Null-vectors in integrable field theory

O. Babelon, Denis Bernard, F.A. Smirnov

SPhT/96/063
LPTHE-96-23

Null-vectors in Integrable Field Theory.

O. Babelon [a 0], D. Bernard [b 0] and F.A. Smirnov [a 1]

[a] Laboratoire de Physique Théorique et Hautes Energies [2]
Université Pierre et Marie Curie, Tour 16 1^{er} étage, 4 place Jussieu
75252 Paris cedex 05-France

[b] Service de Physique Théorique de Saclay [3]
F-91191, Gif-sur-Yvette, France.

Abstract. The form factor bootstrap approach allows to construct the space of local fields in the massive restricted sine-Gordon model. This space has to be isomorphic to that of the corresponding minimal model of conformal field theory. We describe the subspaces which correspond to the Verma modules of primary fields in terms of the commutative algebra of local integrals of motion and of a fermion (Neveu-Schwarz or Ramond depending on the particular primary field). The description of null-vectors relies on the relation between form factors and deformed hyper-elliptic integrals. The null-vectors correspond to the deformed exact forms and to the deformed Riemann bilinear identity. In the operator language, the null-vectors are created by the action of two operators $\mathcal{Q}$ (linear in the fermion) and $\mathcal{C}$ (quadratic in the fermion). We show that by factorizing out the null-vectors one gets the space of operators with the correct character. In the classical limit, using the operators $\mathcal{Q}$ and $\mathcal{C}$ we obtain a new, very compact, description of the KdV hierarchy. We also discuss a beautiful relation with the method of Whitham.

[0]Membre du CNRS
[1]On leave from Steklov Mathematical Institute, Fontanka 27, St. Petersburg, 191011, Russia
[2]*Laboratoire associé au CNRS.*
[3]*Laboratoire de la Direction des Sciences de la Matière du Commissariat à l'Energie Atomique.*

1 Introduction.

In this article we present a synthesis of the ideas of the papers [1] and [2]. In the first of these papers the space of fields for the sine-Gordon model (SG) was described in terms of the form factors previously obtained in the bootstrap approach [3]. This description is based on rather special properties of form factors for the SG model. Namely, it uses the fact that the form factors were written in terms of deformed hyper-elliptic differentials, allowing deformations of all the nice properties of the usual hyper-elliptic differentials: the notion of deformed exact forms and of the deformed Riemann bilinear identity are available for them [4]. Using these facts it has been shown in [1] that the same number of local operators can be constructed in the generic case of the sine-Gordon model as at the free fermion point. The deformed exact forms and the deformed Riemann bilinear identity are necessary in order to reduce the space of fields to the proper size because in its original form factor description the space is too big. The description of [1] is basically independent of the coupling constant, but for rational coupling constant there is a possibility to find additional degenerations.

The problem with the description of the space of fields obtained in [1] is due to the fact that it is difficult to compare it with the description coming from Conformal Field Theory (CFT) or from the classical theory. The latter two are closely connected because the Virasoro algebra can be considered as a quantization of the second Poisson structure of KdV. It is even difficult to distinguish the descendents with respect to the two chiral Virasoro algebras.

On the other hand in [2] the semi-classical limit of the form factor formulae has been understood. This opens the possibility of identifying all the local operators by their classical analogues. Using this result we decided to try to construct the module of the descendents of the primary fields with respect to the chiral Virasoro algebra. The result of this study happened to be quite interesting.

Let us formulate more precisely the problems discussed in this paper. The sine-Gordon model is described by the action:

$$S = \frac{\pi}{\gamma}\int \left((\partial_\mu\varphi)^2 + m^2(\cos(2\varphi)-1)\right)\ d^2x$$

where γ is the coupling constant. In the quantum theory, the relevant coupling constant is: $\xi = \frac{\pi\gamma}{\pi-\gamma}$.

The sine-Gordon theory contains two subalgebras of local operators which, as operator algebras, are generated by $\exp(i\varphi)$ and $\exp(-i\varphi)$ respectively. We shall consider one of them, say the one generated by $\exp(i\varphi)$. It is known that this subalgebra can be considered independently of the rest of the operators, as the operator algebra of the theory with the modified energy-momentum tensor:

$$T^{mod}_{\mu\nu} = T^{SG}_{\mu\nu} + i\alpha\epsilon_{\mu,\mu'}\epsilon_{\nu,\nu'}\partial_{\mu'}\partial_{\nu'}\varphi$$

where $\alpha = \pi\sqrt{\frac{6}{\xi(\pi+\xi)}}$. This modification changes the trace of the energy-momentum tensor which is now: $T^{mod}_{\mu\mu} = m^2\exp(2i\varphi)$. This modified energy-momentum tensor corresponds to the restricted sine-Gordon theory (RSG). For rational $\frac{\xi}{\pi}$, the RSG model describes the $\Phi_{[1,3]}$-perturbations of the minimal models of CFT. In this paper we consider only the RSG model.

It is natural from a physical point of view of integrable perturbations [5] to expect that the space of fields for the perturbed model is the same as for its conformal limit. The latter consists of the primary fields and their descendents with respect to the two chiral Virasoro algebras. In this paper we shall consider the descendents with respect to one of these algebras, the possibility of considering the descendents with respect to the other one is explained in Subsection 2.2. The Verma module of the Virasoro algebra is generated by the action of the generators L_{-k} of the Virasoro algebra on the primary field. The irreducible representation corresponding to a given primary field is obtained by factorizing out the null-vectors [6, 7].

On the other hand the very possibility of integrable deformations is due to the fact that there exists a commutative subalgebra of the universal enveloping algebra of the Virasoro algebra: the algebra of local integrals [5, 8, 9]. The local integrals I_{2k-1} have odd spins. Logically it must be possible to present the Verma module as a result of the action of the local integrals of motion on a smaller module: the quotient over the local integrals. The latter is isomorphic to the module created from the vacuum by the action of bosons of even spins J_{2k}. The important observation is that the form factor formulae allow us to give a description of this space. Certainly, such a realization of the Verma module is very useful for integrable applications, but there is a difficulty. It is not trivial to describe the null-vectors in this realization. This question, of the description of the null-vectors, is the main problem solved in this paper.

Briefly, the result is as follows. It is useful to fermionize the bosons J_{2k} by introducing Neveu-Schwarz (ψ_{2k-1}) or Ramond (ψ_{2k}) fermions, depending on which primary field we consider. As in [1] the null-vectors are due to the deformed exact forms and the deformed Riemann bilinear identity. In the fermionic language these null-vectors are created by the action of two operators: a linear one in the fermion ($\mathcal{Q}$) and a quadratic one ($\mathcal{C}$). By calculating the characters we show that, after factorizing these null-vectors, we find exactly the same number of descendents of a given primary field as in the corresponding irreducible representation of the Virasoro algebra.

In Conformal Field Theory, the existence of null vectors provide differential equations for the correlation functions. The null vectors built out of $\mathcal{Q}$ and $\mathcal{C}$ in our approach will also lead to a system of equations for the correlation functions in the massive case. These equations will be similar to the hierarchies of equations in classical integrable systems. In spite of the fact that such an infinite set of equations seems *a priori* untractable, our hope is that, as in the classical case, large classes of solutions will be constructed. Hence we hope that these equations will be really useful for the study of correlation functions in the massive case.

We also present a detailed analysis of the classical limit of our constructions. The description of the null-vectors in terms of $\mathcal{Q}$ and $\mathcal{C}$ implies in the classical limit, a new and very compact description of the KdV hierarchy. This new description is not the same as the description in terms of τ-functions [10], still the technics we use are close to those of the Kyoto school. Finally, we discuss an amazing analogy between the quantum theory and the results of the Whitham method for small perturbations around a given quasi-periodic classical solution of KdV.

Let us make one general remark on the exposition. In this paper we restrict ourselves to the reflectionless case of the sine-Gordon model. This is done only in order to simplify the reading of the paper. All our conclusions are valid for a generic value of the coupling constant as well. For this reason, we chose to present the final formulae in the situation of a generic coupling constant. This leads, from time to time, to a contradictory situation. We hope that we shall be forgiven for this because if we wrote generic formulae everywhere the understanding of the paper would have been much more difficult.

2 The space of fields.

2.1 The description of the space of fields in the A, B-variables.

At the reflectionless points ($\xi = \frac{\pi}{\nu}$, $\nu = 1, 2, \cdots$) the form factors of an arbitrary local operator $\mathcal{O}$, in the RSG-model, corresponding to a state with n-solitons and n-anti-solitons are given by

$$\begin{aligned} &f_{\mathcal{O}}(\beta_1,\beta_2,\cdots,\beta_{2n})_{-\cdots-+\cdots+} = \\ &= c^n \prod_{i<j}\zeta(\beta_i-\beta_j)\prod_{i=1}^{n}\prod_{j=n+1}^{2n}\frac{1}{\sinh\nu(\beta_j-\beta_i-\pi i)}\exp(-\frac{1}{2}(\nu(n-1)-n)\sum_j\beta_j) \\ &\qquad \times\ \widehat{f}_{\mathcal{O}}(\beta_1,\beta_2,\cdots,\beta_{2n})_{-\cdots-+\cdots+} \end{aligned} \tag{1}$$

The function $\zeta(\beta)$, without poles in the strip $0 < Im\ \beta\ < 2\pi$, satisfies $\zeta(-\beta) = S(\beta)\zeta(\beta)$ and $\zeta(\beta - 2\pi i) = \zeta(-\beta)$: the S-matrix $S(\beta)$ and the constant c are given in the Appendix A. The most essential part of the form factor is given by

$$\begin{aligned} &\widehat{f}_{\mathcal{O}}(\beta_1,\beta_2,\cdots,\beta_{2n})_{-\cdots-+\cdots+} \quad = \\ &= \frac{1}{(2\pi i)^n}\int dA_1\cdots\int dA_n\prod_{i=1}^{n}\prod_{j=1}^{2n}\psi(A_i,B_j)\prod_{i<j}(A_i^2-A_j^2)\ L_{\mathcal{O}}^{(n)}(A_1,\cdots,A_n|B_1,\cdots,B_{2n})\prod_{i=1}^{n}a_i^{-i} \end{aligned} \tag{2}$$

where $B_j = e^{\beta_j}$ and

$$\psi(A,B) = \prod_{j=1}^{\nu-1}(B - Aq^{-j}), \qquad \text{with}\ \ q = e^{i\pi/\nu}$$

As usual we define $a = A^{2\nu}$. Here and later if the range of integration is not specified the integral is taken around 0. Notice that the operator dependence of the form factors (1) only enters in $\widehat{f}_{\mathcal{O}}$.

Different local operators $\mathcal{O}$ are defined by different functions $L_{\mathcal{O}}^{(n)}(A_1,\cdots,A_n|B_1,\cdots,B_{2n})$. These functions are symmetric polynomials of $A_1,\cdots,A_n$. For the primary operators $\Phi_{2k} = \exp(2ki\varphi)$ and their Virasoro descendants, $L_{\mathcal{O}}$ are symmetric Laurent polynomials of $B_1,\cdots,B_{2n}$. For the primary operators $\Phi_{2k+1} = \exp((2k+1)i\varphi)$, they are symmetric Laurent polynomials of $B_1,\cdots,B_{2n}$ multiplied by $\prod B_j^{\frac{1}{2}}$. Our definition of the fields Φ_m is related to the notations coming from CFT as follows: Φ_m corresponds to $\Phi_{[1,m+1]}$. The requirement of locality is guaranteed by the following simple recurrent relation for the polynomials $L^{(n)}_{\mathcal{O}}$:

$$L^{(n)}_{\mathcal{O}}(A_1,\cdots,A_n|B_1,\cdots,B_{2n})\Big|_{B_{2n}=-B_1,\ A_n=\pm B_1} = \\ = -\epsilon^{\pm}L^{(n-1)}_{\mathcal{O}}(A_1,\cdots,A_{n-1}|B_2,\cdots,B_{2n-1}) \tag{3}$$

where $\epsilon = +$ or $-$ respectively for the operators Φ_{2k} and their descendents, or for Φ_{2k+1} and their descendents. We explain this condition in Appendix A.

The explicit form of the polynomials $L_{\mathcal{O}}$ for the primary operators is as follows

$$L^{(n)}_{\Phi_m}(A_1,\cdots,A_n|B_1,\cdots,B_{2n}) = \prod_{i=1}^{n} A_i^m \prod_{j=1}^{2n} B_j^{-\frac{m}{2}}$$

In this paper we shall consider the Virasoro descendents of the primary fields. We shall restrict ourselves by considering only one chirality. Obviously, the locality relation (3) is not destroyed if we multiply the polynomial $L^{(n)}_{\mathcal{O}}(A|B)$ either by $I_{2k-1}(B)$ or by $J_{2k}(A|B)$ with

$$I_{2k-1}(B) = \left(\frac{1+q^{2k-1}}{1-q^{2k-1}}\right) s_{2k-1}(B), \qquad k=1,2,\cdots \tag{4}$$

$$J_{2k}(A|B) = s_{2k}(A) - \frac{1}{2}s_{2k}(B), \qquad k=1,2,\cdots \tag{5}$$

Here and later we shall use the following definition:

$$s_k(x_1,\cdots,x_m) = \sum_{j=1}^{m} x_j^k$$

The multiplication by I_{2k-1} corresponds to the application of the local integrals of motion. The normalization factor $\frac{1+q^{2k-1}}{1-q^{2k-1}}$ is introduced for further convenience. Since the boost operator acts by dilatation on A and B, I_{2k-1} has spin $(2k-1)$ and J_{2k} has spin $2k$.

The crucial assumption which we make is that the space of local fields descendents of the operator Φ_m is generated by the operators obtained from the generating function

$$\mathcal{L}_m(t,y|A|B) = \exp\Big(\sum_{k\geq 1} t_{2k-1}I_{2k-1}(B) + y_{2k}J_{2k}(A|B)\Big) \left(\prod_{i=1}^{n} A_i^m \prod_{j=1}^{2n} B_j^{-\frac{m}{2}}\right) \tag{6}$$

This is our main starting point. This assumption follows from the classical meaning of the variables A, B [2]. We shall give classical arguments later, and the connection with CFT is explained in the next subsection.

2.2 The relation with CFT.

The RSG model coincides with the $\Phi_{[1,3]}$-perturbation of the minimal models of CFT. For the coupling constants which we consider the minimal models in question are not unitary, but this fact is of no importance for us. It is natural from a physical point of view, to conjecture that the number of local operators in the perturbed model is same as in its conformal limit.

We first need to recall a few basic facts concerning the minimal conformal field theories. At the coupling constant ξ, the conformal limit of the RSG models have central charge

$$c = 1 - 6\frac{\pi^2}{\xi(\xi+\pi)}$$

The conformal limit of the fields $\Phi_m = e^{im\varphi}$ is identified with the operators $\Phi_{[1,m+1]}$ in the minimal models. They have conformal dimensions:

$$\Delta_m = \frac{m(m\xi - 2\pi)}{4(\xi+\pi)}$$

The reflectionless points, $\xi = \frac{\pi}{\nu}$ with $\nu = 1, 2, \cdots$, correspond the non-unitary minimal models $\mathcal{M}_{(1,\nu+1)}$, which are degenerate cases. At these points one has to identify Φ_m with $\Phi_{2\nu-m}$.

