HAL
open science

# Logarithmic corrections and universal amplitude ratios in the 4-state Potts model

Bertrand Berche, Paolo Butera, Lev Shchur

## To cite this version:

Bertrand Berche, Paolo Butera, Lev Shchur. Logarithmic corrections and universal amplitude ratios in the 4-state Potts model. 2007. hal-00164633

**HAL Id: hal-00164633**

**https://hal.science/hal-00164633**

Preprint submitted on 22 Jul 2007

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Logarithmic corrections and universal amplitude ratios in the 4-state Potts model

B. Berche[1], P. Butera[2], L.N. Shchur[3]

[1] Laboratoire de Physique des Matériaux, Université Henri Poincaré, Nancy I, BP 239, F-54506 Vandœuvre les Nancy Cedex, France
berche@lpm.u-nancy.fr

[2] Istituto Nazionale di Fisica Nucleare, Universitá Milano-Bicocca, Piazza delle Scienze 3, 20126, Milano, Italia
paolo.butera@mib.infn.it

[3] Landau Institute for Theoretical Physics, Russian Academy of Sciences, Chernogolovka 142432, Russia
lev@landau.ac.ru

**Summary.** Monte Carlo and series expansion data for the energy, specific heat, magnetisation and susceptibility of the 4-state Potts model in the vicinity of the critical point are analysed. The role of logarithmic corrections is discussed. Estimates of universal ratios $A_+/A_-$, $\Gamma_+/\Gamma_L$, $\Gamma_T/\Gamma_L$ and $R_c^+$ are given.

July 23, 2007

hal-00164633, version 1 - 22 Jul 2007

## 1 Introduction

The study of critical phenomena and phase transitions is a traditional subject of statistical physics which has known its "modern age", since powerful approaches have been developed (renormalization group, conformal invariance, sophisticated simulation algorithms, ...). Simplified models attracted a lot of attention. This is essentially due to a spectacular property of continuous phase transitions at their critical point, *scale invariance*, which leads to an extreme robustness of some quantities, like the critical exponents which are thus referred to as *universal quantities*. Only very general properties (space or spin dimension, symmetry, range of interaction, ...) determine the universality class. This makes the theory of critical phenomena a very efficient and predictive tool: As soon as one knows the general characteristics of a physical system from general symmetry arguments, it is possible in principle to predict exactly the "shape" of the singularities which are developed at the critical point. The term "exact" is here understood rigorously, for example a two-dimensional system with the symmetries of an Ising model, should it be a magnet, an alloy or anything else, will exhibit a diverging susceptibility $\chi \sim |T - T_c|^{-7/4}$ (see a sketch in Fig. 1) with the precise value 7/4 for the exponent.

On a theoretical ground, the major two-dimensional problems (Ising model and its generalizations like the Potts model and the percolation model,

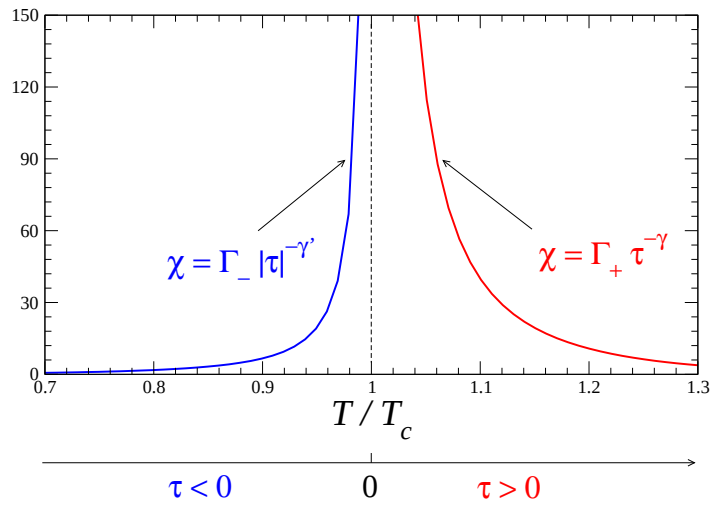


**Fig. 1.** Typical behaviour of the susceptibility at a second order phase transition. The quantities $\gamma = \gamma'$ and $\Gamma_+/\Gamma_-$ are universal.

$XY$ model, Heisenberg model and so on) are essentially solved at least for their critical singularities (when a second-order phase transition is indeed present), but critical exponents are not the only universal quantities at a critical point. The universal character of appropriate combinations of critical amplitudes [1] is also an important prediction of scaling theory but in some cases these combinations remain uncompletely determined and subject to controversies.

The Potts model [2, 3], as one of the paradigmatic models exhibiting continuous phase transitions is a good frame to consider the question of universal combinations of amplitudes. The universality class of the Potts model at its critical point is parametrized by the number of states $q$. The two-dimensional Potts model with three and four states can be experimentally realized as strongly chemisorbed atomic adsorbates on metallic surfaces at sub-monolayer concentrations [4]. Although critical exponents could be measured quite accurately for adsorbed sub-monolayers, confirming that these systems actually belong to the three-state [5] or to the four-state Potts model classes [6], it is unlikely that the low temperature LEED results can be pushed [7] to determine also the critical amplitudes. Therefore, the numerical analysis of these models is the only available tool to check analytic predictions.

