}$ is built up as follows: the exceptional isomorphism of Lie algebras $\mathfrak{sl}_4 \cong \mathfrak{so}_6$ induces a natural vector bundle isomorphism

$$(9) \quad \text{End}_0(E_\kappa) \xrightarrow{\sim} \Lambda^2(\Lambda^2 E_\kappa).$$

Then $\Phi_{\kappa'}$ is the dual of the linear map given by the wedge product of global sections

$$\Lambda^2 H^0(\Lambda^2 E_\kappa \otimes \kappa') \longrightarrow H^0(\Lambda^2(\Lambda^2 E_\kappa) \otimes K) = H^0(\text{End}_0(E_\kappa) \otimes K).$$

Proposition 3.8. *The 0-dimensional scheme \mathcal{B}_4 is reduced.*

Proof. Since E_κ is a smooth point of \mathcal{M}_4 and $\dim T_{E_\kappa} \mathcal{M}_4 = 15$, it is sufficient to show that for any theta-characteristic $\kappa' \neq \kappa$ the divisor $\Delta_{\kappa'}$ is smooth at E_κ and that the 15 hyperplanes $\ker \Phi_{\kappa'} \subset T_{E_\kappa} \mathcal{M}_4$ are linearly independent: using the isomorphism (6) we obtain that for $\kappa' \neq \kappa$

$$h^0(\Lambda^2 E_\kappa \otimes \kappa') = \sharp S(\kappa) \cap S(\kappa') = 2$$

and using the isomorphism (9) we obtain that

$$\text{End}_0(E_\kappa) = \bigoplus_{\alpha \in J[2] \setminus \{0\}} \alpha.$$

On the other hand one easily sees that if $\gamma, \delta \in J[2]$ are the two 2-torsion points in the intersection $S(\kappa) \cap S(\kappa')$, then $\kappa' = \kappa\gamma\delta$, hence $\Lambda^2 H^0(\Lambda^2 E_\kappa \otimes \kappa') \cong H^0(K\gamma\delta)$. This implies that the linear form

$$\Phi_{\kappa'} : \bigoplus_{\alpha \in J[2] \setminus \{0\}} H^1(\alpha) \longrightarrow H^0(K\gamma\delta)^* = H^1(\beta)$$

is projection onto the direct summand $H^1(\beta)$, where $\beta = \kappa^{-1}\kappa' \in J[2]$. This description of the linear forms $\Phi_{\kappa'}$ clearly shows that they are non-zero and linearly independent. \square

This completes the proof of Theorem 1.2.

4. PROOF OF COROLLARY 1.3

Since by Theorem 1.2 \mathcal{B}_4 is a reduced 0-dimensional scheme of length 16, the degree of the theta map θ is given by the formula

$$\deg \theta + 16 = c_{15},$$

where $\frac{c_{15}}{15!}$ is the leading coefficient of the Hilbert polynomial

$$P(n) = \chi(\mathcal{M}_4, \mathcal{L}^n) = \frac{c_{15}}{15!} n^{15} + \text{lower degree terms.}$$

In order to compute the polynomial P we write

$$(10) \quad P(X) = \sum_{k=0}^{15} \alpha_k Q_k(X), \quad \text{with} \quad Q_k(X) = \frac{1}{k!} (X+7)(X+6) \cdots (X+8-k)$$

and $Q_0(X) = 1$. Note that $\deg Q_k = k$ and that $c_{15} = \alpha_{15}$. The canonical bundle of \mathcal{M}_4 equals \mathcal{L}^{-8} . By the Grauert-Riemenschneider vanishing theorem we obtain that $h^i(\mathcal{M}_4, \mathcal{L}^n) = 0$ for any $i \geq 1$ and $n \geq -7$. Hence $P(n) = h^0(\mathcal{M}_4, \mathcal{L}^n)$ for $n \geq -7$. Moreover $P(n) = 0$ for $n = -7, -6, \dots, -1$ and $P(0) = 1$. The values $P(n)$ for $n = 1, 2, \dots, 8$ can be computed by the Verlinde formula and with the use of MAPLE. They are given in the following table.

n	1	2	3	4	5	6	7	8
$P(n)$	16	140	896	4680	21024	83628	300080	984539

Using the expression (10) of P one straightforwardly deduces the coefficients α_k by increasing induction on k : $\alpha_k = 0$ for $k = 0, 1, \dots, 6$ and the values α_k for $k = 7, \dots, 15$ are given in the following table.

k	7	8	9	10	11	12	13	14	15
α_k	1	8	32	96	214	328	324	184	46

Hence $\deg \theta = \alpha_{15} - 16 = 30$.

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