The Virasoro Verma module corresponding to the primary field Φ_m is generated by the vectors

$$L_{-k_1} L_{-k_2} \cdots L_{-k_N} \Phi_m$$

At $\xi = \frac{\pi}{\nu}$ their structure is the same as for generic value of ξ. This means that for m integer, the Verma module possesses only one submodule. The so-called basic null-vectors, which we shall denote by Γ_m, are the generators of these submodules. They appear at level $m+1$. The first few are given by:

$$\begin{aligned}
\Gamma_0 &= L_{-1}\Phi_0, \\
\Gamma_1 &= (L_{-2} + \kappa L_{-1}^2)\Phi_1, \qquad \kappa = -1 - \frac{\pi}{\xi} \\
\Gamma_3 &= (L_{-3} + \kappa_1 L_{-1} L_{-2} + \kappa_2 L_{-1}^3)\Phi_2, \quad \kappa_1 = -2\frac{\pi+\xi}{\pi+3\xi}, \kappa_2 = \frac{(\pi+\xi)^2}{2\xi(\pi+3\xi)} \\
& \quad etc \ \cdots
\end{aligned}$$

Other null-vectors, whose set form the submodule, are created from the basic ones as follows

$$L_{-k_1} L_{-k_2} \cdots L_{-k_N} \quad \Gamma_m$$

As a consequence, the character of the irreducible Virasoro representation with highest weight Φ_m is:

$$\chi_m(p) = \frac{1 - p^{m+1}}{\prod_{j\geq 1}(1 - p^j)}$$

There exists an alternative description of these modules which is more appropriate for our purposes. Indeed, as is well known the integrability of the $\Phi_{[1,3]}$-perturbation is related to the existence of a certain commutative subalgebra of the Virasoro universal enveloping algebra generated by elements I_{2k-1}, polynomial of degree k in L_n, such that $[L_0, I_{2k-1}] = (2k-1)I_{2k-1}$ and

$$[I_{2k-1}, I_{2l-1}] = 0, \qquad k, l = 1, 2, \cdots$$

The first few are given by:

$$\begin{aligned}
I_1 &= L_{-1}, \\
I_3 &= 2\sum_{n\geq 0} L_{-n-2} L_{n-1}, \\
& \quad etc \ \ \cdots
\end{aligned}$$

This is the subalgebra of the local integrals of motion. The meaning of these operators and of the $I_{2k-1}(B)$ which we had above is the same, so we denote them by the same letters.

Of course acting only with this subalgebra on Φ_m does not generate the whole Verma module. But the left quotient of the Verma module by the ideal generated by the integrals of motion produces a space whose character is $\prod_{j\geq 0}(1-p^{2j})^{-1}$. This space can be thought of as generated by some J_{2k} with $[L_0, J_{2k}] = 2kJ_{2k}$.

We expect that there is the following alternative way of generating the Virasoro module. Namely, in a spirit similar to the Feigin-Fuchs construction [6], we expect that there exists an appropriate completion of the subalgebra of the integrals of motion by elements J_{2k} such that (i) the Verma module is isomorphic, as a graded space, to the space generated by the vectors

$$\exp(\sum_{k\geq 1} t_{2k-1} I_{2k-1}) : \exp(\sum_{k\geq 1} y_{2k} J_{2k}) : \Phi_m \tag{7}$$

where the double dots refer to an appropriate normal ordering, and (ii) that for certain normalization of I_{2k-1} and certain choice of J_{2k} and their normal ordering the generating functions (6) and (7) provide different realizations of the same object. We shall call the description given by (6) the A, B-representation of the Virasoro module. The existence of this alternative description of the Virasoro module is very important for integrable applications.

The main problem for this I, J description arises from the non-trivial construction of the null-vectors. This problem has two aspects. First one has to construct the null-vectors Γ_m in terms of I, J. Then one also has to construct the submodule associated to it. This is not as trivial as in the standard representation since, while acting on Γ_m with the I_{2k-1} still produces a null-vector, acting with the J_{2k} does not necessarily leads to a null-vector. We shall show that the whole null-vector submodule can be described in the A, B-representation. The proof is based on rather delicate properties of the form-factor integrals, the main of them being the deformed Riemann bilinear identity [4].

Let us discuss briefly the problem of the second chirality. In [2] we have explained that the formula (1) has to be understood as a result of light-cone quantization in which x_- is considered as space and x_+ as time. We must be able to consider the alternative possibility (x_+ - space, x_- - time), and the results of quantization must coincide. Where exactly the choice of hamiltonian picture manifests itself in our formulae? Consider the formula (2). As it is explained in [2] the fact that $L_{\mathcal{O}}^{(n)}$ is a polynomial in A_i corresponds to the choice of x_- as space direction while the multiplier $\prod_{i=1}^{n} a_i^{-i}$ corresponds semi-classically to the choice of trajectories under x_+-flow. These are the two ingredients which change when the hamiltonian picture is changed. Indeed, analyzing the results of [2], we find the following alternative description of the form factors. The form factors are given by the formulae (1) with $\widehat{f}_{\mathcal{O}}$ replaced by

$$\widehat{h}_{\mathcal{O}}(\beta_1, \beta_2, \cdots, \beta_{2n})_{-\cdots-+\cdots+} =$$
$$= \frac{1}{(2\pi i)^n} \int dA_1 \cdots \int dA_n \prod_{i=1}^{n}\prod_{j=1}^{2n} \psi(A_i, B_j) \prod_{i<j}(A_i^2 - A_j^2)\ K_{\mathcal{O}}^{(n)}(A_1, \cdots, A_n|B_1, \cdots, B_{2n}) \prod_{i=1}^{n} a_i^{-i+1} \prod_{j=1}^{2n} b_j^{-\frac{1}{2}}$$

where $b_j = B_j^{2\nu}$, $K_{\mathcal{O}}^{(n)}$ is a polynomial in A_i^{-1} and the replacement of $\prod a_i^{-i}$ by $\prod a_i^{-i+1} \prod b_j^{-\frac{1}{2}}$ corresponds to the change of x_+-trajectories to x_--trajectories. In particular for the primary fields we have

$$K_{\Phi_m}^{(n)}(A_1, \cdots, A_n|B_1, \cdots, B_{2n}) = \prod_{i=1}^{n} A_i^{-m} \prod_{j=1}^{2n} B_j^{\frac{m}{2}}$$

The $\overline{L}_{-k}$ descendents are obtained by multiplying $K_{\Phi_m}^{(n)}$ by $I_{-(2k-1)}(B)$ and $J_{-2k}(A|B)$. The consistency of the two pictures requires that for the primary fields they give the same result:

$$\widehat{f}_{\Phi_m}(\beta_1, \beta_2, \cdots, \beta_{2n})_{-\cdots-+\cdots+} = \widehat{h}_{\Phi_m}(\beta_1, \beta_2, \cdots, \beta_{2n})_{-\cdots-+\cdots+} \tag{8}$$

This is a complicated identity which nevertheless can be proven. We do not present the proof here because it goes beyond the scope of this paper. It should be said, however, that the proof is based on the same technics as used below (deformed Riemann bilinear identity *etc*). Using the equivalence of the two representations of the form factors we can consider the descendents with respect to $\overline{L}_{-k}$ and also mixed $L_{-k}, \overline{L}_{-k}$ descendents. A formula similar to (8) holds for any coupling constant. However, at the reflectionless points there is another consequence of (8): it also shows that the operators Φ_m and $\Phi_{2\nu-m}$ are identified as it should be.

3 Null-vectors.

3.1 The null-polynomials.

Null-vectors correspond to operators with all the form factors vanishing. Consider the integral

$$\frac{1}{(2\pi i)^n} \int dA_1 \cdots \int dA_n \prod_{i=1}^{n}\prod_{j=1}^{2n} \psi(A_i, B_j) \prod_{i<j}(A_i^2 - A_j^2)\ L_{\mathcal{O}}^{(n)}(A_1, \cdots, A_n|B_1, \cdots, B_{2n}) \prod_{i=1}^{n} a_i^{-i} \tag{9}$$

Instead of $L_{\mathcal{O}}^{(n)}$, we shall often use the anti-symmetric polynomials $M_{\mathcal{O}}^{(n)}$:

$$M_{\mathcal{O}}^{(n)}(A_1, \cdots, A_n|B_1, \cdots, B_{2n}) = \prod_{i<j}(A_i^2 - A_j^2)\ L_{\mathcal{O}}^{(n)}(A_1, \cdots, A_n|B_1, \cdots, B_{2n})$$

The dependence on $B_1, \cdots, B_{2n}$ in the polynomials $M_{\mathcal{O}}^{(n)}$ will often be omitted.

There are several reasons why this integral can vanish. Some of them depend on a particular value of the coupling constant or on a particular number of solitons. We should not consider these occasional situations. There are three general reasons for the vanishing of the integral, let us present them.

1. Residue. The integral (9) vanishes if vanishes the residue with respect to A_n at the point $A_n = \infty$ of the expression

$$\prod_{j=1}^{2n} \psi(A_n, B_j) a_n^{-n} M_{\mathcal{O}}^{(n)}(A_1, \cdots, A_n)$$

Of course the distinction of the variable A_n is of no importance because $M_{\mathcal{O}}^{(n)}(A_1, \cdots, A_n)$ is anti-symmetric.

2. "Exact forms." The integral (9) vanishes if $M_{\mathcal{O}}^{(n)}(A_1, \cdots, A_n)$ happens to be an "exact form". Namely, if it can be written as:

$$M_{\mathcal{O}}^{(n)}(A_1, \cdots, A_n) = \sum_k (-1)^k M(A_1, \cdots, \widehat{A_k}, \cdots, A_n) \left(Q(A_k)P(A_k) - qQ(qA_k)P(-A_k)\right), \quad (10)$$

with

$$P(A) = \prod_{j=1}^{2n} (B_j + A)$$

for some anti-symmetric polynomial $M(A_1, \cdots, A_{n-1})$. Here and later $\widehat{A_k}$ means that A_k is omitted. This is a direct consequence of the functional equation satisfied by $\psi(A, B)$ (See eq.(65) in Appendix B). For $Q(A)$ one can take in principle any Laurent polynomial, but since we want $M_{\mathcal{O}}^{(n)}$ to be a polynomial the degree of $Q(A)$ has to be greater or equal -1.

3. Deformed Riemann bilinear relation. The integral (9) vanishes if

$$M_{\mathcal{O}}^{(n)}(A_1, \cdots, A_n) = \sum_{i<j} (-1)^{i+j} M(A_1, \cdots, \widehat{A_i}, \cdots, \widehat{A_j}, \cdots A_n) C(A_i, A_j)$$

where $M(A_1, \cdots, A_{n-2})$ is an anti-symmetric polynomial of $n-2$ variables, and $C(A_1, A_2)$ is given by

$$C(A_1, A_2) = \frac{1}{A_1 A_2} \left\{ \frac{A_1 - A_2}{A_1 + A_2} (P(A_1)P(A_2) - P(-A_1)P(-A_2)) + (P(-A_1)P(A_2) - P(A_1)P(-A_2)) \right\} \quad (11)$$

This property needs some comments. For the case of generic coupling constant its proof is rather complicated. It is a consequence of the so called deformed Riemann bilinear identity [4]. The name is due to the fact that in the limit $\xi \to \infty$ the deformed Riemann bilinear identity happens to be the same as the Riemann bilinear identity for hyper-elliptic integrals. The formula for $C(A_1, A_2)$ given in [11] differs from (11) by simple "exact forms". Notice that the formula for $C(A_1, A_2)$ does not depend on the coupling constant. For the reflectionless case a very simple proof is available which is given in Appendix B.

In order to apply these restrictions to the description of the null-vectors we have to make some preparations. The expression for $C(A_1, A_2)$ given above is economic in a sense that, as a polynomial of A_1, A_2 it has degree $2n-1$, but it is not appropriate for our goals because it mixes odd and even degrees of A_1, A_2 while the descendents of primary fields contain only odd or only even polynomials. So, by adding an "exact forms" we want to replace $C(A_1, A_2)$ by equivalent expressions of higher degrees which contain only odd or only even degrees.

Proposition 1. *The following equivalent forms of $C(A_1, A_2)$ exist:*

$$C(A_1, A_2) \simeq C_e(A_1, A_2) \simeq C_o(A_1, A_2), \quad (12)$$

here and later $\simeq$ means equivalence up to "exact forms". The formulae for $C_e(A_1, A_2)$ and $C_o(A_1, A_2)$ are as follows:

$$C_e(A_1, A_2) = \int\limits_{\substack{|D_2|>|D_1| \\ |D_1|>|A_1|,|A_2|}} dD_2 dD_1 \, \tau_e\left(\frac{D_1}{D_2}\right) \left(\frac{P(D_1)P(D_2)}{(D_1^2 - A_1^2)(D_2^2 - A_2^2)} - \frac{P(D_1)P(D_2)}{(D_2^2 - A_1^2)(D_1^2 - A_2^2)} \right) \quad (13)$$

where

$$\tau_e(x) = \sum_{k=1}^{\infty} \frac{1 - q^{2k-1}}{1 + q^{2k-1}} x^{2k-1} - \sum_{k=1}^{\infty} \frac{1 + q^{2k}}{1 - q^{2k}} x^{2k}$$

and

$$C_o(A_1,A_2)=A_1A_2\int\limits_{\substack{|D_2|>|D_1|\\|D_1|>|A_1|,|A_2|}} dD_2dD_1\tau_o\left(\frac{D_1}{D_2}\right)\left(\frac{P(D_1)P(D_2)}{(D_1^2-A_1^2)(D_2^2-A_2^2)}-\frac{P(D_1)P(D_2)}{(D_2^2-A_1^2)(D_1^2-A_2^2)}\right)$$

where

$$\tau_o(x)=\sum_{k=1}^{\infty}\frac{1-q^{2k}}{1+q^{2k}}x^{2k}-\sum_{k=1}^{\infty}\frac{1+q^{2k-1}}{1-q^{2k-1}}x^{2k-1}$$

The proof of this proposition is given in Appendix B.

The functions $\tau_e(x)$ and $\tau_o(x)$ are not well defined when $q^r=1$ because certain denominators vanish. However the formula (12) in which the LHS is independent of q implies that in the dangerous places we always find "exact forms" . So, for our applications these singularities are harmless.

3.2 The fermionization.

The descendents of the local operators are created by I_{2k-1} and J_{2k}. This generates a bosonic Fock space. It is very convenient to fermionize J_{2k}. Let us introduce Neveu-Schwarz and Ramond fermions: $\psi_{2k-1},\psi^*_{2k-1}$ and ψ_{2k},ψ^*_{2k}. The commutation relations are as follows

$$\psi_l\psi^*_m+\psi^*_m\psi_l=\delta_{l,m}$$

We prefer to follow the notations from [10] than those coming from CFT, the reader used to CFT language has to replace ψ^*_m by ψ^*_{-m}.

The vacuum vectors for the spaces with different charges are defined as follows. In the Neveu-Schwarz sector we have:

$$\begin{aligned}\psi_{2k-1}|2m-1\rangle=0,\quad for\ k>m,\qquad &\psi^*_{2k-1}|2m-1\rangle=0,\quad for\ k\le m;\\ \langle 2m-1|\psi_{2k-1}=0,\quad for\ k\le m,\qquad &\langle 2m-1|\psi^*_{2k-1}=0,\quad for\ k>m\end{aligned}$$

For the Ramond sector we have:

$$\begin{aligned}\psi_{2k}|2m\rangle=0,\quad for\ k>m,\qquad &\psi^*_{2k}|2m\rangle=0,\quad for\ k\le m;\\ \langle 2m|\psi_{2k}=0,\quad for\ k\le m\qquad &\langle 2m|\psi^*_{2k}=0,\quad for\ k>m\end{aligned}$$

We shall never mix the Neveu-Schwarz and Ramond sectors. The spaces spanned by the right action of an equal number of ψ's and ψ^*'s on the vector $\langle p|$ will be called H^*_p. The right action of ψ sends H^*_p to H^*_{p+2}. It is useful to think of the vector $\langle p|$ as a semi-infinite product

$$\langle p|=\cdots\psi_{p-4}\psi_{p-2}\psi_p$$

Let us introduce generating functions for the fermions. The operators $\psi(A),\psi^*(A)$ are defined for the Neveu-Schwarz and the Ramond sectors respectively as follows

$$\begin{aligned}\psi(A)&=\sum_{k=-\infty}^{\infty}A^{-2k+1}\psi_{2k-1},\qquad \psi^*(A)=\sum_{k=-\infty}^{\infty}A^{2k-1}\psi^*_{2k-1};\\ \psi(A)&=\sum_{k=-\infty}^{\infty}A^{-2k}\psi_{2k},\qquad \psi^*(A)=\sum_{k=-\infty}^{\infty}A^{2k}\psi^*_{2k}\end{aligned}$$

We shall use the decomposition of $\psi(A),\psi^*(A)$ into the regular and singular parts (at zero):

$$\psi(A)=\psi(A)_{reg}+\psi(A)_{sing},\qquad \psi^*(A)=\psi^*(A)_{reg}+\psi^*(A)_{sing}$$

where $\psi(A)_{reg}$ and $\psi^*(A)_{reg}$ contain all the terms with non-negative degrees of A.