The critical amplitudes and critical exponents describe the behaviour of the magnetization $m$, the susceptibility $\chi$, the specific heat $C$ and the correlation length $\xi$ for a spin system in zero external field[4] in the vicinity of the critical point

[4] In this paper we only deal with the physical properties in zero magnetic field.

$$M(\tau) \approx B_-(-\tau)^\beta, \ \tau < 0, \tag{1}$$

$$\chi(\tau) \approx \Gamma_\pm |\tau|^{-\gamma}, \tag{2}$$

$$\chi_T(\tau) \approx \Gamma_T(-\tau)^{-\gamma}, \ \tau < 0, \tag{3}$$

$$C(\tau) \approx \frac{A_\pm}{\alpha}|\tau|^{-\alpha}, \tag{4}$$

$$\xi(\tau) \approx \xi_0^\pm |\tau|^{-\nu}. \tag{5}$$

Here $\tau$ is the reduced temperature $\tau = (T - T_c)/T$ and the labels $\pm$ refer to the high-temperature and low-temperature sides of the critical temperature $T_c$. For the Potts models with $q > 2$ a transverse susceptibility $\chi_T$ can be defined in the low-temperature phase[5].

Critical exponents are known exactly for 2D Potts model [8–11] through the relation $x_\epsilon = (1-\alpha)/\nu$ to the thermal scaling dimension

$$x_\epsilon = \frac{1+y}{2-y} \tag{6}$$

and the relation $x_\sigma = \beta/\nu$ to the magnetic scaling dimension

$$x_\sigma = \frac{1-y^2}{4(2-y)}, \tag{7}$$

where the parameter $y$ is related to the number of states $q$ of the Potts variable by the expression

$$\cos\frac{\pi y}{2} = \frac{1}{2}\sqrt{q} \tag{8}$$

The central charge of the corresponding conformal field theory is also simply expressed [9] in terms of $y$

$$c = 1 - \frac{3y^2}{2-y}. \tag{9}$$

Analytical estimates of critical amplitude ratios for the $q$-state Potts models with $q = 1, 2, 3$, and 4 were recently obtained by Delfino and Cardy [12]. They used the two-dimensional scattering field theory of Chim and Zamolodchikov [13] and estimated the central charge $c = 0.985$ for 4-state Potts model, for which the exactly known value is $c = 1$. Reporting these approximate values in (9), one can calculate the scaling dimensions from (6)-(7) and get the values $x_\sigma = 0.13016$ and $x_\epsilon = 0.577$, to be compared respectively to the exact values 1/8 and 1/2. The discrepancy is around 4 and 15 per cent, emphasizing the difficulty of the $q = 4$ case (to give an idea, in the case of the 3-state Potts model, a similar analysis leads to a very good agreement with less than one percent deviation).

The universal susceptibility amplitude ratios $\Gamma_+/\Gamma_L$ and $\Gamma_T/\Gamma_L$ were also calculated in [12] and [14]. The figures obtained are the following,

[5] In the following we will use equally the notations $\Gamma_L$ or $\Gamma_-$ for the longitudinal susceptibility amplitude in the low temperature phase.

$$q = 3 : \Gamma_+/\Gamma_L = 13.848, \Gamma_T/\Gamma_L = 0.327, \tag{10}$$

$$q = 4 : \ \Gamma_+/\Gamma_L = 4.013, \ \Gamma_T/\Gamma_L = 0.129. \tag{11}$$

These results have been confirmed numerically in the case $q = 3$ by several groups, $\Gamma_+/\Gamma_L \approx 10$ and $\Gamma_T/\Gamma_L \approx 0.333(7)$ in Ref. [14] (Monte Carlo (MC) simulations), $\Gamma_+/\Gamma_L = 14\pm1$ in Ref. [15] (MC and series expansion (SE) data) and quite recently, these results were confirmed and substantially improved, $\Gamma_+/\Gamma_L = 13.83(9)$, $\Gamma_T/\Gamma_L = 0.325(2)$, by Enting and Guttmann [16] who analysed new longer series expansions.

The 4-state Potts model was also studied through MC simulations in Ref. [14], but the authors considered that their data were not conclusive. Another MC contribution is reported by Caselle, et al [17], $\Gamma_+/\Gamma_L = 3.14(70)$, and Enting and Guttmann [16] also analysed SE data for the 4-state Potts model and found $\Gamma_+/\Gamma_L = 3.5(4)$, $\Gamma_T/\Gamma_L = 0.11(4)$ in relatively good agreement with the predictions of [12] and [14]. The situation thus seems to be clear, although the use of the logarithmic corrections in the fitting procedure of MC data was questioned, e.g. in [16]: *[Caselle et al] estimates depend critically on the assumed form of the sub-dominant terms, and on the further assumption that the other sub-dominant terms, which include powers of logarithms, powers of logarithms of logarithms etc, can all be neglected. We doubt that this is true.*

Let us recall that the existence of logarithmic corrections to scaling in the 4-state Potts model was pointed out in the pioneering works of Cardy, Nauenberg and Scalapino [18, 19], where a set of non-linear RG equations were proposed. Their discussion was later extended by Salas and Sokal [20].

Generically, the logarithmic corrections appear as corrections to scaling. We mentioned above that in the vicinity of a critical point, a susceptibility for example diverges like $\chi(\tau) \approx \Gamma_\pm|\tau|^{-\gamma}$. This is true, but this singular behaviour can be superimposed to a regular signal (e.g. $D_0+D_1|\tau|+\ldots$), and the leading singular behaviour itself needs to be corrected when we consider the physical quantity away from the transition temperature. The expression for $\chi(\tau)$ then takes a form which can become "terrific":

$$\begin{array}{lll}
\chi(\tau) = D_0 + D_1|\tau| + \ldots & & \text{regular background} \\
\quad + \quad \Gamma|\tau|^{-\gamma}(1+ & & \text{leading singularity} \\
 & +a^{(1)}|\tau|^{\Delta} + a^{(2)}|\tau|^{2\Delta} + \ldots & \text{leading corrections} \\
 & +a'^{(1)}|\tau|^{\Delta'} + a'^{(2)}|\tau|^{2\Delta'} + \ldots & \text{next corrections} \\
 & +b^{(1)}|\tau| + b^{(2)}|\tau|^2 + \ldots) & \text{analytic corrections} \\
 & \times(-\ln|\tau|)^{\star} \times \left(1 + \triangleleft\frac{\ln(-\ln|\tau|)}{-\ln|\tau|}\right) \times \ldots & \text{logarithmic corrections}
\end{array}$$

Together with the amplitude and exponent associated to the leading singularity, $\Gamma$ and $\gamma$, appear corrections to scaling due to the presence of irrelevant scaling fields ($a^{(n)}$ and $\Delta$, $a'^{(n)}$ and $\Delta'$, ...), analytic corrections due to non-linearities of the relevant scaling fields ($b^{(n)}$), or multiplicative logarithmic

corrections ($\triangleleft$ and $\star$, ...). These logarithmic coefficients may have different origins (see e.g. in Ref. [1] and references therein). They can be due to the upper critical dimension, to poles in the expansion of regular and singular amplitudes, or to the presence of marginal scaling fields. The 4-state Potts model belongs to this latter category.