Let us introduce the bosonic commuting operators h_{-2k} for $k\ge 1$:

$$h_{-2k}=\sum_{j=-\infty}^{\infty}\psi_{2j-1}\psi^*_{2k+2j-1}\qquad for\ Neveu-Schwarz\ sector,$$

$$h_{-2k}=\sum_{j=-\infty}^{\infty}\psi_{2j}\psi^*_{2k+2j}\qquad for\ Ramond\ sector,$$

They satisfy the commutation relations: $[h_{-2k}, h^*_{-2l}] = -k\delta_{k,l}$. We also have the following commutation relations between the fermions and the bosons:

$$\psi(A)h_{-2k} = (h_{-2k} - A^{-2k})\psi(A) \tag{14}$$

The bosonic generating function $\mathcal{L}_m(t,y|A|B)$ for the descendents of the operators Φ_m can be rewritten as:

$$\mathcal{L}_m(t,y|A|B) = \exp\left(\sum_{k\geq 1} t_{2k-1} I_{2k-1}(B)\right) \ \langle m-1| \exp\big(\sum_{k\geq 1} y_{2k} h^*_{-2k}\big) \ \widehat{\mathcal{L}}_m(A|B)|m-1\rangle \tag{15}$$

where

$$\widehat{\mathcal{L}}_m(A|B) = \exp\left(-\sum_{k\geq 1}\frac{1}{k} h_{-2k} J_{2k}(A,B)\right) \ \prod_i A_i^m \prod_j B_j^{-\frac{m}{2}}$$

In other words, to have a particular descendent one has to take in the expression

$$\exp\left(\sum_{k\geq 1} t_{2k-1} I_{2k-1}(B)\right) \ \widehat{\mathcal{L}}_m(A|B)|m-1\rangle$$

the coefficient in front of some monomial in t_{2k-1} and to calculate the matrix element with some vector from the fermionic space H^*_{m-1}.

We can replace this bosonic expression by a fermionic one. Recall that as a direct result of the boson-fermion correspondence one has:

$$\prod_{i<j}(A_i^2 - A_j^2)\exp\left(-\sum_{k\geq 1}\frac{1}{k} h_{-2k}\sum_{i=1}^{n} A_i^{2k}\right)|m-1\rangle = \psi^*(A_1)\cdots\psi^*(A_n)|m-2n-1\rangle \prod_{i=1}^{n} A_i^{-m+2n-1}$$

where the fermions are Neveu-Schwarz or Ramond ones depending on the parity of m. Using this fact the formula (15) can be rewritten for any n as follows

$$\prod_{i<j}(A_i^2 - A_j^2) \ \ \widehat{\mathcal{L}}_m(A|B)|m-1\rangle = \ g(B) \ \ \psi^*(A_1)\cdots\psi^*(A_n)|m-2n-1\rangle \ \ \prod_i A_i^{2n-1}\prod_j B_j^{-\frac{m}{2}}$$

where

$$g(B) = \exp\left(\sum_{k\geq 1}\frac{1}{2k} h_{-2k} s_{2k}(B)\right) \tag{16}$$

The occurrence of this operator is similar to that of the spin field in the description [12] of CFT on hyperelliptic curves.

Let us now concentrate on the operator $\Phi_0 = 1$ and its descendents. It means that we are working with the H^*_{-1} subspace of the Neveu-Schwarz sector. The polynomials corresponding to the operators in this sector are even in A_i. Let us describe the null-vectors due to the restrictions 1, 2, 3, from the previous subsection. To do that we shall need the results of the following three propositions.

Proposition 2. *The set of polynomials of the form*

$$M^{(n)}_{\mathcal{O}}(A_1,\cdots,A_n) = \sum_{i<j}(-1)^{i+j} M(A_1,\cdots,\widehat{A_i},\cdots,\widehat{A_j},\cdots A_n) C_e(A_i,A_j) \tag{17}$$

coincides up to "exact forms" with the set of the matrix elements:

$$\langle\Psi_{-5}|\widehat{\mathcal{C}} \ \psi^*(A_1)\cdots\psi^*(A_n)|-2n-1\rangle\prod A_i^{2n-1}, \qquad \forall \ \Psi_{-5}\in H^*_{-5} \tag{18}$$

where

$$\widehat{\mathcal{C}} = \int\limits_{|D_2|>|D_1|} dD_2 \int dD_1 P(D_1)P(D_2)D_1^{-2n-1}D_2^{-2n-1}\tau_e\left(\frac{D_1}{D_2}\right)\psi(D_1)\psi(D_2)$$

Proof.
The vectors from the space H^* can be written as

$$\langle\Psi| = \langle N|\psi_{k_1}\cdots\psi_{k_p}$$

where $k_p > \cdots > k_1 > N+1$. We shall call N the depth of $\langle\Psi|$. There are three possibilities for the matrix element (18) to differ from zero:

1. The depth of $\langle\Psi_{-5}|$ is greater than $-2n-1$
2. The vector $\langle\Psi_{-5}|$ is obtained from a vector $\langle\Psi_{-1}|$ whose depth is greater than $-2n-1$ by application of $\psi^*_{-2p-1}\psi^*_{-2q-1}$ with $q > p \geq n$ (i.e. there are two holes below $-2n-1$).
3. The vector $\langle\Psi_{-5}|$ is obtained from a vector $\langle\Psi_{-3}|$ whose depth is greater than $-2n-1$ by application of ψ^*_{-2p-1} with $p \geq n$ (i.e. there is one hole below $-2n-1$).

In the first case using the formula

$$\langle -2n-1|\psi(D)\psi^*(A)|-2n-1\rangle = \left(\frac{D}{A}\right)^{2n-1}\frac{D^2}{(D^2-A^2)}, \qquad |D|>|A|$$

and (13) one finds

$$\langle\Psi_{-5}|\widehat{\mathcal{C}}\ \psi^*(A_1)\cdots\psi^*(A_n)|-2n-1\rangle\prod A_i^{2n-1} =$$
$$= \sum_{i<j}(-1)^{i+j}M(A_1,\cdots,\widehat{A_i},\cdots,\widehat{A_j},\cdots A_n)C_e(A_i,A_j)$$

where

$$M(A_1,\cdots,A_{n-2}) = \langle\Psi_{-3}|\psi^*(A_1)\cdots\psi^*(A_{n-2})|-2n-1\rangle\prod A_i^{2n-1}$$

In the second case it is necessary that in the expression $\langle\Psi_{-1}|\psi^*_{-2p-1}\psi^*_{-2q-1}\ \widehat{\mathcal{C}}$ the two holes below $-2n-1$ are annihilated by $\widehat{\mathcal{C}}$, the result is

$$\langle\Psi_{-1}|\int\limits_{|D_2|>|D_1|} dD_2\int dD_1 P(D_1)P(D_2)D_1^{-2n+2p}D_2^{-2n+2q}\tau_e\left(\frac{D_1}{D_2}\right) = 0$$

because the integrand is a regular function of D_1 for $p \geq n$.

In the third case it is necessary that in the expression $\langle\Psi_{-3}|\psi^*_{-2p-1}\ \widehat{\mathcal{C}}$ the hole below $-2n-1$ is annihilated by $\widehat{\mathcal{C}}$, the result is

$$\langle\Psi_{-3}|\int\limits_{|D_2|>|D_1|} dD_2\int dD_1 P(D_1)P(D_2)D_1^{-2n-1}D_2^{-2n+2p}\tau_e\left(\frac{D_1}{D_2}\right)\psi(D_1)$$

where the pairing of $\psi(D_1)$ and ψ^*_{-2p-1} is not considered because it produces zero for the same reason as above. In the matrix element we shall have the polynomials

$$\langle -2n-1|\int\limits_{|D_2|>|D_1|} dD_2\int dD_1 P(D_1)P(D_2)D_1^{-2n-1}D_2^{-2n+2p}\tau_e\left(\frac{D_1}{D_2}\right)\psi(D_1)\psi^*(A_j)|-2n-1\rangle A_j^{2n-1} =$$
$$= \int\limits_{|D_2|>|D_1|} dD_2\int dD_1 P(D_1)P(D_2)\frac{1}{D_1^2-A_j^2}D_2^{-2n+2p}\tau_e\left(\frac{D_1}{D_2}\right) \equiv R_{2n+2p}(A_j)$$

The polynomial $R_{2n+2p}(A)$ is an even polynomial of degree $2n+2p$. Let us show that it is an "exact form". From the calculations of Appendix B we have

$$R_{2n+2p}(A) = \int\limits_{|D_2|>|D_1|} dD_2\int dD_1 P(D_1)P(D_2)\frac{1}{D_1^2-A^2}D_2^{-2n+2p}\tau_e\left(\frac{D_1}{D_2}\right) \simeq$$

$$\simeq A \int\limits_{|D_2|>|D_1|} dD_2 \int dD_1 P(D_1)P(D_2)\frac{1}{D_1^2-A^2}D_2^{-2n+2p}\left(\frac{1}{D_1+D_2}\right)=$$

$$= A \int\limits_{|D_1|>|D_2|} dD_2 \int dD_1 P(D_1)P(D_2)\frac{1}{D_1^2-A^2}D_2^{-2n+2p}\left(\frac{1}{D_1+D_2}\right)=0 \tag{19}$$

where we have changed the integral over $|D_2|>|D_1|$ by the integral over $|D_1|>|D_2|$ because the residue at $D_2=-D_1$ gives the integral over D_1 of an even function, such integral equals zero. The last integral in (19) vanishes because the integrand is a regular function of D_2 for $p\geq n$. Let us emphasize that our construction is self-consistent because for every n the polynomials of too high degree (greater than $4n-2$) are "exact forms".

Thus we have non-trivial matrix elements only in the first case which obviously exhausts the polynomial of the kind (17). ∎

Consider now the restriction 2 of the previous subsection. It is easy to figure out that there is only one uniform way to write for all n polynomials of the type (10) which are even in all variables A_i. Namely:

$$M_{\mathcal{O}}^{(n)}(A_1,\cdots,A_n)=\sum_k(-1)^kM(A_1^2,\cdots,\widehat{A_k^2},\cdots,A_n^2)\left(P(A_k)-P(-A_k)\right)A_k^{-1} \tag{20}$$

where $M(A_1^2,\cdots,\cdots,A_{n-1}^2)$ is arbitrary anti-symmetric polynomial.

The following two simple propositions are given without proof.

Proposition 3. *The set of polynomials (20) coincides with the set of matrix elements:*

$$\langle\Psi_{-3}|\widehat{\mathcal{Q}}\ \psi^*(A_1)\cdots\psi^*(A_n)|-2n-1\rangle\prod A_i^{2n-1}\qquad\forall\Psi_{-3}\in H_{-3}^*$$

where

$$\widehat{\mathcal{Q}}=\int dDD^{-2n-1}P(D)\psi(D)$$

Proposition 4. *The set of polynomials $M_{\mathcal{O}}^{(n)}(A_1,\cdots,A_n)$ such that*

$$res_{A_n=\infty}\prod_j\psi(A_n,B_j)a_n^{-n}M_{\mathcal{O}}^{(n)}(A_1,\cdots,A_n)=0$$

(we hope that the same letter ψ used for the function $\psi(A,B_j)$ and for the fermion is not confusing) coincides with the set of matrix elements:

$$\langle\Psi_1|\widehat{\mathcal{Q}}^\dagger\psi^*(A_1)\cdots\psi^*(A_n)|-2n-1\rangle\prod A_i^{2n-1}\qquad\forall\Psi_1\in H_1^*$$

where

$$\widehat{\mathcal{Q}}^\dagger=res_{A=\infty}\prod_j\psi(A,B_j)a^{-n}\int\limits_{|D|>|A|}\psi^*(D)\frac{1}{D^2-A^2}D^{2n}dD$$

Let us apply these results to the description of null-vectors. We need to introduce the following notations:

$$X(D)=\sum_{k\geq1}\frac{1}{2k-1}D^{-2k+1}s_{2k-1}(B)=\sum_{k\geq1}D^{-2k+1}\frac{1}{2k-1}\left(\frac{1-q^{2k-1}}{1+q^{2k-1}}\right)I_{2k-1}(B), \tag{21}$$

$$Y(D)=\sum_{k\geq1}\frac{1}{2k}D^{-2k}s_{2k}(B) \tag{22}$$

Obviously

$$P(D)=D^{2n}e^{X(D)-Y(D)} \tag{23}$$

The null-vectors will be produced by acting with some operators $\mathcal{C}$, $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ on $\widehat{\mathcal{L}}_0(A|B)|-1\rangle$. In view of the bosonization formulae, these operators are obtained from $\widehat{\mathcal{C}}$, $\widehat{\mathcal{Q}}$ and $\widehat{\mathcal{Q}}^\dagger$ by conjugation with $g(B)$:

$$\mathcal{C}\ g(B)=g(B)\ \widehat{\mathcal{C}},\qquad\mathcal{Q}\ g(B)=g(B)\ \widehat{\mathcal{Q}},\qquad\mathcal{Q}^\dagger\ g(B)=g(B)\ \widehat{\mathcal{Q}}^\dagger$$

The formulae for $\mathcal{C}$ and $\mathcal{Q}$ are given in the following two propositions:

Proposition 2'. *From the Proposition 2 we find that the null-vectors due to the deformed Riemann bilinear identity are of the form:*

$$\exp(\sum_{k\geq 1} t_{2k-1} I_{2k-1}(B)) \quad \langle \Psi_{-5}| \ \mathcal{C} \ \exp(-\sum_{k\geq 1} \frac{1}{k} h_{-2k} J_{2k}(A,B))| -1\rangle$$

where

$$\mathcal{C} = \int\limits_{|D_2|>|D_1|} \frac{dD_2}{D_2} \int \frac{dD_1}{D_1} e^{X(D_1)} e^{X(D_2)} \tau_e \left(\frac{D_1}{D_2}\right) \psi(D_1)\psi(D_2) \tag{24}$$

Proposition 3'. *From the Proposition 3 one gets the following set of null-vectors due to the "exact forms":*

$$\exp(\sum_{k\geq 1} t_{2k-1} I_{2k-1}(B)) \quad \langle \Psi_{-3}| \mathcal{Q} \exp(-\sum_{k\geq 1} \frac{1}{k} h_{-2k} J_{2k}(A,B))| -1\rangle$$

where

$$\mathcal{Q} = \int \frac{dD}{D} e^{X(D)} \psi(D) \tag{25}$$

Notice a very important feature in these formulae: The operators $\mathcal{Q}$ and $\mathcal{C}$ are independent of n.

These propositions are direct consequences of the previous ones and of the following conjugation property of the fermions:

$$\psi(D)\ g(B) = g(B)\ \psi(D)\ e^{-Y(D)}, \qquad \psi^*(D)\ g(B) = g(B)\ \psi^*(D)\ e^{Y(D)}$$

Before dealing with $\mathcal{Q}^\dagger$ let us discuss the operator $\mathcal{C}$, $\mathcal{Q}$ in more details. It will be convenient to rewrite them in terms of another set of fermions $\widetilde{\psi}$ and $\widetilde{\psi}^\dagger$. To understand the purpose of introducing a new basis for the fermions consider the formula (24). In this formula the fermion $\psi(D_2)$ can be replaced by its regular part $\psi(D_2)_{reg}$ because other multipliers in the integrand contain only negative powers of D_2. That is why $\mathcal{C}$ can be rewritten in the form

$$\mathcal{C} = \int \frac{dD}{D} \widetilde{\psi}(D)_{sing} \widetilde{\psi}(D)_{reg} \tag{26}$$

where the modified fermion $\widetilde{\psi}$ is defined as follows

$$\widetilde{\psi}(D)_{reg} = \psi(D)_{reg}, \qquad \widetilde{\psi}(D)_{sing} = U\psi\ (D),$$

with U the following operator

$$Uf\ (D) = \left[\ \int\limits_{|D|>|D_1|} \frac{dD_1}{D_1} e^{X(D_1)} e^{X(D)} \tau_e \left(\frac{D_1}{D}\right) f(D_1) \right]_{odd}$$

where $[\cdots]_{odd}$ means that only odd degrees of the expression with respect to D are taken because only those count in the integral (26). It is quite obvious that this transformation is triangular, namely

$$\widetilde{\psi}_{2k-1} = \left(\frac{1-q^{2k-1}}{1+q^{2k-1}}\right) \psi_{2k-1} + (terms\ with\ \psi_{2l-1}, l<k), \qquad k\geq 1$$

Altogether we can write $\widetilde{\psi}(D) = \widehat{U}\psi\ (D)$ where $\widehat{U}$ is triangular.

Introduce the fermions $\widetilde{\psi}^\dagger$ satisfying canonical commutation relations with $\widetilde{\psi}$:

$$\widetilde{\psi}^\dagger(D) = \left(\widehat{U}^{-1}\right)^T \psi^*\ (D)$$

Since the operator $\widehat{U}$ is not unitary, we do not use $*$ but $\dagger$ for $\widetilde{\psi}$. The triangularity of the operator $\widehat{U}$ guaranties that the Fock space H^* constructed in terms of $\widetilde{\psi}, \widetilde{\psi}^\dagger$ coincides with the original one.

Thus, we can rewrite (26) as follows

$$\mathcal{C} = \sum_{j=1}^{\infty} \widetilde{\psi}_{-2j+1}\widetilde{\psi}_{2j-1}$$

The important property of this formula is that for given number of solitons n the summation can be taken from 1 to n because the operators $\widetilde{\psi}_{2j-1}$ with $j > n$ produce "exact forms" when plugged into the matrix elements (see the proof of Proposition 2).

Similarly, we can express the operator $\mathcal{Q}$, defined in (25), in terms of $\widetilde{\psi}$:

$$\mathcal{Q} = \int \frac{dD}{D} e^{X(D)} \widetilde{\psi}(D)$$

This equality is due to the fact that only the regular part of $\psi(D)$ contributes into the integral which does not change under the transformation to $\widetilde{\psi}(D)$.

Now we are ready to consider the operator $\mathcal{Q}^\dagger$.

Proposition 4'. *From Proposition 4 one gets the following set of null-vectors due the vanishing of the residues:*

$$\exp(\sum_{k\geq 1} t_{2k-1} I_{2k-1}(B)) \quad \langle \Psi_1 | \mathcal{Q}^\dagger \exp(-\sum_{k\geq 1} \frac{1}{k} h_{-2k} J_{2k}(A,B)) | -1 \rangle$$

where

$$\mathcal{Q}^\dagger = \int \frac{dD}{D} e^{X(D)} \widetilde{\psi}^\dagger(D)$$

Proof.
Directly from Proposition 4 one gets the following formula for $\mathcal{Q}^\dagger$:

$$\mathcal{Q}^\dagger = res_{A=\infty} \prod_j \psi(A, B_j) a^{-n} \int\limits_{|D|>|A|} dD D^{2n} e^{-Y(D)} \psi^*(D) \frac{1}{D^2 - A^2} \tag{27}$$

This formula looks much simpler in terms of $\widetilde{\psi}^\dagger$. By definition we have

$$\psi^*(D) = \widehat{U}^T \widetilde{\psi}^\dagger \ (D) = \left[\int\limits_{|D_1|>|D|} \frac{dD_1}{D_1} e^{X(D_1)} e^{X(D)} \tau_e\left(\frac{D}{D_1}\right) \widetilde{\psi}^\dagger(D_1) \right]_{odd} + \widetilde{\psi}^\dagger(D)_{sing}$$

The last term does not contribute to the residue because

$$\int\limits_{|D|>|A|} dD D^{2n} e^{-Y(D)} \frac{1}{D^2 - A^2} \widetilde{\psi}^\dagger(D)_{sing} = \mathcal{O}(A^{2n-2})$$

and

$$\prod_j \psi(A, B_j) a^{-n} = A^{-2n}(1 + \mathcal{O}(A^{-1})) \tag{28}$$

Substituting the rest into (27) one has

$$\mathcal{Q}^\dagger = \int \frac{dD_1}{D_1} \widetilde{\psi}^\dagger(D_1) e^{X(D_1)} res_{A=\infty} \prod_j \psi(A, B_j) a^{-n} \int\limits_{|D_1|>|D|} dD P(D) \frac{1}{D^2 - A^2} \tau_e\left(\frac{D}{D_1}\right)$$

Using the formulae from Appendix B one can show that

$$\int\limits_{|D_1|>|D|>|A|} dD P(D) \frac{1}{D^2 - A^2} \tau_e\left(\frac{D}{D_1}\right) \simeq \frac{1}{2}\left(\frac{P(A)}{A + D_1} + \frac{P(A)}{A - D_1}\right) + \mathcal{O}(A^{-1}) = A^{2n-1}(1 + \mathcal{O}(A^{-1}))$$

Here the equality is up to "exact form" in A; such "exact form" never contribute to the residue. Now the formula (28) gives

$$res_{A=\infty}\prod_j \psi(A,B_j)a^{-n}\int\limits_{|D_1|>|D|>|A|} dDP(D)\frac{1}{D^2-A^2}\tau_e\left(\frac{D}{D_1}\right)=1$$

which proves the proposition. ■

This alternative expression for $\mathcal{Q}^\dagger$ shows that it is independent of n, as $\mathcal{Q}$ and $\mathcal{C}$ are.

Notice that in the formulae for $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ only the holomorphic parts of $\widetilde{\psi}(D)$ and $\widetilde{\psi}^\dagger(D)$ are relevant. This leads to the important commutation relation:

$$[\mathcal{C},\mathcal{Q}^\dagger]=\mathcal{Q} \tag{29}$$

Notice also that $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ are nilpotent operators, $\mathcal{Q}^2=(\mathcal{Q}^\dagger)^2=0$, and $[\mathcal{C},\mathcal{Q}]=0$.

3.3 Algebraic definition of the null-vectors.

Let us summarize our study of the null-vectors for the descendents of Φ_0. It is quite convenient to present the null-vectors as vectors from the dual space. In this section we shall write $\langle\Psi|\overset{w}{=}0$ if the matrix element of $\langle\Psi|$ with the fermionic generating function of local fields vanishes under the integral (w means "in a weak sense"). We have found three types of null-vectors:

$$\begin{aligned}
&\langle\Psi_{-5}|\mathcal{C}\overset{w}{=}0, &&\forall\ \langle\Psi_{-5}|\in H^*_{-5}, &&(i)\\
&\langle\Psi_{-3}|\mathcal{Q}\overset{w}{=}0, &&\forall\ \langle\Psi_{-3}|\in H^*_{-3}, &&(ii)\\
&\langle\Psi_{1}|\mathcal{Q}^\dagger\overset{w}{=}0, &&\forall\ \langle\Psi_{1}|\in H^*_{1}, &&(iii)
\end{aligned}$$

where the operators $\mathcal{C}$, $\mathcal{Q}^\dagger$, $\mathcal{Q}$ are given respectively by

$$\mathcal{C}=\int\frac{dD}{D}\widetilde{\psi}(D)_{reg}\widetilde{\psi}(D)_{sing},\qquad \mathcal{Q}=\int\frac{dD}{D}e^{X(D)}\widetilde{\psi}(D),\qquad \mathcal{Q}^\dagger=\int\frac{dD}{D}e^{X(D)}\widetilde{\psi}^\dagger(D)$$

Let us show that these three conditions for the null-vectors are not independent.

It is easy to show that the operator $\mathcal{C}$ identifies the spaces H^*_{-3} and H^*_1:

$$Ker(\mathcal{C}|_{H^*_{-3}\to H^*_1})=0,\qquad Im(\mathcal{C}|_{H^*_{-3}\to H^*_1})=H^*_1$$

Hence every $\langle\Psi_1|\in H^*_1$ can be presented as $\langle\Psi_{-3}|\mathcal{C}$ for some $\langle\Psi_{-3}|\in H^*_{-3}$. Let us show that the null-vectors (iii) are linear combinations of (i) and (ii). We have:

$$\langle\Psi_1|\mathcal{Q}^\dagger=\langle\Psi_{-3}|\mathcal{C}\mathcal{Q}^\dagger=\langle\Psi_{-3}|\mathcal{Q}+\langle\Psi_{-3}|\mathcal{Q}^\dagger\mathcal{C}$$

where we have used the commutation relation (29). Thus we have proven the following

Proposition 5. *In the space of descendents of $\Phi_0=1$ which is $H(I)\otimes H^*_{2m}$ (where $H(I)$ is the space of polynomials of $\{I_{2k-1}\}$) the null-vectors coincide with the vectors*

$$\begin{aligned}
&\langle\Psi_{-5}|\mathcal{C}\overset{w}{=}0, &&\forall\ \langle\Psi_{-5}|\in H^*_{-5}, &&(i)\\
&\langle\Psi_{-3}|\mathcal{Q}\overset{w}{=}0, &&\forall\ \langle\Psi_{-3}|\in H^*_{-3}, &&(ii)
\end{aligned}$$

and their descendents with respect to I's.

The consideration of the other operators Φ_{2m} is based on the same formulae, but involves additional complications. We do not want to go into details, and we only present the final result.

Proposition 6. *For the operator Φ_{2m} whose descendents are counted by the vectors from $H(I)\otimes H^*_{2m-1}$ we have two types of independent null-vectors:*

$$\begin{aligned}
&\langle\Psi_{-5-2m}|(\mathcal{C})^{m+1}\overset{w}{=}0, &&\forall\ \langle\Psi_{-5-2m}|\in H^*_{-5-2m}, &&(i)\\
&\langle\Psi_{-3-2m}|(\mathcal{C})^{m}\mathcal{Q}\overset{w}{=}0, &&\forall\ \langle\Psi_{-3-2m}|\in H^*_{-3-2m}, &&(ii)
\end{aligned}$$

and their descendents with respect to I's.

For the operators Φ_{2m+1} one has the following picture. There is no uniform "exact form" which is an odd polynomial, so, the analogue of the operator $\mathcal{Q}$ does not exist for Φ_{2m+1}, and the null-vectors are either due to the deformed Riemann bilinear identity or due to the vanishing of the residue. To construct the null-vectors in terms of fermions one has to introduce first the operator $\mathcal{C}$:

$$\mathcal{C} = \int\limits_{|D_2|>|D_1|} \frac{dD_2}{D_2} \int \frac{dD_1}{D_1} e^{X(D_1)} e^{X(D_2)} \tau_o \left(\frac{D_1}{D_2}\right) \psi(D_1)\psi(D_2)$$

where the fermions are from the Ramond sector. This operators can be rewritten in a form similar to (26):

$$\mathcal{C} = \int \frac{dD}{D} \widetilde{\psi}(D)_{reg} \widetilde{\psi}(D)_{sing}$$

The fermions $\widetilde{\psi}$ are related to ψ by triangular transformation.

The consideration of the operator Φ_1 is absolutely parallel to the consideration of Φ_0. The null-vectors are created either by the action of $\mathcal{C}$ (Riemann identity) or by the action of $\widetilde{\psi}_0^\dagger$ (residue). Notice that

$$[\mathcal{C}, \widetilde{\psi}_0^\dagger] = 0$$

which guaranties the consistency. For higher operators Φ_{2m+1} there are additional problems which we would not like to discuss here. The general result is given in the following

Proposition 7. *The descendents of the operator Φ_{2m+1} are counted by the vectors from the space $H(I) \otimes H^*_{2m}$. We have two types of null-vectors*

$$\langle \Psi_{-4-2m}|(\mathcal{C})^{m+1} \overset{w}{=} 0, \qquad \forall \quad \langle \Psi_{-4-2m}| \in H^*_{-4-2m}, \qquad (i)$$
$$\langle \Psi_{2-2m}|(\mathcal{C})^{m}\widetilde{\psi}_0^\dagger \overset{w}{=} 0, \qquad \forall \quad \langle \Psi_{2-2m}| \in H^*_{2-2m}, \qquad (ii)$$

and their descendents with respect to I's

3.4 Examples of null-vectors and the characters.

Let us present the simplest examples of null-vectors for the operators Φ_0, Φ_1 and Φ_2.

For the operator Φ_0 the simplest null-vector is created by

$$\langle -3|\mathcal{Q} = s_1(B)\langle -1|$$

This null-vector is

$$\left(\frac{1-q}{1+q}\right) I_1 \Phi_0$$

This null-vector is to be compared with

$$L_{-1}\Phi_0$$

For the operator Φ_1 the simplest null-vector is created by

$$\langle 2|\widetilde{\psi}_0^\dagger = \langle -2|\widetilde{\psi}_2 = \left(\frac{1-q^2}{1+q^2}\right)\left(\langle -2|\psi_2 - \frac{1}{2}\left(\frac{1+q}{1-q}\right)^2 s_1(B)^2\langle 0|\right)$$

which gives the null-vector

$$\left(\frac{1-q^2}{1+q^2}\right)(J_2 - \frac{1}{2}I_1^2)\Phi_1$$

which has to be compared with

$$(L_{-2} + \kappa L_{-1}^2)\Phi_1$$

For the operator Φ_2 the simplest null-vector is created by $\langle -5|\mathcal{C}\mathcal{Q}$. It yields

$$\frac{1}{3}\left(\frac{1-q^3}{1+q^3}\right)(I_3 - 3I_1J_2 + \frac{1}{2}I_1^3)\Phi_2$$

which has to be compared with

$$(L_{-3} + \kappa_1 L_{-1} L_{-2} + \kappa_2 L_{-1}^3)\Phi_2$$

Notice that the relative coefficients in our paramatrization of the null-vectors are independent of ξ. So, they are exactly of the same form as the classical one. However, this is not always the case.

Let us show that generally the number of our null-vectors is the same as for the representations of the Virasoro algebra. Recall that we consider the null-vectors which do not depend on the arithmetical properties of $\frac{\xi}{\pi}$, so, there is one basic null-vector in every Verma module of Virasoro algebra. The character of the irreducible module associated with Φ_m is

$$\chi_m(p) = (1 - p^{m+1}) \frac{1}{\prod_{j\geq 1}(1 - p^j)} \tag{30}$$

where we omitted the multiplier with the scaling dimension of the primary field. We can not control this scaling dimension, the dimensions of the descendents are understood relatively to the dimension of the primary field. The character (30) is obtained from the character of the Verma module by omitting the module of descendents of the null-vector on the level $m + 1$.

Let us consider the character of the module which we constructed in terms of I, J. The dimensions of I_{2k-1} and J_{2k} are naturally $2k - 1$ and $2k$. If we do not take into account the null-vectors, the characters of all the modules associated with Φ_m are the same:

$$\chi(p) = \frac{1}{\prod_{j\geq 1}(1 - p^j)}$$

Let us take into account the null-vectors. They are described in terms of fermions. By consistency with the dimensions of I_{2k-1} and J_{2k} one finds that the dimensions of ψ_l and $\psi^\dagger_{-l}$ equal l.

Technically it is easier to start with Φ_{2m+1}. The space of descendents is $H(I) \otimes H^*_{2m}$ where $H(I)$ is the space of polynomials of $\{I_{2k-1}\}$.