Some of the quantities indicated above are universal. This is the case of the exponents as well as of many combinations of coefficients. In the present paper we are interested in the amplitude of the leading singular term, but its precise determination can be affected by the form of the corrections. We shall be concerned with the following universal combinations of critical amplitudes

$$\frac{A_+}{A_-}, \quad \frac{\Gamma_+}{\Gamma_L}, \quad \frac{\Gamma_T}{\Gamma_L}, \quad R_C^+ = \frac{A_+\Gamma_+}{B_-^2}. \tag{12}$$

We present, for the 4-state Potts model, more accurate Monte Carlo data supplemented by a reanalysis of the extended series made available by Enting and Guttmann [16] and we address the following question: Is it possible to devise some procedure in which the role of these logarithmic corrections is properly taken into account?

## 2 Amplitudes and universal combinations

The scaling hypothesis states that the singular part of the free energy density can be written in terms of the deviation from the critical point, $\tau = (T-T_c)/T$ and $h = H - H_c$,

$$f_{\rm sing}(\tau, h) = b^{-D} F_\pm(\kappa_\tau b^{y_\tau}\tau, \kappa_h b^{y_h} h) \tag{13}$$

where $F_\pm(x, y)$ is a universal function (actually there is one universal function for each side $\tau > 0$ or $\tau < 0$ of the critical point) and $\kappa_\tau$ and $\kappa_h$ are "metric factors" which contain all the non universal aspects of the critical behaviour. $D$ is the space dimension. Let us stress that the functions $F_\pm$ are universal in the sense that some details of the model are irrelevant (e.g. the coordination number of the lattice (so long as it remains finite), the presence of next nearest neighbour interactions, etc) but they depend on the boundary conditions or the shape of the system. The metric factors on the other hand depend on these details, and the universal combinations are obtained when the metric factors are eliminated from some combinations.

The connection with scaling relations can be shown with an example. From Eq. (13), we also deduce similar homogeneous expressions for the magnetization, $M(\tau, h) = b^{-D+y_h}\kappa_h M_\pm(x, y)$ and the susceptibility, $\chi(\tau, h) = b^{-D+2y_h}\kappa_h^2 X_\pm(x, y)$. The choice $b = (\kappa_\tau|\tau|)^{-1/y_\tau}$ and $h = 0$ leads (for example below the transition temperature) for the following combination of quantities

$$\alpha \frac{C(\tau,0)\chi(\tau,0)}{m^2(\tau,0)}|\tau|^2 \equiv (\kappa_\tau|\tau|)^{2-\alpha-2\beta-\gamma}\alpha\frac{C_-(1,0)X_-(1,0)}{M_-^2(1,0)} \equiv R. \qquad (14)$$

The prefactor takes the value 1 thanks to the well known scaling relation between critical exponents $\alpha + 2\beta + \gamma = 2$. Thus it follows that the above combination is a *universal number*. From the definition of magnetization, specific heat and susceptibility amplitudes in zero magnetic field, e.g. $M(\tau,0) = |\tau|^{\frac{D-y_h}{y_\tau}}\kappa_h M_-(1,0) \equiv B_-|\tau|^\beta$ by virtue of Eq. (1), this universal number is in fact a combination of amplitudes, $R \equiv A_-\Gamma_-/B_-^2$. We have similar universal combinations above the critical temperature or associated to other scaling relations.

## 3 RG approach for the Potts model and logarithmic corrections at $q = 4$

Let us remind that the $q$-state Potts model is an extension of the usual lattice Ising model in which the site variables $s_i$ (abusively called spins) can have $q$ different values, $s_i = 0, 1, \ldots\ q-1$ but the nearest neighbour interaction energy $-J\delta_{s_i,s_j}$ only takes two possible values, e.g. $-J$ and 0 depending whether the neighbouring spins are in the same state or not. The Hamiltonian of the model reads as

$$H = -J\sum_{\langle ij\rangle}\delta_{s_i s_j}\ . \qquad (15)$$

At the early times of real-space renormalization, the application to the pure Potts model led to some difficulties: the impossibility to affect a particular value for the spin of a cell after decimation due to a too large number of states (see Fig. 2).

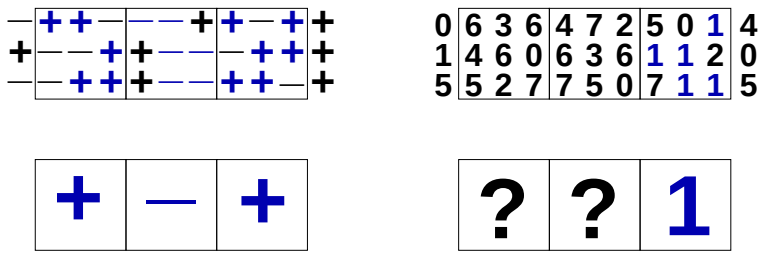


**Fig. 2.** Decimation of spin blocks for the Ising model (left) and high-$q$ Potts model (right). In the latter case, the state of many cells cannot be decided by a simple majority rule.

It was thus necessary to extend the parameter space, introducing new variables, called vacancies, to replace the question marks in Fig. (2) and

to study an annealed disordered model. The RG equations satisfied by the model, written in terms of the relevant thermal and magnetic fields $\tau$ and $h$, with corresponding RG eigenvalues $y_\tau$ and $y_h$, and the marginal dilution field $\psi$, are given by $\frac{d\tau}{d\ln b} = y_\tau \tau$, $\frac{dh}{d\ln b} = y_h h$, $\frac{d\psi}{d\ln b} = q - q_c$,where $b$ is the length rescaling factor and $l = \ln b$. When $q > q_c$, the dilution field $\psi$ is relevant (and the phase transition is of first order), while in the regime $q < q_c$, $\psi$ is irrelevant and the system exhibits a second-order phase transition. The case $q = q_c$ is marginal. This picture is qualitatively correct, and in fact the critical value of the number of states which discriminates between the two regimes is $q_c = 4$. In the $q$ direction, $q_c = 4$ appears as the end of a line of fixed points where logarithmic corrections are expected. At $q_c = 4$, the RG equations were extended by Cardy, Nauenberg and Scalapino (CNS) [18, 19] and then by Salas and Sokal (SS) [20]. As a result of the coupling between the dilution field $\psi$, and $\tau$ and $h$, they were led to non-linear equations,

$$\frac{d\tau}{d\ln b} = (y_\tau + y_{\tau\psi}\psi)\tau, \tag{16}$$

$$\frac{dh}{d\ln b} = (y_h + y_{h\psi}\psi)h, \tag{17}$$

$$\frac{d\psi}{d\ln b} = g(\psi). \tag{18}$$

The function $g(\psi)$ may be Taylor expanded, $g(\psi) = y_{\psi^2}\psi^2(1 + \frac{y_{\psi^3}}{y_{\psi^2}}\psi + \ldots)$. Accounting for marginality of the dilution field, there is no linear term at $q = 4$. Comparing to the available results (for example the expression of the latent heat for $q \geq q_c$ by Baxter [21] or the den Nijs and Pearson's conjectures for the RG eigenvalues for $q \leq q_c$ [10, 11]), the parameters were found to take the values $y_{\tau\psi} = 3/(4\pi)$, $y_{h\psi} = 1/(16\pi)$, $y_{\psi^2} = 1/\pi$ and $y_{\psi^3} = -1/(2\pi^2)$, while the relevant scaling dimensions are $y_\tau = \nu^{-1} = 3/2$ and $y_h = 15/8$.

The fixed point is at $\tau = h = 0$. Starting from initial conditions $\tau$, $h$, the relevant fields grow exponentially with $l$ up to some $\tau = O(1)$, $h = O(1)$ outside the critical region. Notice also that the marginal field $\psi$ remains of order of its initial value, $\psi \sim O(\psi_0)$. In zero magnetic field, under a change of length scale, the singular part of the free energy density transforms according to

$$f(\psi_0, \tau) = e^{-Dl} f(\psi, 1). \tag{19}$$

Solving Eqs. (16-18) leads to

$$l = -\frac{1}{y_\tau}\ln\tau + \frac{y_{\tau\psi}}{y_\tau y_{\psi^2}}\ln\left(\frac{\psi_0}{\psi}G(\psi_0, \psi)\right), \tag{20}$$

(for brevity we will denote $\nu = 1/y_\tau = \frac{2}{3}$, $\mu = \frac{y_{\tau\psi}}{y_\tau y_{\psi^2}} = \frac{1}{2}$). Note that $G(\psi_0, \psi)$ would take the value 1 in Ref. [19] and the value $\frac{y_{\psi^2} + y_{\psi^3}\psi}{y_{\psi^2} + y_{\psi^3}\psi_0}$ in Ref. [20]. We can thus deduce the following behaviour for the free energy density in zero magnetic field in terms of the thermal and dilution fields,

$$f(\tau,\psi_0)=\tau^{D\nu}\left(\frac{\psi}{\psi_0}\frac{y_{\psi^2}+y_{\psi^3}\psi_0}{y_{\psi^2}+y_{\psi^3}\psi}\right)^{D\mu}f(1,\psi). \tag{21}$$

A similar expression would be obtained if the magnetic field $h$ were also included. The other thermodynamic properties follow from derivatives with respect to the scaling fields. The quantity between parentheses is the only one where the log terms are hidden in the 4-state Potts model, and thus we may infer that *not only the leading log terms*, but *all the log terms hidden in the dependence on the marginal dilution field* disappear in the conveniently defined effective ratios[6]. Now we proceed by iterations of Eq. (20), and eventually we get for the full correction to scaling variable the *heavy* expression

$$\frac{\psi_0}{\psi}G(\psi_0,\psi)=\text{const}\times\underbrace{(-\ln|\tau|)}_{\text{CNS}}\underbrace{\left(1+\frac{3}{4}\frac{\ln(-\ln|\tau|)}{-\ln|\tau|}\right)\left(1-\frac{3}{4}\frac{\ln(-\ln|\tau|)}{-\ln|\tau|}\right)^{-1}}_{1+\frac{3}{2}\frac{\ln(-\ln|\tau|)}{-\ln|\tau|}\text{ in SS}}$$
$$\times\left(1+\frac{3}{4}\frac{1}{(-\ln|\tau|)}\right)\times\underbrace{\left(1+\frac{\text{const}}{-\ln|\tau|}+O\left(\frac{1}{-\ln|\tau|^2}\right)\right)}_{F(-\ln|\tau|)} \tag{22}$$

where CNS and SS refer to the results previously obtained in the literature [18–20] and $F(-\ln|\tau|)$ is the only factor where *non universality* enters through the dilution field $\psi_0$. This allows to write down the behaviour of the magnetization for example

$$M(\tau)=B_-|\tau|^{1/12}(-\ln|\tau|)^{-1/8}\left[\left(1+\frac{3}{4}\frac{\ln(-\ln|\tau|)}{-\ln|\tau|}\right)\right.$$
$$\left.\left(1-\frac{3}{4}\frac{\ln(-\ln|\tau|)}{-\ln|\tau|}\right)^{-1}\left(1+\frac{3}{4}\frac{1}{-\ln|\tau|}\right)F(-\ln|\tau|)\right]^{-1/8} \tag{23}$$