Proposition 8. *The character of the space of descendents of Φ_{2m+1}, modulo the null vectors, equals*

$$\chi_{2m+1}(p) = (1 - p^{2(m+1)}) \frac{1}{\prod_{j\geq 1}(1 - p^j)}$$

Proof.
The null-vectors are defined in the Proposition 7. It is easy to eliminate the null-vectors (ii): we have to consider the subspace $H^*_{-2m,\ 0}$ in which the level $\widetilde{\psi}_0$ is always occupied. Consider the sequence

$$H^*_{-2m-4,\ 0} \xrightarrow{\mathcal{C}} H^*_{-2m,\ 0} \xrightarrow{\mathcal{C}^m} H^*_{2m,\ 0}$$

The operator $\mathcal{C}^m$ identifies the spaces $H^*_{-2m,\ 0}$ and $H^*_{2m,\ 0}$:

$$Ker(\mathcal{C}^m|_{H^*_{-2m,\ 0} \to H^*_{2m,\ 0}}) = 0, \qquad Im(\mathcal{C}^m|_{H^*_{-2m,\ 0} \to H^*_{2m,\ 0}}) = H^*_{2m,\ 0}$$

and $\langle -2m - 2|\widetilde{\psi}_0^\dagger \mathcal{C}^m = \langle 2m|$. Hence we can count the descendents by vectors of the space $H(I) \otimes H^*_{-2m,\ 0}$ with the null-vectors:

$$\langle \Psi_{-4-2m}|\mathcal{C} \overset{w}{=} 0, \qquad \forall \quad \langle \Psi_{-4-2m}| \in H^*_{-4-2m,\ 0},$$

Notice that the operator $\mathcal{C}$ is dimensionless and

$$Ker(\mathcal{C}|_{H^*_{-4-2m,\ 0} \to H^*_{-2m,\ 0}}) = 0$$

That is why for the character of the space of descendents without null-vectors we have:

$$\chi_{2m+1}(p) = \frac{1}{\prod_{j\geq 1}(1 - p^{2j-1})} p^{-m(m-1)} \left(\chi_{\ H^*_{-2m,\ 0}}(p) - \chi_{\ H^*_{-2m-4,\ 0}}(p) \right) \tag{31}$$

where the first multiplier comes from $H(I)$ the multiplier $p^{-m(m-1)}$ is needed in order to cancel the dimension of the vacuum vector in H^*_{-2m}. Let us evaluate the expression in brackets:

$$
\begin{aligned}
&\chi_{\ H^*_{-2m,\ 0}}(p)-\chi_{\ H^*_{-2m-4,\ 0}}(p)= \\
&=\int\prod_{j\geq 1}(1+p^{2j}x)(1+p^{2j}x^{-1})x^{-m}\frac{dx}{x}-\int\prod_{j\geq 1}(1+p^{2j}x)(1+p^{2j}x^{-1})x^{-m-2}\frac{dx}{x}= \\
&=\int(1+x^{-1})\prod_{j\geq 1}(1+p^{2j}x)(1+p^{2j}x^{-1})x^{-m}\frac{dx}{x}-\int(1+x^{-1})\prod_{j\geq 1}(1+p^{2j}x)(1+p^{2j}x^{-1})x^{-m-1}\frac{dx}{x}= \\
&=(1-p^{2(m+1)})\int(1+x^{-1})\prod_{j\geq 1}(1+p^{2j}x)(1+p^{2j}x^{-1})x^{-m}\frac{dx}{x}= \\
&=(1-p^{2(m+1)})\chi_{\ H^*_{-2m}}(p)=p^{m(m-1)}(1-p^{2(m+1)})\frac{1}{\prod_{j\geq 1}(1-p^{2j})}
\end{aligned}
$$

where we have changed the variable of integration $x\rightarrow xp^{-2}$ in the second integral when passing from third to forth line. Substituting this result into (31) we get the correct character:

$$\chi_{2m+1}(p)=(1-p^{2(m+1)})\frac{1}{\prod_{j\geq 1}(1-p^{j})}$$

■

Let us consider now the operators Φ_{2m}. We parametrize the descendents of Φ_{2m} by the vectors from $H(I)\otimes H^*_{2m-1}$,

Proposition 9. *The character of the space of descendents of Φ_{2m}, modulo the null-vectors, equals*

$$\chi_{2m}(p)=(1-p^{2m+1})\frac{1}{\prod_{j\geq 1}(1-p^{j})}$$

Proof.
The null-vectors are defined in the Proposition 6. We have

$$
\begin{aligned}
H^*_{-2m-5}&\xrightarrow{\mathcal{C}}H^*_{-2m-1}\xrightarrow{\mathcal{C}^m}H^*_{2m-1}\\
H^*_{-2m-3}&\xrightarrow{\mathcal{Q}}H^*_{-2m-1}\xrightarrow{\mathcal{C}^m}H^*_{2m-1}
\end{aligned}
$$

The operator $\mathcal{C}^m$ identifies H^*_{-2m-1} and H^*_{2m-1}:

$$Ker(\mathcal{C}^m|_{H^*_{-2m-1}\rightarrow H^*_{2m-1}})=0,\qquad Im(\mathcal{C}^m|_{H^*_{-2m-1}\rightarrow H^*_{2m-1}})=H^*_{2m-1}$$

and $\langle -2m-1|\ \mathcal{C}^m=\langle 2m-1|$. Hence we can replace the space $H(I)\otimes H^*_{2m-1}$ with these null-vectors by $H(I)\otimes H^*_{-2m-1}$ with null-vectors

$$
\begin{aligned}
&\langle\Psi_{-5-2m}|\mathcal{C}\overset{w}{=}0,\qquad &\forall\ \ \langle\Psi_{-5-2m}|\in H^*_{-5-2m},\qquad &(i)\\
&\langle\Psi_{-3-2m}|\mathcal{Q}\overset{w}{=}0,\qquad &\forall\ \ \langle\Psi_{-3-2m}|\in H^*_{-3-2m},\qquad &(ii)
\end{aligned}
$$

So, the character in question is

$$\chi_{2m}(p)=\frac{1}{\prod_{j\geq 1}(1-p^{2j-1})}p^{-m^2}\left(\chi_{\ H^*_{-2m-1,\ 0}}(p)-\chi_{\ H^*_{-2m-5,0}}(p)\right)$$

where $H^*_{-2l-1,\ 0}=H^*_{-2l-1}/H^*_{-2l-3}\mathcal{Q}$. In order to calculate the character $\chi_{\ H^*_{-2l-1,\ 0}}(p)$ one has to take into account that $\mathcal{Q}$ is a nilpotent operator, $\mathcal{Q}^2=0$, with a trivial cohomology. Hence

$$Ker(\mathcal{Q}|_{H^*_{-2j-3}\rightarrow H^*_{-2j-1}})=Im(\mathcal{Q}|_{H^*_{-2j-5}\rightarrow H^*_{-2j-3}})$$

Summing up over this complex we obtain:

$$\chi_{\ H^*_{-2l-1,\ 0}}(p)=\int\limits_{|x|>1}\prod_{j\geq 1}(1+p^{2j-1}x)(1+p^{2j-1}x^{-1})x^{-l}\frac{x}{x+1}\frac{dx}{x}$$

Hence

$$
\begin{aligned}
&\chi_{\ H^*_{-2m-1,\ 0}}(p)-\chi_{\ H^*_{-2m-5,\ 0}}(p)= \\
&=\int\limits_{|x|>1}\prod_{j\geq 1}(1+p^{2j-1}x)(1+p^{2j-1}x^{-1})x^{-m}\frac{x}{x+1}\frac{x^2-1}{x^2}\frac{dx}{x}= \\
&=\int\prod_{j\geq 1}(1+p^{2j-1}x)(1+p^{2j-1}x^{-1})x^{-m}\frac{dx}{x}-\int\prod_{j\geq 1}(1+p^{2j-1}x)(1+p^{2j-1}x^{-1})x^{-m-1}\frac{dx}{x}= \\
&=(1-p^{2m+1})\chi_{\ H^*_{-2m-1}}(p)=p^{m^2}(1-p^{2m+1})\frac{1}{\prod_{j\geq 1}(1-p^{2j})}
\end{aligned}
$$

Thus the character is given by

$$\chi_{2m}(p)=(1-p^{2m+1})\frac{1}{\prod_{j\geq 1}(1-p^j)}$$

as it should be. ∎

4 Classical case.

4.1 Local fields and null-vectors in the classical theory.

The classical limit of the light-cone component T_{--} of the energy-momentum tensor gives the KdV field $u(x_-)$. When working with the multi-time formalism we shall identify x_- with t_1. Local fields in the KdV theory, descendents of the identity operators, are simply polynomials in $u(t)$ and its derivatives with respect to t_1:

$$\mathcal{O}\quad=\quad\mathcal{O}(u,u',u'',...) \tag{32}$$

We shall use both notations ∂_1 and $'$ for the derivatives with respect to $x_-=t_1$.

Instead of the variables $u,u',u'',...$, it will be more convenient to replace the odd derivatives of u by the higher time derivatives $\partial_{2k-1}u$, according to the equations of motion

$$\frac{\partial L}{\partial t_{2k-1}}\quad=\quad\left[\left(L^{\frac{2k-1}{2}}\right)_+,L\right]=u^{(2k-1)}+\cdots$$

Here

$$L=\partial_1^2-u$$

is the Lax operator of KdV. We have used the pseudo-differential operator formalism. We follow the book [13].

The even derivatives of $u(x)$ will be replaced by the densities S_{2k} of the local integrals of motion,

$$S_{2k}\quad=\quad res_{\partial_1}L^{\frac{2k-1}{2}}=-\frac{1}{2^{2k-1}}u^{(2k-2)}+\cdots,$$

In particular on level 2 we have $S_2=-\frac{1}{2}u$. For a reader who prefers the τ-function language $S_{2k}=\partial_1\partial_{2k-1}\log\tau$.

From analogy with the conformal case we put forward the following main conjecture underlying the classical picture:

Conjecture. *We can write any local fields as*

$$\mathcal{O}(u,u',u'',...)\quad=\quad F_{\mathcal{O},0}(S_2,S_4,\cdots)+\sum_{\nu>1}\partial^\nu F_{\mathcal{O},\nu}(S_2,S_4,\cdots) \tag{33}$$

where $\nu=(i_1,i_3,\cdots)$ is a multi index, $\partial^\nu=\partial_1^{i_1}\partial_3^{i_3}\cdots$, $|\nu|=i_1+3i_3+\cdots$.

We have checked this conjecture up to very high levels. To see that this conjecture is a non trivial one, let us compute the character of the space of local fields eq.(32). Attributing the degree 2 to u and 1 to ∂_1, we find that

$$\chi_1\quad=\quad\prod_{j>2}\frac{1}{1-p^j}=(1-p)\prod_{j>1}\frac{1}{1-p^j}=1+p^2+p^3+2p^4+2p^5+\cdots$$

On the other hand the character of the elements in the right hand side of eq.(33) is

$$\chi_2 \quad = \quad \prod_{j>1} \frac{1}{1-p^{2j-1}} \prod_{j>1} \frac{1}{1-p^{2j}} = \prod_{j>1} \frac{1}{1-p^j} = 1 + p + 2p^2 + 3p^3 + 5p^4 + 7p^5 + \cdots$$

Hence $\chi_1 < \chi_2$, and this is precisely why null-vectors exist. Let us give some examples of null-vectors

$$\begin{array}{lcl} \textit{level } 1 & : & \partial_1 \cdot 1 = 0 \\ \textit{level } 2 & : & \partial_1^2 \cdot 1 = 0 \\ \textit{level } 3 & : & \partial_1^3 \cdot 1 = 0, \quad \partial_3 \cdot 1 = 0 \\ \textit{level } 4 & : & \partial_1^4 \cdot 1 = 0, \quad \partial_1 \partial_3 \cdot 1 = 0, \quad (\partial_1^2 S_2 - 4S_4 + 6S_2^2) \cdot 1 = 0 \\ \textit{level } 5 & : & \partial_1^5 \cdot 1 = 0, \quad \partial_1^2 \partial_3 \cdot 1 = 0, \quad \partial_5 \cdot 1 = 0, \\ & & \partial_1(\partial_1^2 S_2 - 4S_4 + 6S_2^2) \cdot 1 = 0, \quad (\partial_3 S_2 - \partial_1 S_4) \cdot 1 = 0 \end{array} \tag{34}$$

We have written all the null-vectors explicitly to show that their numbers exactly match the character formulae. The non trivial null-vector at level 4 expresses S_4 in terms of the original variable u: $4S_4 = -\frac{1}{2}u'' + \frac{3}{2}u^2$. With this identification the non-trivial null-vector at level 5, $\partial_3 S_2 - \partial_1 S_4$, gives the KdV equation itself

$$\partial_3 u + \frac{3}{2} u u' - \frac{1}{4} u''' = 0$$

More generally one can consider the descendents of the fields $e^{im\varphi}$ where φ is related to u by the Miura transformation

$$u = -\varphi'^2 + i\varphi''$$

Here, the presence of i is a matter of convention. The reality problems have been discussed at length in [2].

For this consideration and for other purposes we need certain information about the Baker-Akhiezer function. The Baker-Akhiezer function $w(t, A)$ is a solution of the equation

$$Lw(t, A) = A^2 w(t, A) \tag{35}$$

which admits an asymptotic expansion at $A = \infty$ of the form

$$w(t, A) = e^{\zeta(t,A)}(1 + 0(1/A)); \quad \zeta(t, A) = \sum_{k \geq 1} t_{2k-1} A^{2k-1}$$

In these formulae, higher times are considered as parameters. The second solution of equation (35), denoted by $w^*(t, A)$, has the asymptotics

$$w^*(t, A) = e^{-\zeta(t,A)}(1 + 0(1/A))$$

These definitions do not fix completely the Baker-Akhiezer functions since we can still multiply them by constant asymptotic series of the form $1 + O(1/A)$. Since normalizations will be important to us, let us give a more precise definition. We first introduce the dressing operator

$$L = \Phi \partial_1^2 \Phi^{-1}; \quad \Phi = 1 + \sum_{i>1} \Phi_i \partial_1^{-i}$$

and we define

$$w(t, A) = \Phi e^{\zeta(t,A)}, \quad w^*(t, A) = (\Phi^*)^{-1} e^{-\zeta(t,A)}$$

where $\Phi^* = 1 + \sum_{i>1} (-\partial_1)^i \Phi_i$ is the formal adjoint of Φ.

Proposition 10. *With the above definitions, one has*
1) The wronskian $W(A) = w(t, A) w^*(t, A)' - w^*(t, A) w(t, A)'$ takes the value

$$W(A) = 2A$$

2) The generating function of the local densities $S(A) = 1 + \sum_{k>0} S_{2k} A^{-2k}$ *is related to the Baker-Akhiezer function by*

$$S(A) = w(t,A) w^*(t,A)$$

3) The function $S(A)$ *satisfies the Ricatti equation*

$$2S(A)S(A)'' - (S(A)')^2 - 4uS(A)^2 - 4A^2S(A)^2 + 4A^2 = 0 \tag{36}$$

Proof.
Let us prove the wronskian identity. This amounts to showing that $res_A(W(A)A^i) = 2\delta_{i,-2}$. But we have

$$res_A\Big(W(A)A^i\Big) = res_A\left\{\Big(\partial_1\Phi\partial_1^i e^{\zeta(t,A)}\Big)\Big((\Phi^*)^{-1}e^{-\zeta(t,A)}\Big) - \Big(\Phi e^{\zeta(t,A)}\Big)\Big(\partial_1(\Phi^*)^{-1}(-\partial_1)^i e^{-\zeta(t,A)}\Big)\right\}$$

We can transform the residue in A in a residue in ∂_1 using the formula

$$res_A\left\{\Big(Pe^{\zeta(t,A)}\Big)\cdot\Big(Qe^{-\zeta(t,A)}\Big)\right\} = res_{\partial_1}\Big(PQ^*\Big)$$

Hence we find

$$res_A\Big(W(A)A^i\Big) = res_{\partial_1}\Big\{\partial_1\Phi\partial_1^i\Phi^{-1} + \Phi\partial_1^i\Phi^{-1}\partial_1\Big\} = res_{\partial_1}\Big\{\partial_1 L^{\frac{i}{2}} + L^{\frac{i}{2}}\partial_1\Big\}$$

If i is even positive, the residue is zero because the $L^{\frac{i}{2}}$ is a purely differential operator. If $i = -2$ the residue is obviously 2, and if $i < -2$ it is zero. If i is odd, then $(L^{\frac{i}{2}})^* = -L^{\frac{i}{2}}$ so that the operator $\partial_1 L^{\frac{i}{2}} + L^{\frac{i}{2}}\partial_1$ is formally self-adjoint and it cannot have a residue.

The proof of 2) is simple [13]

$$res_A\Big(A^{2k-1}w(t,A)w^*(t,A)\Big) = res_{\partial_1}\Big(\Phi\partial_1^{2k-1}\Phi^{-1}\Big) = res_{\partial_1}\Big(L^{\frac{2k-1}{2}}\Big) = S_{2k}$$

The Ricatti equation follows immediately from 1),2) and eq.(35). ∎

Let us return to the descendents of the primary fields. For the descendents of the fields $e^{im\varphi}$, our conjecture states that

$$\mathcal{O}(u,u',u'',...)e^{im\varphi} \quad = \quad \sum_{\nu\geq 0}\partial^\nu\left(F_{\mathcal{O},\nu}(S_2,S_4,\cdots)e^{im\varphi}\right) \tag{37}$$

Let us consider for example $e^{i\varphi}$. For a true solution of the KdV equation, the Baker-Akhiezer function is a true function on the spectral curve, and it can be analytically continued at $A=0$. From the definition of $e^{i\varphi}$ we have $Le^{i\varphi} = 0$. Comparing with eq.(35), we see that $e^{i\varphi} = w(t,A)|_{A=0}$. To check eq.(37), at least on the first few levels, we need the time derivatives of $e^{i\varphi}$. They can be obtained as follows. The time evolution of the Baker-Akhiezer function is well known.

$$\frac{\partial w}{\partial t_{2k-1}} = \Big(L^{\frac{2k-1}{2}}\Big)_+ w$$

By analytical continuation at $A=0$, we obtain the evolution equations for $e^{i\varphi}$.