## 4 Numerical techniques

In the Monte Carlo simulations we use the Wolff algorithm [22] for studying square lattices of linear size $L$ (between $L=20$ and $L=200$) with periodic boundary conditions. Starting from an ordered state, we let the system equilibrate in $10^5$ steps measured by the number of flipped Wolff clusters. The averages are computed over $10^6$—$10^7$ steps. The random numbers are produced by an exclusive-XOR combination of two shift-register generators with the taps (9689,471) and (4423,1393), which are known [23] to be safe for the Wolff algorithm.

The order parameter of a microstate `M(t)` is evaluated during the simulations as

[6] i.e. effective ratios which eventually tend towards universal limits when $|\tau|\to 0$

$$\mathtt{M} = \frac{qN_m/N - 1}{q-1}, \tag{24}$$

where $N_m$ is the number of sites $i$ with $s_i = m$ at the time $\mathtt{t}$ of the simulation and $m \in [0, 1, ..., (q-1)]$ is the spin value of the majority of the sites. $N = L^2$ is the total number of spins. The thermal average is denoted $M = \langle \mathtt{M} \rangle$. Thus, the longitudinal susceptibility in the low-temperature phase is measured by the fluctuation of the majority of the spins

$$\chi_L = \beta(\langle N_m^2 \rangle - \langle N_m \rangle^2) \tag{25}$$

and the transverse susceptibility is defined in the low-temperature phase as the fluctuations of the minority of the spins

$$\chi_T = \frac{\beta}{(q-1)} \sum_{\mu \neq m} (\langle N_\mu^2 \rangle - \langle N_\mu \rangle^2), \tag{26}$$

while in the high-temperature phase $\chi_+$ is given by the fluctuations in all $q$ states,

$$\chi_+ = \frac{\beta}{q} \sum_{\mu=0}^{q-1} (\langle N_\mu^2 \rangle - \langle N_\mu \rangle^2), \tag{27}$$

where $N_\mu$ is the number of sites with the spin in the state $\mu$. The internal energy density of a microstate is calculated as

$$\mathtt{E} = -\frac{1}{N} \sum_{\langle ij \rangle} \delta_{s_i s_j} \tag{28}$$

its ensemble average denoted as $E = \langle \mathtt{E} \rangle$ and the specific heat per spin measures the energy fluctuations,

$$C = -\beta^2 \frac{\partial E}{\partial \beta} = \beta^2 \left( \langle \mathtt{E}^2 \rangle - \langle \mathtt{E} \rangle^2 \right). \tag{29}$$

Our MC study of the critical amplitudes is supplemented by an analysis of the high-temperature (HT) and low-temperature (LT) expansions for $q = 4$ recently calculated through remarkably high orders by Enting, Guttmann and coworkers [16, 24]. In terms of these series, we can compute the effective critical amplitudes for the susceptibilities and the magnetization and extrapolate them by the current resummation techniques, namely simple Padé approximants (PA) and differential approximants (DA) properly biased with the exactly known critical temperatures and critical exponents. The LT expansions, expressed in terms of the variable $z = \exp(-\beta)$, extend through $z^{59}$ for the longitudinal susceptibility and through $z^{47}$ in the case of the transverse susceptibility. The magnetization and energy expansions extend through $z^{43}$. The HT expansion is computed in terms of the variable $v = (1-z)/(1+(q-1)z)$. The susceptibility expansion has been computed up to $v^{24}$ and the energy expansion up to $v^{43}$.

## 5 Analysis of the magnetization behaviour

For the sake of simplification of the notations, we group all the terms containing logs in Eq. (22) into a single function $H(-\ln|\tau|) = E(-\ln|\tau|) \times F(-\ln|\tau|)$ where

$$E(-\ln|\tau|) = \left[(-\ln|\tau|)\left(1+\frac{3}{4}\frac{\ln(-\ln|\tau|)}{-\ln|\tau|}\right)\right.$$
$$\left.\times\left(1-\frac{3}{4}\frac{\ln(-\ln|\tau|)}{-\ln|\tau|}\right)^{-1}\left(1+\frac{3}{4}\frac{1}{-\ln|\tau|}\right)\right]. \quad (30)$$

The function $E$ contains all leading logarithms with universal coefficients, it is known exactly while the function $F$ needs to be fitted. We thus obtain a closed expression for the dominant logarithmic corrections which is more suitable than previously proposed forms to describe an observable (Obs.) in the temperature range accessible in a numerical study:

$$\text{Obs.}(\tau) \simeq \text{Ampl.} \times |\tau|^{\bullet} \times H^{\sharp}(-\ln|\tau|) \times (1+\text{Corr. terms}), \quad (31)$$
$$\text{Corr. terms} = a|\tau|^{2/3} + b_{\pm}|\tau| + \ldots, \quad (32)$$

where $\bullet$ and $\sharp$ are exponents which depend on the observable considered, and take the values $1/12$ and $-1/8$ respectively in the case of the magnetization. Here we stress that the inclusion of a correction in $|\tau|^{2/3}$ seems to be necessary according to previous work of Joyce on the Baxter-Wu model [25, 26] (of 4-state Potts model universality class), where the magnetization is shown to obey an expression of the form $M(\tau) = B_{-}|\tau|^{1/12}(1 + \text{const} \times |\tau|^{2/3} + \text{const}' \times |\tau|^{4/3})$. The exponent $2/3$ comes out from the conformal scaling dimensions of Dotsenko and Fateev [9], and its presence is needed in order to account for the numerical results (see also Ref. [27]). Caselle et al. [17] also considered $|\tau|^{2/3}$ term to fit the magnetization. Here we also allow inclusion of a linear correction in $b_{\pm}|\tau|$ to account for possible non linearities of the relevant scaling fields [1]. The next term in $|\tau|^{4/3}$ will be forgotten.

In Fig. 3 we plot effective magnetization amplitudes $B_{eff}(\tau)$ vs $|\tau|^{2/3}$. From the available data for the magnetization, we define the following quantities,

$$B_{CNS}(\tau) = M(\tau)|\tau|^{-1/12}(-\ln|\tau|)^{1/8}, \quad (33)$$
$$B_{SS}(\tau) = M(\tau)|\tau|^{-1/12}(-\ln|\tau|)^{1/8}\left(1-\frac{3}{16}\frac{\ln(-\ln|\tau|)}{-\ln|\tau|}\right)^{-1}, \quad (34)$$
$$B_{E}(\tau) = M(\tau)|\tau|^{-1/12}E^{1/8}(-\ln|\tau|), \quad (35)$$

which are expected to behave according to the corrections to scaling

$$B_{-}(1 + a|\tau|^{2/3} + b_{\pm}|\tau| + \ldots) \quad (36)$$

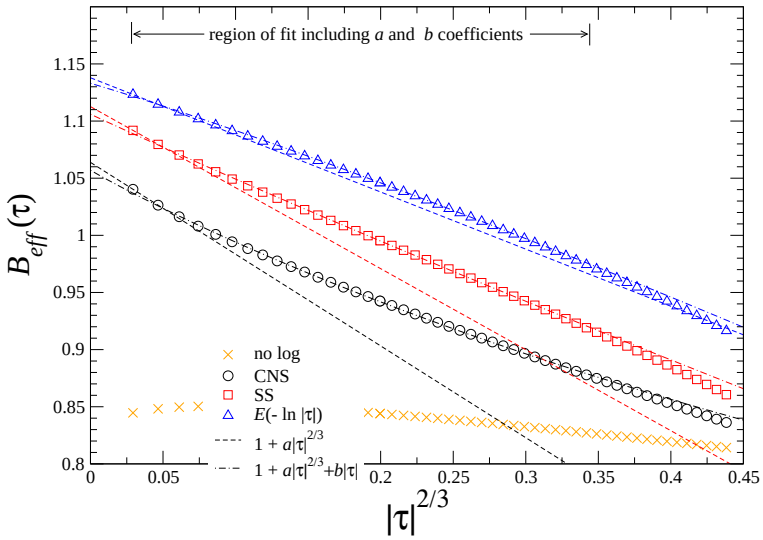


**Fig. 3.** Effective amplitudes (from MC data) as deduced from different assumptions for the logarithmic corrections. Symbols correspond to MC data, dashed and dotted-dashed lines are fits as explained in the text.

The numerical results are shown in Fig. 3. From bottom to top the various symbols indicate the effective amplitudes with "no-log" at all, then with the CNS and the SS corrections and finally the effective amplitude where the known universal logarithmic terms have been included. The dashed lines correspond to a rough determination of the corrections to scaling including only the terms in $a|\tau|^{2/3}$ in the limit $|\tau| \to 0$, and the dot-dashed lines include also the terms in $b|\tau|$. From this plot, we deduce that *none of the three effective amplitudes in Eqs. (33-35) can be correctly fitted by Eq. (36)*, since the coefficients of the correction terms (e.g. the coefficient $a$ which is estimated directly by the slope at small $|\tau|$ values) strongly depend on the range of fit. This undesirable dependence of the coefficients on the width of the temperature window is shown in the first six lines of table 1.

In order to improve the quality of the fits, one has to take into account the correction function $F(-\ln|\tau|)$ and to extract an effective function $F_{eff}(-\ln|\tau|)$ which mimics the real one in the convenient temperature range. This is done by fitting $B_{eff}(\tau)$ to a more complicated expression,

$$B_-(1 + a|\tau|^{2/3} + b_\pm|\tau| + \ldots) \times \left(1 + \frac{C_1}{-\ln|\tau|} + \frac{C_2 \ln(-\ln|\tau|)}{(-\ln|\tau|)^2}\right)^{1/8}, \quad (37)$$

which means that we include the corrections to scaling *and* the non universal function function $F(-\ln|\tau|)$ taking the approximate expression

$$F_{eff}(-\ln|\tau|) \simeq \left(1 + \frac{C_1}{-\ln|\tau|} + \frac{C_2 \ln(-\ln|\tau|)}{(-\ln|\tau|)^2}\right)^{-1}. \quad (38)$$

**Table 1.** Fits of the effective amplitude of the magnetization.

| $B_{eff}(\tau)$ | $\lvert\tau\rvert^{2/3}$-window | $B_-$ | $a$ | $b$ |
|---|---|---|---|---|
| $B_{CNS}(\tau)$ | $[0, 0.15]$ | 1.07 | $-0.98$ | 0.94 |
| | $[0, 0.45]$ | 1.05 | $-0.66$ | 0.29 |
| $B_{SS}(\tau)$ | $[0, 0.15]$ | 1.11 | $-0.77$ | 0.56 |
| | $[0, 0.45]$ | 1.10 | $-0.45$ | $-0.06$ |
| $B_E(\tau)$ | $[0, 0.15]$ | 1.14 | $-0.47$ | 0.16 |
| | $[0, 0.45]$ | 1.13 | $-0.25$ | $-0.27$ |
| $B_{EF}(\tau)$ | $[0, 0.15]$ | 1.16 | $-0.20$ | 0.02 |
| | $[0, 0.45]$ | 1.16 | $-0.18$ | $-0.02$ |