Let us give some examples of these null vectors. We show below the first null vector associated to the primary fields $e^{im\varphi}$.

$$\begin{aligned}
m=1 &\quad:\quad (\partial_1^2 + 2S_2)e^{i\varphi} = 0\\
m=2 &\quad:\quad (2\partial_3 + \partial_1^3 + 6\partial_1 S_2)e^{2i\varphi} = 0\\
m=3 &\quad:\quad (8\partial_1\partial_3 + \partial_1^4 + 12\partial_1^2 S_2 + 24S_4)e^{3i\varphi} = 0\\
m=4 &\quad:\quad (24\partial_5\partial_1 + 20\partial_3\partial_1^3 + \partial_1^6 + 20\partial_1^4 S_2 + 40\partial_3\partial_1 S_2 + 120\partial_1^2 S_4)e^{4i\varphi} = 0
\end{aligned}$$

In these formulae, the derivatives act on everything on their right i.e. $\partial_1 S_2 e^{2i\varphi} = \partial_1(S_2 e^{2i\varphi})$.

4.2 Finite-zone and soliton solutions.

For the finite-zone solutions, the Baker-Akhiezer function is an analytical function on the spectral curve which is an algebraic Riemann surface. Let us recall briefly the construction [14, 15].

We start with an hyperelliptic curve Γ of genus n described by the equation

$$\Gamma : Y^2 \quad = \quad X\mathcal{P}(X), \quad \mathcal{P}(X) = \prod_{j=1}^{2n}(X - B_j^2), \quad B_{2n} > \cdots > B_2 > B_1 > 0$$

For historical reasons we prefer to work with the parameter A such that $X = A^2$. The surface is realized as the A-plane with cuts on the real axis over the intervals $c_i = (B_{2i-1}, B_{2i})$ and $\overline{c}_i = (-B_{2i}, -B_{2i-1})$, $i = 1, \cdots, n$, the upper (lower) bank of c_i is identified with the upper (lower) bank of $\overline{c}_i$. The square root $\sqrt{\mathcal{P}(A^2)}$ is chosen so that $\sqrt{\mathcal{P}(A^2)} \to A^{2n}$ as $A \to \infty$. The canonical basis of cycles is chosen as follows: the cycle a_i starts from B_{2i-1} and goes in the upper half-plane to $-B_{2i-1}$, b_i is an anti-clockwise cycle around the cut c_i.

Let us consider in addition a divisor of order n on the surface Γ:

$$\mathcal{D} = (P_1, \cdots, P_n)$$

With these data we construct the Baker-Akhiezer function which is the unique function with the following analytical properties:

- It has an essential singularity at infinity: $w(t, A) = e^{\zeta(t,A)}(1 + O(1/A))$.
- It has n simple poles outside infinity. The divisor of these poles is $\mathcal{D}$.

Considering the quantity $-\partial_1^2 w + A^2 w$, we see that it has the same analytical properties as w itself, apart for the first normalization condition. Hence, because w is unique, there exists a function $u(t)$ such that

$$-\partial_1^2 w + u(t)w + A^2 w = 0 \tag{38}$$

We recognize eq.(35). One can give various explicit constructions of the Baker-Akhiezer function. Let us introduce the divisor $\mathcal{Z}(t)$ of the zeroes of the Baker-Akhiezer function. It is of degree n:

$$\mathcal{Z}(t) = (A_1(t), \cdots, A_n(t))$$

The equations of motion for the divisor $\mathcal{Z}(t)$ read [15].

$$\partial_1 A_i(t) = \frac{\sqrt{\mathcal{P}(A_i^2(t))}}{\prod_{j\neq i}(A_i^2(t) - A_j^2(t))} \tag{39}$$

The normalization of the Baker-Akhiezer function corresponds to a particular choice of the divisor of its poles $\mathcal{D}$. Later we shall specify the divisor which corresponds to the normalization of the Baker-Akhiezer function which was required in the previous subsection, for the moment we give a formula in which the normalization is irrelevant. Consider two sets of times t and $t^{(0)}$, differing only by the value of t_1. Then we can write

$$\frac{w(t, A)}{w(t^{(0)}, A)} = \sqrt{\frac{Q(A^2, t)}{Q(A^2, t^{(0)})}} \exp\left(\int_{t_1^{(0)}}^{t_1} \frac{A\sqrt{\mathcal{P}(A^2)}}{Q(A^2, t)} dt_1\right) \tag{40}$$

where the polynomial $Q(A^2, t)$ is defined as $Q(A^2, t) = \prod_i (A^2 - A_i^2(t))$.

The ratio of two dual Baker-Akhiezer functions $\frac{w^*(t,A)}{w^*(t^{(0)},A)}$ is obtained by applying the hyperelliptic involution. This amounts to the reflection $A \to -A$ in eq.(40). Let us prove the following simple proposition.

Proposition 11. *For the Baker-Akhiezer functions $w(t, A)$, $w^*(t, A)$ normalized by*

$$w(t, A)w^*(t, A)' - w^*(t, A)w(t, A)' = 2A$$

we have

$$S(A) = \frac{Q(A^2)}{\sqrt{\mathcal{P}(A^2)}} \equiv \exp\left(-\sum_k \frac{1}{k} J_{2k} A^{-2k}\right) \tag{41}$$

the latter equality is the definition of J_{2k}*. We recall that* $Q(A^2)$ *and* $\mathcal{P}(A^2)$ *are the polynomials*

$$Q(A^2) = \prod_{i=1}^{n}(A^2 - A_i^2), \quad \mathcal{P}(A^2) = \prod_{i=1}^{2n}(A^2 - B_i^2)$$

Proof.
To prove the proposition we use the Wronskian identity

$$w(t,A)w^*(t,A)' - w^*(t,A)w(t,A)' = w(t,A)w^*(t,A)\partial_1\left(\log\frac{w(t,A)}{w^*(t,A)}\right) = 2A$$

but using eq.(40) and the fact that $\frac{w(t,A)}{w(t^{(0)},A)}$ and $\frac{w^*(t,A)}{w^*(t^{(0)},A)}$ differ by the sign of the square root, we have

$$\partial_1\left(\log\frac{w(t,A)}{w^*(t,A)}\right) = 2\frac{A\sqrt{\mathcal{P}(A^2)}}{Q(A^2)}$$

and the result follows. ∎

Notice that for J_{2k} defined in (41) we have

$$J_{2k} = \sum_i A_i^{2k} - \frac{1}{2}\sum_i B_i^{2k}$$

From eq.(41) we see that the normalization of $w(t,A)$ and $w^*(t,A)$ which corresponds to the proper value of the Wronskian is such that the divisors $\mathcal{D}$ and $\mathcal{D}^*$ are composed of Weierstrass points and $\mathcal{D}+\mathcal{D}^* = (B_1,\cdots,B_{2n})$. Actually, it is this quite unique normalization which was used by Akhiezer in his original paper.

Now we are in position to describe the dynamics of $S(A)$ with respect to all times. It is very useful to define the following strange object

$$dI(D) = \sum_{k\geq 1} D^{-2k}\frac{\partial}{\partial t_{2k-1}}dD \tag{42}$$

$dI(D)$ is a 1-form in the D-plane and a vector field with respect to times. We have

Proposition 12.

$$dI(D)\cdot S(A) = \frac{S(D)S(A)' - S(A)S(D)'}{D^2 - A^2}dD \tag{43}$$

Proof.
We give a proof of this proposition for the finite zone solutions, which are our main concern here, but clearly the formula is quite general. We are sure that a general proof of equation (43) exists, but it must be based on manipulations with asymptotic formulae. We prefer to work with analytical functions. Anyway, every solution of KdV can be obtained from the finite-zone ones by a suitable limiting procedure, so considering finite-zone solutions is not a real restriction.

Let us describe the motion, under the time t_l, of the divisor $\mathcal{Z}(t)$ of the zeroes of the Baker-Akhiezer function. Introduce the normalized holomorphic differentials $d\omega_i$ for $i = 1,\cdots,n$ and the normalized second kind differentials with singularity at infinity $d\widetilde{\omega}_{2i-1}$, $i \geq 1$

$$\int_{a_j} d\omega_i = \delta_{i,j}$$

$$\int_{a_j} d\widetilde{\omega}_{2i-1} = 0, \qquad d\widetilde{\omega}_{2i-1}(A) = d(A^{2i-1}) + \mathcal{O}(A^{-2})dA \; for \; A\sim\infty$$

It is well known that, by the Abel map, this motion is transformed into a linear flow on the Jacobi variety.

$$\frac{\partial}{\partial t_{2l-1}}\sum_j \int^{A_j} d\omega_k = \int_{b_k} d\widetilde{\omega}_{2l-1}$$

Form this equation one easily finds:

$$dI(D)\cdot\sum_j\int^{A_j}d\omega_k=\int_{b_k}d\widetilde{\omega}_D \tag{44}$$

where $d\widetilde{\omega}_D$ is a 2-differential defined $\Gamma\times\Gamma_0$ (Γ_0 is the Riemann sphere) parametrized by A and D respectively. It is useful to think of Γ_0 as a realization of the curve $Y^2 = X$ similar to Γ. The a-periods of $d\widetilde{\omega}_D$ on Γ vanish. The only singularities of the differential $d\widetilde{\omega}_D$ are the second order poles at the two points $A = \pm D$:

$$d\widetilde{\omega}_D(A)=\Big(\frac{A^2+D^2}{(A^2-D^2)^2}+\mathcal{O}(1)\Big)dA\,dD$$

By Riemann's bilinear relations one easily finds:

$$\int_{b_k}d\widetilde{\omega}_D=d\omega_k(D)$$

Eq.(44) then takes the form

$$dI(D)\cdot\sum_j\int^{A_j}d\omega_k=d\omega_k(D) \tag{45}$$

The normalized differentials are linear combinations of the differentials

$$d\sigma_k(A)=\frac{A^{2k-2}}{\sqrt{\mathcal{P}(A^2)}}dA,\quad k=1\cdots n$$

with coefficients depending on B_i. They do not depend on times. Hence by linearity we can write for $d\sigma_k$ the same equation as (45). Differentiating explicitly we get the following system of equations

$$\sum_{j=1}^n\frac{A_j^{2k-2}}{\sqrt{\mathcal{P}(A_j^2)}}dI(D)\cdot A_j=\frac{D^{2k-2}}{\sqrt{\mathcal{P}(D^2)}}dD;\qquad k=1\cdots n$$

Solving this linear system of equations gives

$$dI(D)\cdot A_j^2=A_j\,\frac{Q(D^2)}{\sqrt{\mathcal{P}(D^2)}}\frac{1}{D^2-A_j^2}\frac{\sqrt{\mathcal{P}(A_j^2)}}{\prod_{i\neq j}(A_j^2-A_i^2)}dD=\frac{Q(D^2)}{\sqrt{\mathcal{P}(D^2)}}\frac{1}{D^2-A_j^2}\partial_1A_j^2\;dD$$

where we have used eq.(39) in the last step. Finally, using eq.(41) we find

$$dI(D)\cdot S(A)=S(D)S(A)\sum_j\frac{1}{D^2-A_j^2}\frac{1}{A^2-A_j^2}\partial_1(A_j^2)dD=\frac{1}{D^2-A^2}S(D)S(A)\left(\log\frac{S(D)}{S(A)}\right)'\;dD$$

■

The soliton solutions correspond to a rational degeneration of the finite-zone solutions such that

$$B_{2j-1}\to\widetilde{B}_j\leftarrow B_{2j},\qquad j=1,\cdots,n$$

The points of the divisor A_i and the points $\widetilde{B}_j$ are the coordinates in the n-solitons phase space. In [2] we gave a detailed discussion of the Hamiltonian structure. In particular we have

$$\frac{\partial}{\partial t_{2k-1}}=\{I_{2k-1},\ \cdot\};\quad I_{2k-1}=\frac{1}{2k-1}\sum_i\widetilde{B}_i^{2k-1}$$

The expressions for the local observables also follow easily from the finite-zone case. In particular

$$J_{2k}(A,\widetilde{B})=\sum_{i=1}^n\Big(A_i^{2k}-\widetilde{B}_i^{2k}\Big) \tag{46}$$

The expressions for S_{2k} follow from here.

4.3 Classical limit of $\mathcal{Q}$ and $\mathcal{C}$.

Let us consider the classical limit of the operators $\mathcal{Q}$ and $\mathcal{C}$. To this aim, we have to understand the relation between the quantum and the classical descriptions of the observables. In the quantum case we considered the form factors i.e. matrix elements of the form

$$f_{\mathcal{O}}(\beta_1, \cdots, \beta_{2n})_{-\cdots-+\cdots+} = \langle 0|\mathcal{O}(0)|\beta_1, \cdots, \beta_n;\ \beta_{n+1}, \cdots, \beta_{2n}\rangle$$

where $\beta_1, \cdots, \beta_n$ are rapidities of anti-solitons, $\beta_{n+1}, \cdots, \beta_{2n}$ are rapidities of solitons. The matrix elements of this form do not allow a direct semi-classical interpretation, it is necessary to perform a crossing transformation to the matrix elements between two n-soliton states:

$$\langle\beta_1, \cdots, \beta_n|\mathcal{O}(0)|\beta_{n+1}, \cdots, \beta_{2n}\rangle = f_{\mathcal{O}}(\beta_1 - \pi i, \cdots, \beta_n - \pi i, \beta_{n+1}, \cdots, \beta_{2n})_{-\cdots-+\cdots+}$$

In [2] it is explained that the formula (2) for this form factor is a result of quantization of n-soliton solutions in which $A_1, \cdots, A_n$ play the role of coordinates, $B_1, \cdots, B_n$ and $B_{n+1}, \cdots, B_{2n}$ give the collection of eigenvalues for two eigenstates.

Recall that the generating function for the local descendents of the primary field Φ_m was written as follows

$$\mathcal{L}_m(t, y|A|B) = \exp\Big(\sum_{k\geq 1} t_{2k-1}I_{2k-1}(B) + y_{2k}J_{2k}(A|B)\Big)\ \left(\prod_{i=1}^{n} A_i^m \prod_{j=1}^{2n} B_j^{-\frac{m}{2}}\right) \tag{47}$$

where $I_{2k-1}(B)$ and $J_{2k}(A|B)$ are defined in eqs.(4,5). The expression

$$\exp\Big(\sum_{k\geq 1} y_{2k}J_{2k}(A|B)\Big)\left(\prod_{i=1}^{n} A_i^m \prod_{j=1}^{2n} B_j^{-\frac{m}{2}}\right)$$

is practically unchanged under the crossing transformation which corresponds to $B_i \to -B_i$ for $i = 1, \cdots, n$. Comparing it with the classical formulae (46) we see that it corresponds to special symmetric ordering of them, for example

$$J_{2k}(A|B) = \frac{1}{2}\Big(J_{2k}(A_1, \cdots, A_n, B_1, \cdots, B_n) + J_{2k}(A_1, \cdots, A_n, B_{n+1}, \cdots, B_{2n})\Big)$$

This ordering is a prescription which we make for the quantization.

On the other hand the eigenvalues of the Hamiltanians $I_{2k-1}(B)$ under crossing transformation change to

$$-I_{2k-1}(B_1, \cdots, B_n) + I_{2k-1}(B_{n+1}, \cdots, B_{2n})$$

i.e. the descendents with respect to I_{2k-1} correspond to taking commutator of $\mathcal{O}$ with I_{2k-1}. Certainly the classical limit makes sense only for the states with close eigenvalues, so, it is needed that

$$s_{2k-1}(B_1, \cdots, B_n) - s_{2k-1}(B_{n+1}, \cdots, B_{2n}) = \mathcal{O}(\xi)$$

Recall that $\xi = -i\log(q)$ plays the role of Plank's constant.