While $a$ and $b$ are coefficients of corrections to scaling due to irrelevant operators, $C_1$ and $C_2$ are *effective* coefficients of logarithmic terms which, in a given temperature range, mimic a slowly convergent series of logarithmic terms depending on a non universal dilution field. Therefore, we expect that different fits made in different temperature windows will produce different values of $C_1$ and $C_2$ while $a$ and $b$ (and of course also the magnetization amplitude $B_-$) should be relatively less influenced by the window range. The choice of values for $C_1$ and $C_2$ is thus partially arbitrary and the values quoted should be specified together with the temperature window where they are appropriate. In the following, we obtain $C_1 \simeq -0.76$ and $C_2 \simeq -0.52$ in the window $|\tau|^{2/3} \in [0, 0.35]$, which yields an amplitude $B_- \simeq 1.157$. The resulting $a$ and $b$ coefficients now appear very stable. This is checked in Fig. 4 where the quantity

$$B_{EF}(\tau) = M(\tau)|\tau|^{-1/12}[E(-\ln|\tau|)F(-\ln|\tau|)]^{1/8} \tag{39}$$

is reported toghether with the previous curves and fitted as indicated in table 1. As an independent test, we add the SE data which are superimposed to the MC data at small values of $|\tau|$ only for this latter assumption of effective amplitude.

Eventually, our approach confirms expression Eq. (23) for the magnetization, with the function $F(-\ln|\tau|)$ given in Eq. (38) and the parameters $C_1$ and $C_2$ given above for the appropriate temperature window. Nevertheless, we have to stress that the different effective amplitudes should all reach *the same amplitude* $B_-$ *in the limit* $|\tau| \to 0$, since $B_{CNS}(\tau)$ is in fact an approximation of $B_{SS}(\tau)$, which is an approximation of $B_E(\tau)$, which eventually approximates $B_{EF}(\tau)$. An attempt of illustration of this behaviour is shown in Fig. 5 where dotted lines (which are only guides for the eyes) all converge towards the unique value $B_- \simeq 1.157$. Here we stress that we have deleted the points of SE data which are too close to the critical point, since the series are no longer reliable because the current extrapolation procedures are in principle unable to approximate the complicated structure of the singularity involving log corrections.

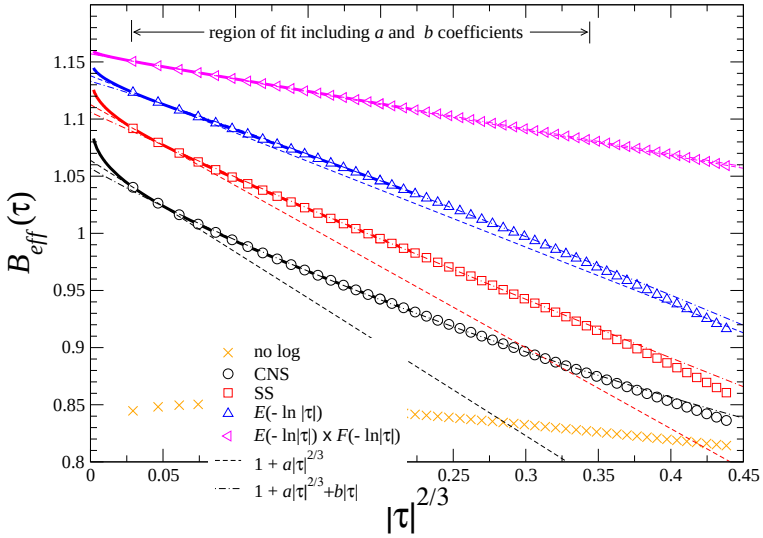


**Fig. 4.** Effective amplitudes (from MC and SE data) as deduced from different assumptions for the logarithmic corrections. The upper curves correspond to $B_{EF}(\tau)$ and the thick solid lines to SE data.

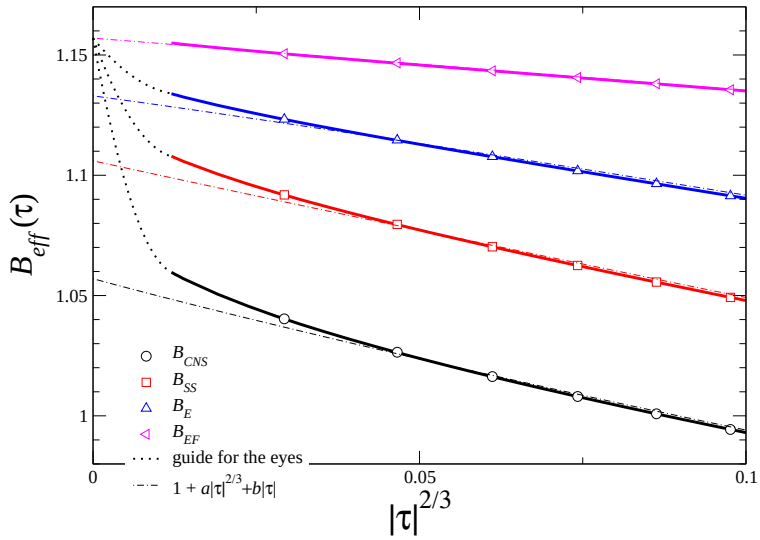


**Fig. 5.** Zoom of the effective amplitudes in the vicinity of the critical point. MC data (symbols) and SE data (thick solid lines).

## 6 Universal combinations for the 4-state Potts model and conclusions

The other quantities can be analyzed along the same lines. The important point is that now the function $F_{eff}(\tau)$ is fixed in the corresponding tempera-

ture scale and thus the remaining freedom for the other physical quantities in Eq. (31) is only through the leading amplitude and the coefficients of $|\tau|^{2/3}$ and $|\tau|$-terms in the corrections plus possibly the background terms. It is still a complicated task to perform this analysis, but the current results for the universal combinations mentioned in the introduction appear in the following table.