Thus comparing the classical and quantum pictures provides the following result. The quantum generating function (47) corresponds to the classical generating function

$$\mathcal{L}^{cl}_m(t, y) = \exp\Big(\sum_{k\geq 1} t_{2k-1}I_{2k-1}\Big) \cdot \exp\Big(\sum_{k\geq 1} y_{2k}J_{2k}\Big)\ e^{im\varphi} \tag{48}$$

where $\cdot$ means the application of Poisson brackets. In fact I_{2k-1} can be replaced by ∂_{2k-1}. The normalization in the formula for $I(B)$ (4) is chosen in order to provide an exact agreement with the classical formulae.

Now let us consider the classical limit of the operators $\mathcal{Q}$ and $\mathcal{C}$ in the Neveu-Schwarz sector. For $\mathcal{Q}$ we had the formula

$$\mathcal{Q} = \int \frac{dD}{D} e^{X(D)}\psi(D) = \int \frac{dD}{D}\sinh(X(D))\psi(D)$$

The latter equation is due to the fact that the fermion is odd. From the definition (21) of $X(D)$, we have in the classical limit

$$X(D) \to -i\xi \sum_{k\geq 1} D^{-2k+1} I_{2k-1}$$

Hence the following expression is finite in the classical limit:

$$\mathcal{Q}_{cl} = \lim_{\xi\to 0} \frac{\mathcal{Q}}{i\xi} = \int \psi(D) dI(D) \tag{49}$$

where $dI(D)$ is the 1-form in D-plane introduced in the previous subsection: $dI(D) = \sum\limits_{k\geq 1} D^{-2k} I_{2k-1} dD$. Remark that $\mathcal{Q}_{cl}$ can be thought of as a generalized Dirac operator.

For $\mathcal{C}$ we had the formula

$$\begin{aligned}
\mathcal{C} = &\int\limits_{|D_2|>|D_1|} \frac{dD_2}{D_2} \int \frac{dD_1}{D_1} e^{X(D_1)} e^{X(D_2)} \tau_e\left(\frac{D_1}{D_2}\right) \psi(D_1)\psi(D_2) = \\
= &\int\limits_{|D_2|>|D_1|} \frac{dD_2}{D_2} \int \frac{dD_1}{D_1} \cosh(X(D_1)) \cosh(X(D_2)) \tau_e^-\left(\frac{D_1}{D_2}\right) \psi(D_1)\psi(D_2) + \\
+ &\int\limits_{|D_2|>|D_1|} \frac{dD_2}{D_2} \int \frac{dD_1}{D_1} \sinh(X(D_1)) \sinh(X(D_2)) \tau_e^+\left(\frac{D_1}{D_2}\right) \psi(D_1)\psi(D_2)
\end{aligned}$$

where τ_e^+ and τ_e^- are even and odd parts of τ_e:

$$\tau_e^-(x) = \sum_{k=1}^{\infty} \frac{1-q^{2k-1}}{1+q^{2k-1}}\, x^{2k-1}, \qquad \tau_e^+(x) = -\sum_{k=1}^{\infty} \frac{1+q^{2k}}{1-q^{2k}}\, x^{2k}$$

Obviously when $\xi \to 0$ one has

$$\tau_e^-(x) \to -i\xi\, x \frac{d}{dx}\left(\frac{x}{1-x^2}\right), \quad \tau_e^+(x) \to (i\xi)^{-1} \log(1-x^2)$$

So, the following expression is finite in the classical limit

$$\begin{aligned}
&\mathcal{C}_{cl} = \lim_{\xi\to 0} \frac{\mathcal{C}}{i\xi} = \\
&= \int \psi(D) \frac{d}{dD}\psi(D) dD + \frac{1}{2\pi i} \int\limits_{|D_2|>|D_1|} dI(D_2) \int dI(D_1) \log\left(1 - \left(\frac{D_1}{D_2}\right)^2\right) \psi(D_1)\psi(D_2)
\end{aligned} \tag{50}$$

In the next subsection we are going to apply these operators to description of the classical KdV hierarchy. Notice that as usual the quantum formulae are far more symmetric than the classical ones.

4.4 The classical equations of motion from $\mathcal{Q}_{cl}$ and $\mathcal{C}_{cl}$.

In this subsection we shall consider only the descendents of the identity, i.e. the pure KdV fields. We have described this space by the generating function (48):

$$\mathcal{L}^{cl}_{m=0}(t,y) = \exp\Bigl(\sum_{k\geq 1} t_{2k-1} I_{2k-1}\Bigr) \cdot \exp\Bigl(\sum_{k\geq 1} y_{2k} J_{2k}\Bigr) \cdot\ 1,$$

Let us fermionize J_{2k} and apply the equations

$$\langle \Psi_{-3}|\ \mathcal{Q}_{cl} \overset{w}{=} 0, \qquad \langle \Psi_{-5}|\ \mathcal{C}_{cl} \overset{w}{=} 0 \tag{51}$$

to the description of the equations of motion. In this section the symbol $\overset{w}{=} 0$ means the vanishing of the scalar product with the generating of local fields.

We give the list of null-vectors following from these two equations up to the level 5 specifying explicitly the vectors $\langle \Psi_{-3}|$ and $\langle \Psi_{-5}|$ from which they come. We do not write the descendents with respect to I's of

the already listed null vectors:
Null vectors coming from $\mathcal{Q}_{cl}$:

$$\begin{aligned} \langle -1|\psi^*_{-1} &\ :\ \partial_1 \cdot 1 \\ \langle -1|\psi^*_{-3} &\ :\ \partial_3 \cdot 1 \\ \langle -1|\psi^*_{-5} &\ :\ \partial_5 \cdot 1 \\ \langle -1|\psi^*_{-3}\psi^*_{-1}\psi_1 &\ :\ (-\partial_3 S_2 + \partial_1 S_4)\cdot 1 \end{aligned}$$

Null vectors coming from $\mathcal{C}_{cl}$:

$$\langle -1|\psi^*_{-3}\psi^*_{-1} \ :\ (\partial_1^2 S_2 - 4S_4 + 6S_2^2 + \frac{1}{2}\partial_1\partial_3)\cdot 1$$

Obviously these null-vectors coincide with (34). So, in particular, equations (51) imply the KdV equation itself. We have verified that the null-vectors coincide with those obtained from the Gelfand-Dickey construction up to level 16. On higher levels we find higher equations of KdV hierarchy.

We have seen that the KdV equation follows from the equations (51). Let us prove the opposite: equations (51) hold on any solution of KdV. We start with the operator $\mathcal{Q}_{cl}$.

Proposition 13. *Let*

$$\mathcal{Q}_{cl} \;=\; \int \psi(D)dI(D) = \sum_{k\geq 1}\psi_{-2k+1}\frac{\partial}{\partial_{t_{2k-1}}}$$

Then if J_{2k} are constructed from a solution of KdV we have

$$\mathcal{Q}_{cl}\exp\left(-\sum_{k\geq 1}\frac{1}{k}J_{2k}h_{-2k}\right)|-1\rangle \;=\; 0$$

Proof.
Let us introduce the notation:

$$T = \exp\left(-\sum_{k\geq 1}\frac{1}{k}J_{2k}h_{-2k}\right) = \exp\left(\int\frac{dA}{A}\log S(A)h(A)\right)$$

where $h(A) = \sum_{k\geq 1} h_{2k}A^{2k}$. We have $\psi(D)\ T = T\ S^{-1}(D)\psi(D)$. So

$$\mathcal{Q}_{cl}\ T\ |-1\rangle \;=\; T\int\frac{1}{S(D)S(A)}\psi(D)h(A)\frac{dA}{A}(dI(D)\cdot S(A))|-1\rangle$$

We now use eq.(43) to get

$$\begin{aligned} &\int\frac{1}{S(D)S(A)}\psi(D)h(A)\frac{dA}{A}(dI(D)\cdot S(A)) = \\ &\quad = \int dD\frac{dA}{A}\frac{1}{D^2-A^2}\left[(\log S(A))' - (\log S(D))'\right]\psi(D)h(A)|-1\rangle \end{aligned}$$

But

$$\psi(D)h(A)|-1\rangle =: \psi(D)\psi(A)\psi^*(A) : |-1\rangle + \frac{AD}{D^2-A^2}\psi(A)|-1\rangle \quad |D|>|A|$$

Let us consider first the integral

$$\int_{|D|>|A|}\frac{dD}{D}\frac{dA}{A}\frac{D}{D^2-A^2}(\log S(A))'\left[:\psi(D)\psi(A)\psi^*(A): + \frac{AD}{D^2-A^2}\psi(A)\right]|-1\rangle$$

One can do the integral over D. Notice that the integrand is regular at $D=0$. Hence, the contributions to the integral come from the poles at $D^2 = A^2$. The simple pole does not contribute because its residue vanishes since we have the product of two fermion fields at the same point in the normal product. The double pole obviously does not contribute either, and the integral is zero. Next we look at the integral

$$\int_{|D|>|A|} \frac{dD}{D}\frac{dA}{A}\frac{D}{D^2-A^2}\left(\log S(D)\right)'\left[:\psi(D)\psi(A)\psi^*(A):+\frac{AD}{D^2-A^2}\psi(A)\right]|-1\rangle$$

This time one can do the integral over A. But it is clear that the integrand is regular at $A=0$, and the integral also vanishes. ∎

Let us consider now the operator $\mathcal{C}_{cl}$.

Proposition 14. *Let*

$$\mathcal{C}_{cl} \quad = \quad \int dD\psi(D)\frac{d}{dD}\psi(D)+\frac{1}{2\pi}\int_{|D_1|<|D_2|}\log\left(1-\frac{D_1^2}{D_2^2}\right)\psi(D_1)\psi(D_2)dI(D_1)dI(D_2)$$

then if J_{2k} are constructed from a solution to KdV

$$\mathcal{C}_{cl}\exp\left(-\sum_{k\geq 1}\frac{1}{k}J_{2k}h_{-2k}\right)|-1\rangle=0$$

Proof.
Let us split $\mathcal{C}_{cl}$ into two pieces $\mathcal{C}_{cl}=\mathcal{C}_1+\mathcal{C}_2$:

$$\begin{aligned}\mathcal{C}_1 &= \int\frac{dD}{D}\psi(D)D\frac{d}{dD}\psi(D)\\ \mathcal{C}_2 &= \frac{1}{2\pi i}\int_{|D_1|<|D_2|}\log\left(1-\frac{D_1^2}{D_2^2}\right)\psi(D_1)\psi(D_2)dI(D_1)dI(D_2)\end{aligned}$$

We use the same notation T as in the previous proposition. We treat first the $\mathcal{C}_2$ term

$$\begin{aligned}\mathcal{C}_2\ T\,|-1\rangle \quad &= \quad \frac{1}{2\pi i}\int dI(D_2)\psi(D_2)\ T\int_{|A_1|<|D_1|<|D_2|}\frac{dD_1}{D_1}\frac{dA_1}{A_1}\log\left(1-\frac{D_1^2}{D_2^2}\right)\frac{D_1}{D_1^2-A_1^2}\left(\log\frac{S(A_1)}{S(D_1)}\right)'\\ &\times\left[:\psi(D_1)\psi(A_1)\psi^*(A_1):+\frac{A_1D_1}{D_1^2-A_1^2}\psi(A_1)\right]|-1\rangle\end{aligned}$$

by the same argument as in the previous proof, we see that only the term $(\log S(A_1))'$ contributes. This time however, the double pole gives a non vanishing contribution

$$\frac{1}{2\pi i}\int_{|D_1|>|A_1|}dD_1\log\left(1-\frac{D_1^2}{D_2^2}\right)\frac{D_1}{(D_1^2-A_1^2)^2}=-\frac{1}{2}\frac{1}{D_2^2-A_1^2}$$

so that

$$\mathcal{C}_2\ T\,|-1\rangle \quad = \quad -\frac{1}{2}\int dI(D_2)\psi(D_2)\ T\int_{|D_2|>|A_1|}dA_1\frac{(\log S(A_1))'}{D_2^2-A_1^2}\psi(A_1)|-1\rangle$$

Commuting again $\psi(D_2)$ and T we finally get after having computed the integral over D_2

$$\mathcal{C}_2\ T\,|-1\rangle \quad = \quad T\int\frac{dA}{A^2}\left[\frac{1}{2}(\log S(A))''+\frac{1}{4}((\log S(A))')^2\right]\psi(A)\frac{d}{dA}\psi(A)|-1\rangle$$

The calculation of the $\mathcal{C}_1$ term is straightforward

$$\mathcal{C}_1\ T\,|-1\rangle \quad = \quad T\int dA\ S^{-2}(A)\ \psi(A)\frac{d}{dA}\psi(A)|-1\rangle$$

Combining the two terms and using the Ricatti equation (36)

$$2(\log S(A))''+((\log S(A))')^2+4A^2S^{-2}(A)=4(u+A^2)$$

we get

$$\mathcal{C}_{cl}\ T\ |-1\rangle = \int \frac{dA}{A^2}(u+A^2)\psi(A)\frac{d}{dA}\psi(A)|-1\rangle = 0$$

the latter integral vanishes since the integrand is regular. ∎

Thus we have shown that the equations (51) provide the complete description of the KdV hierarchy. We find it quite amazing. It is also interesting that this new description came from pure quantum considerations.

5 Connection with the Whitham method.

There is an surprising relation between the methods of this paper and the Whitham equations for KdV [16, 17, 18, 19]. The present section is devoted to the description of this relation.

Let us remind briefly what is the Whitham method about. Suppose we consider the solutions of KdV which are close to a given quasi-periodic solution. The latter is defined by the set of ends of zones $B_1^2, \cdots, B_{2n}^2$. We know that for the finite-zone solution the dynamics is linearized by the Abel transformation to the Jacobi variety of the hyper-elliptic surface $Y^2 = X\mathcal{P}(X)$ for $\mathcal{P}(X) = \prod(X - B_j^2)$. The idea of the Whitham method is to average over the fast motion over the Jacobi variety and to introduce "slow times" T_j which are related to the original KdV times as $T_j = \epsilon t_j$ ($\epsilon \ll 1$), assuming that the ends of zones B_j become functions of these "slow times" (recall that the ends of zones were the integrals of motion for the pure finite-zone solutions).

For the given finite-zone solution the observables can be written in terms of θ-functions on the Jacobi variety, but this kind of formulae is inefficient for writing the averages. One has to undo the Abel transformation, and to write the observables in terms of the divisor $\mathcal{Z} = (A_1, \cdots, A_n)$. The formulae for the observables are much more simple in these variables, and the averages can be written as abelian integrals, the Jacobian due to the Abel transformation is easy to calculate. The result of this calculation is as follows [17]. Every observable $\mathcal{O}$ can be written as an even symmetric function $L_{\mathcal{O}}(A_1, \cdots, A_n)$ (depending on B's as on parameters), and for the average we have

$$\langle\langle\ \mathcal{O}\ \rangle\rangle = \Delta^{-1} \int\limits_{a_1} \frac{dA_1}{\sqrt{\mathcal{P}(A_1^2)}} \cdots \int\limits_{a_n} \frac{dA_n}{\sqrt{\mathcal{P}(A_n^2)}} L_{\mathcal{O}}(A_1, \cdots, A_n) \prod_{i<j}(A_i^2 - A_j^2)$$

where

$$\Delta = det\left(\int\limits_{a_i} \frac{A^{2j-2}dA}{\sqrt{\mathcal{P}(A^2)}}\right)_{i,j=1,\cdots n}$$

The similarity of this formula with the formula for the form factors (1) is the first intriguing fact. Indeed, we have the following dictionary. For the local observables, we have

$$L_{\mathcal{O}} \Longleftrightarrow L_{\mathcal{O}}$$

For the weight of integration, we have

$$\frac{1}{\sqrt{\mathcal{P}(A)}} \Longleftrightarrow \prod_{j=1}^{2n} \psi(A, B_j)$$

But the most striking feature is that the cycles of integration are replaced by functions of $a_i = A_i^{2\nu}$

$$cycle\ a_i \Longleftrightarrow a_i^{-i}$$

The coincidence between the notations for a_i-variables and a_i-cycles is therefore not fortuitous. The explanation of the fact that the cycles are replaced by these functions is given in [2], where it was shown that the factor $\prod_i a_i^{-i}$ selects the classical trajectory in the semi-classical approximation of eq.(2). So, the solution of a non trivial, full fledged, quantum field theory has provided us with a very subtle definition of a *quantum* Riemann surface.