**Table 2.** Rough estimate of the universal combinations of the critical amplitudes in the 4-state Potts model.

| $A_+/A_-$ | $\Gamma_+/\Gamma_L$ | $\Gamma_T/\Gamma_L$ | $R_C^+$ | source |
|---|---|---|---|---|
| 1.[a] | 4.013 | 0.129 | 0.0204 | [12, 14] |
| | 3.14(70) | | 0.021(5) | [17] |
| | 3.5(4) | 0.11(4) | | [16] |
| 1.00(1) | 6.7(4) | 0.161(3) | 0.0307(2) | here |

[a] exact result from duality

Our work is "one more" contribution to the study of this problem and brings some answers, but also raises new questions. Indeed, our results disagree with previous estimates, but we cannot claim for sure that our estimates are more reliable than those of other authors. What is extremely clear is that the groups who studied numerically universal combinations of amplitudes in the 4−state Potts model all noticed the extreme difficulty to take into account properly the logarithmic terms. We believe that our protocol is self-consistent in the sense that our criterion is to obtain a relative stability of the correction to scaling coefficients. The results that we report here, although a rough estimate which calls for deeper analysis, reach a reasonable confidence level. If this is indeed the case, one should identify the reason of the discrepancy from the theoretical predictions of Cardy and Delfino. In the conclusion, and in a footnote of one of their papers, Delfino et al [14] (p.533) explain that their results are sensitive to the relative normalization of the order and disorder operator form factors which could be the origin of some troubles for the ratios $\Gamma_+/\Gamma_L$ and $R_C$. This possible explanation seems nevertheless to be ruled out (as mentioned by Enting and Guttmann already) by the very good agreement between the theoretical predictions and all numerical studies (both MC and SE) in the case of the 3-state Potts model. Eventually let us mention that the two-kink approximation used by Cardy and Delfino is exact for $q = 2$ (Ising model) and quite good for $q = 3$, but probably questionable close to the marginal case $q \to 4$. As a conclusion, we are afraid that this work opens more questions than it bring answers.

## Acknowledgements

Discussions with A. Zamolodchikov, V. Plechko, W. Janke and M. Henkel, and a correspondence with J. Cardy and J. Salas were very helpful.

LNS is grateful to the Statistical Physics group of the University Henri Poincaré Nancy 1 and to the Theoretical group of the University Milano–Bicocca for the kind hospitality. Financial support from the twin research program between the Landau Institute and the Ecole Normale Supérieure de Paris and Russian Foundation for Basic Research are also acknowledged.

## References

1. V. Privman, P.C. Hohenberg, A. Aharony: Universal critical-point amplitude relations. In: *Phase Transitions and Critical Phenomena*, vol. 14, ed. by C. Domb, J.L. Lebowitz (Academic, New York 1991) pp 1–134
2. R.B. Potts: Proc. Camb. Phil. Soc. **48**, 106 (1952)
3. F.Y. Wu: Rev. Mod. Phys. **54**, 235 (1982)
4. M. Sokolowski and H. Pfnür: Phys. Rev. Lett. **49**, 7716 (1994)
5. Y. Nakajima, C. Voges, T. Nagao, S. Hasegawa, G. Klos and H. Pfnür: Phys. Rev. B **55**, 8129 (1997)
6. C. Voges and H. Pfnür: Phys. Rev. B **57**, 3345 (1998)
7. H. Pfnür: private communication
8. B. Nienhuis: J. Stat. Phys. **34**, 731 (1984); B. Nienhuis: Two-dimensional critical phenomena and the Coulomb gas. In: *Phase Transitions and Critical Phenomena*, vol. 11, ed. by C. Domb, J.L. Lebowitz (Academic, New York 1987)
9. Vl.S. Dotsenko and V.A. Fateev: Nucl. Phys. B **240** [FS12], 312 (1984)
10. M.P.M. den Nijs: J. Phys. A **12**, 1857 (1979)
11. R.B. Pearson: Phys. Rev. B **22**, 2579 (1980)
12. G. Delfino and J.L. Cardy: Nucl. Phys. B **519**, 551 (1998)
13. L. Chim and A.B. Zamolodchikov: Int. J. Mod. Phys. A **7**, 5317 (1992)
14. G. Delfino, G.T. Barkema and J.L. Cardy: Nucl. Phys. B **565**, 521 (2000)
15. L.N. Shchur, P. Butera, and B. Berche: Nucl. Phys. B **620**, 579 (2002)
16. I.G. Enting and A.J. Guttmann: Physica A **321**, 90 (2003)
17. M. Caselle, R. Tateo, and S. Vinci: Nucl. Phys. B **562**, 549 (1999)
18. M. Nauenberg abd D.J. Scalapino: Phys. Rev. Lett. **44**, 837 (1980)
19. J. L. Cardy, N. Nauenberg and D.J. Scalapino: Phys. Rev B **22**, 2560 (1980)
20. J. Salas and A. Sokal: J. Stat. Phys. **88**, 567 (1997)
21. R.J. Baxter: J. Phys. C **6**, 445 (1973)
22. U. Wolff: Phys. Rev. Lett. **62**, 361 (1989)
23. L.N. Shchur: Comp. Phys. Comm. **121-122**, 83 (1999)
24. K.M. Briggs, I.G. Enting, and A.J. Guttmann: J. Phys. A **27**, 1503 (1994)
25. G.S. Joyce: Proc. R. Soc. Lond. **A 343**, 45 (1975)
26. G.S. Joyce: Proc. R. Soc. Lond. **A 345**, 277 (1975)
27. L.N. Shchur, B. Berche and P. Butera, arXiv:0707.2430 [cond-mat.stat-mech]