The main result of the Whitham theory is that the averaged equations of motion can be written in the following form:

$$\frac{\partial}{\partial T_{2p+1}} d\widetilde{\omega}_{2q+1}(A) = \frac{\partial}{\partial T_{2q+1}} d\widetilde{\omega}_{2p+1}(A)$$

where, as earlier, $d\widetilde{\omega}_{2q+1}, d\widetilde{\omega}_{2p+1}$ are normalized second kind differentials with prescribed singularity at infinity:

$$d\widetilde{\omega}_{2l+1}(A) = d(A^{2l+1}) + \mathcal{O}(A^{-2})dA, \qquad \int\limits_{a_j} d\widetilde{\omega}_{2l+1} = 0$$

In fact every equation of this type contains many partial differential equations for B_j because one can decompose it with respect to the parameter A. To our knowledge these equations have never been deduced directly from the averaging integrals except for the special case $p = 1, q = 2$ which was considered in the pioneering paper [17]. We want to show that the methods of this paper allow to do that, also we shall also explain how to describe other equations of the hierarchy.

As it has been shown in this paper the classical observables can be obtained from the generating function:

$$\exp(\sum_{k\geq 1} t_{2k-1}I_{2k-1}) \cdot \exp(\sum_{k\geq 1} y_{2k}J_{2k}) \tag{52}$$

In the Whitham case the KdV times become fast which means that $I_{2p-1} = \epsilon \widehat{I}_{2p-1}$ where $\widehat{I}_{2p-1}$ denotes the Poisson bracket with corresponding Hamiltonian of the averaged hierarchy. To keep the first multiplier in (52) finite one has to pass to the slow times T_{2k-1}. So, the observables for the averaged hierarchy are generated by

$$\langle\langle \mathcal{L}_0(T, y|B)\rangle\rangle = \exp(\sum_{k\geq 1} T_{2k-1}\widehat{I}_{2k-1}) \cdot \langle\langle \ \exp(\sum_{k\geq 1} y_{2k}J_{2k}) \ \rangle\rangle$$

Let us repeat that the average in this formulae depends only on B_j which are supposed to be functions of slow times, the multiplier with $\widehat{I}_{2k-1}$ corresponds to derivations with respect to the slow times.

The averaged formulae becomes really beautiful with the fermionic generating function:

$$\langle\langle \widehat{\mathcal{L}}_0(B)\rangle\rangle| - 1\rangle = \Delta^{-1}\ g(B)\ \langle\langle\psi^*\rangle\rangle_1 \cdots \langle\langle\psi^*\rangle\rangle_n| - 1 - 2n\rangle \tag{53}$$

where

$$\langle\langle\psi^*\rangle\rangle_j = \int\limits_{a_j} \frac{dA A^{2n-1}}{\sqrt{\mathcal{P}(A^2)}}\psi^*(A),$$

$g(B)$ is the same as in quantum case (16).

Remind that we have rewritten the equations of KdV hierarchy in the weak sense as follows

$$\langle\Psi_{-5}|\ \mathcal{C}_{cl} \overset{w}{=} 0, \qquad \langle\Psi_{-3}|\ \mathcal{Q}_{cl} \overset{w}{=} 0$$

For the averaged hierarchy these equations have to be replaced by

$$\langle\Psi_{-5}|\ \mathcal{C}_0 \overset{w}{=} 0, \qquad \langle\Psi_{-3}|\ \mathcal{Q}_0 \overset{w}{=} 0 \tag{54}$$

In this section, the symbol $\overset{w}{=} 0$ means vanishing of the matrix elements with the averaged generating function (53). The averaged $\mathcal{C}_0$ and $\mathcal{Q}_0$ are

$$\mathcal{C}_0 = \int \psi(D)\frac{d}{dD}\psi(D)dD, \qquad \mathcal{Q}_0 = \int \psi(D)d\widehat{I}(D)$$

where the terms with $I(D)$ are omitted in the definition of $\mathcal{C}_0$ comparing with $\mathcal{C}_{cl}$ (50) because $dI(D)$ is of order ϵ, the definition of $\mathcal{Q}_0$ is practically the same as in (49) because the latter is homogeneous in $dI(D)$. Let us explain the implications of the equations (54) for the averaged hierarchy.

Proposition 15. *The equation*

$$\langle\Psi_{-5}|\mathcal{C}_0 \overset{w}{=} 0$$

for the averaged hierarchy follows from the Riemann bilinear relation for hyper-elliptic integrals.

Proof.
The Riemann bilinear relation for the hyper-elliptic integrals is equivalent to the formula

$$\int\limits_{c_1} \frac{dA_1}{\sqrt{\mathcal{P}(A_1^2)}} \int\limits_{c_2} \frac{dA_2}{\sqrt{\mathcal{P}(A_2^2)}}\ C(A_1, A_2) = c_1 \circ c_2$$

where $c_1 \circ c_2$ is the intersection number of cycles, the anti-symmetric polynomial $C(A_1, A_2)$ is given by

$$C(A_1, A_2) = C'(A_1, A_2) - C'(A_2, A_1)$$

where

$$C'(A_1, A_2) = \sqrt{\mathcal{P}(A_1^2)}\frac{d}{dA_1}\left(\sqrt{\mathcal{P}(A_1^2)}\frac{A_1}{A_1^2 - A_2^2}\right)$$

(see, for example, [11] for a relevant discussion).

On the other hand we have

$$\mathcal{C}_0\ g(B) = g(B)\ \widehat{\mathcal{C}}_0$$

where

$$\widehat{\mathcal{C}}_0 = \int \mathcal{P}(D^2)D^{-4n}\psi(D)\frac{d}{dD}\psi(D)dD$$

Consider the formula for averaged observables (53). Suppose that we undo the averaging considering instead of

$$\langle \Psi_{-1}|\langle\langle\psi^*\rangle\rangle_1 \cdots \langle\langle\psi^*\rangle\rangle_n| - 2n - 1\rangle$$

the polynomial

$$\langle \Psi_{-1}|\psi^*(A_1)\cdots\psi^*(A_n)| - 2n - 1\rangle \prod_i A_i^{2n-1}$$

The averaging corresponds to integrating over closed cycles on the surface, so the latter polynomial is defined up to exact forms of the following kind

$$\sum_{i=1}^{n}(-1)^i\ M(A_1^2, \cdots, \widehat{A_i^2}, \cdots, A_n^2)\ \sqrt{\mathcal{P}(A_i^2)}\frac{d}{dA_i}\left(\sqrt{\mathcal{P}(A_i^2)}Q(A_i)\right)$$

where $Q(A)$ is an arbitrary polynomial and $M(A_1^2, \cdots, \cdots, A_{n-1}^2)$ is anti-symmetric.

It is clear that the statement of the proposition will be proven if we show that for every $\langle\Psi_{-5}|$ up to exact forms one has

$$\langle \Psi_{-5}|\widehat{\mathcal{C}}_0\psi^*(A_1)\cdots\psi^*(A_n)| - 1 - 2n\rangle \prod_j A_j^{2n-1} \simeq$$
$$\simeq \sum_{i<j}(-1)^{i+j}M(A_1^2, \cdots, \widehat{A_i^2}, \cdots, \widehat{A_j^2}, \cdots A_n^2)C(A_i, A_j) \quad (55)$$

for some anti-symmetric $M(A_1^2, \cdots, A_{n-2}^2)$.

Just like in the proof of Proposition 2 from Section 3 we have three possibilities for $\langle\Psi_{-5}|$ which can give a non-trivial result:

1. The depth of $\langle\Psi_{-5}|$ is greater than $-2n-1$
2. The vector $\langle\Psi_{-5}|$ is obtained from a vector $\langle\Psi_{-1}|$ whose depth is greater than $-2n-1$ by application of $\psi^*_{-2p-1}\psi^*_{-2q-1}$ with $q > p \geq n$ (i.e. there are two holes below $-2n-1$).
3. The vector $\langle\Psi_{-5}|$ is obtained from a vector $\langle\Psi_{-3}|$ whose depth is greater than $-2n-1$ by application of ψ^*_{-2p-1} with $p \geq n$ (i.e. there is one hole below $-2n-1$).

In the first case using the identity

$$2\int\limits_{|D|>|A_1|,|A_2|}\frac{\sqrt{\mathcal{P}(D^2)}}{D^2 - A_1^2}\frac{d}{dD}\left(\frac{\sqrt{\mathcal{P}(D^2)}}{D^2 - A_2^2}\right)DdD = C(A_1, A_2)$$

one easily gets the formula (55) with

$$M(A_1^2, \cdots, A_{n-2}^2) = \langle\Psi_{-5}|\psi^*(A_1)\cdots\psi^*(A_{n-2})| - 1 - 2n\rangle \prod_j A_j^{2n-1}$$

In the second case it is necessary that in the expression $\langle\Psi_{-1}|\psi^*_{-2p-1}\psi^*_{-2q-1}\ \widehat{\mathcal{C}}_0$ the operator $\widehat{\mathcal{C}}_0$ annihilates two holes. Hence one finds the integral

$$\langle\Psi_{-1}|2(p-q)\int \mathcal{P}(D^2)D^{-4n+2p+2q+1}dD = 0$$

It vanishes because $p, q \geq n$.
Finally, in the third case it is necessary that in the expression $\langle \Psi_{-3}|\psi^*_{-2p-1}\ \widehat{\mathcal{C}}_0$ the operator $\widehat{\mathcal{C}}_0$ annihilates the hole. This gives

$$\langle \Psi_{-3}| \int \left(\mathcal{P}(D^2) + D\frac{d}{dD}\mathcal{P}(D^2) \right) D^{-4n+2p}\psi(D)dD$$

So, in the matrix elements we shall find the polynomials

$$\int\limits_{|D|>|A_j|} \left(\mathcal{P}(D^2) + D\frac{d}{dD}\mathcal{P}(D^2) \right) D^{-4n+2p} \frac{1}{D^2 - A_j^2} dD = 2\sqrt{\mathcal{P}(A_j^2)} \frac{d}{dA_j} \left(\sqrt{\mathcal{P}(A_j^2)} A_j^{2p-2n+1} \right)$$

which corresponds to an exact form. ■

The similarity of the proof of this proposition with that of Proposition 2 of Section 3 is quite impressive.
Let us consider now the operator $\mathcal{Q}_0$. It is responsible for the equation of motion as shows the following proposition.

Proposition 16. *The equations*

$$\frac{\partial}{\partial T_{2p+1}} d\widetilde{\omega}_{2q+1}(A) = \frac{\partial}{\partial T_{2q+1}} d\widetilde{\omega}_{2p+1}(A) \tag{56}$$

follow from

$$\langle \Psi_{-3}|\mathcal{Q}_0 \overset{w}{=} 0 \qquad \langle \Psi_{-5}|\mathcal{C}_0 \overset{w}{=} 0 \tag{57}$$

Proof.
We will show that the Whitham equations (56) follow by considering the vectors

$$\langle \Psi_{-3}| = \langle -1|\psi^*_{-2p-1}\psi^*_{-2q-1}\psi_{2s+1}$$

The proof goes in two steps. First we shall show that the equation (57) implies that:

$$(2p+1)\widehat{I}_{2q+1}\langle -1|\psi_{2p+1}\psi^*(A) \ - \ (2q+1)\widehat{I}_{2p+1}\langle -1|\psi_{2q+1}\psi^*(A) \overset{w}{=} 0 \tag{58}$$

Indeed, applying $\mathcal{Q}_0$ to this vector $\langle \Psi_{-3}|$ gives

$$\langle -1|\psi^*_{-2p-1}\psi^*_{-2q-1}\psi_{2s+1}\ \mathcal{Q}_0 = \widehat{I}_{2p+1}\langle -1|\psi^*_{-2q-1}\psi_{2s+1} \ - \ \widehat{I}_{2q+1}\langle -1|\psi^*_{-2p-1}\psi_{2s+1} \overset{w}{=} 0$$

Now notice that

$$(2s+1)\langle -1|\psi^*_{-2p-1}\psi_{2s+1} = (2p+1)\langle -1|\psi^*_{-2s-1}\psi_{2p+1} \ + \ \langle -1|\psi^*_{-2s-1}\psi^*_{-2p-1}\mathcal{C}_0$$

Hence having in mind the equation $\langle \Psi_{-5}|\ \mathcal{C}_0 \overset{w}{=} 0$ one gets

$$(2p+1)\widehat{I}_{2q+1}\langle -1|\psi_{2p+1}\psi^*_{-2s-1} \ - \ (2q+1)\widehat{I}_{2p+1}\langle -1|\psi_{2q+1}\psi^*_{-2s-1} \overset{w}{=} 0$$

Since it is true for every s we can write it for the generating function as in the eq.(58).
The second step consists in computing the following average

$$\Delta^{-1}\langle -1|\psi_{2p+1}\psi^*(A)\ g(B)\ \langle\langle \psi^* \rangle\rangle_1 \cdots \langle\langle \psi^* \rangle\rangle_n| - 1 - 2n\rangle$$

Noticing that

$$\langle -1|\psi_{2p+1}\psi^*(A)\ g(B) = \langle -1| \int dD D^{-2n+2p}\sqrt{\mathcal{P}(D^2)}\frac{A^{2n}}{\sqrt{\mathcal{P}(A^2)}}\psi(D)\psi^*(A)$$

and calculating the matrix element in a usual way we get the answer:

$$\Delta^{-1}\langle -1|\psi_{2p+1}\psi^*(A)\ g(B)\ \langle\langle \psi^* \rangle\rangle_1 \cdots \langle\langle \psi^* \rangle\rangle_n| - 1 - 2n\rangle = \Delta^{-1} A\ det(M(A))$$

where $M(A)$ is $(n+1)\times(n+1)$ matrix with the following matrix elements

$$M(A)_{i,j} = \int_{a_j} \frac{D^{2(i-1)}}{\sqrt{\mathcal{P}(D^2)}} dD, \quad i,j=1,\cdots,n$$

$$M(A)_{i,n+1} = \frac{A^{2(i-1)}}{\sqrt{\mathcal{P}(A^2)}}, \quad i=1,\cdots,n$$

$$M(A)_{n+1,j} = \int_{a_j} \frac{Q_p(D^2)}{\sqrt{\mathcal{P}(D^2)}}, \quad j=1,\cdots,n$$

$$M(A)_{n+1,n+1} = \frac{Q_p(A^2)}{\sqrt{\mathcal{P}(A^2)}}$$

where

$$Q_p(A^2) = \int_{|D|>|A|} dD \frac{D^{2p+1}}{D^2-A^2}\sqrt{\mathcal{P}(D^2)} = \left[\sqrt{\mathcal{P}(A^2)}A^{2p}\right]_+$$

where $[\cdots]_+$ means taking the polynomial part in the expansion around infinity. It is quite obvious that the normalized differential $d\widetilde{\omega}_{2p+1}(A)$ is given by

$$d\widetilde{\omega}_{2p+1}(A) = (2p+1)\Delta^{-1}\ det(M(A))dA$$

which finishes the proof of the proposition. Returning to the beginning of the proof one finds that the expression

$$(2p+1)\langle -1|\psi_{2p+1}\psi^*(A)\frac{dA}{A}$$

can be considered as "symbol" of the normalized differential $d\widetilde{\omega}_{2p+1}(A)$. ■

The equation $\langle\Psi_{-3}|\ \mathcal{Q}_0 \stackrel{w}{=} 0$ for more complicated states $\langle\Psi_{-3}|$ than those considered in the Proposition 16 implies other linear partial differential equations for B_j, so, we get the whole Whitham hierarchy. However the equations (56) are the only ones with derivatives with respect to only two times. We shall not go further into the study of the Whitham hierarchy, because it is not our goal. What we really wanted to do was to show the remarkable parallel between the Whitham method and the quantum form factor formulae. We hope that this goal is achieved.

6 Appendix A

In this appendix we explain why the condition

$$\begin{aligned} L^{(n)}_{\mathcal{O}}(A_1,\cdots,A_n|B_1,\cdots,B_{2n})\Big|_{B_{2n}=-B_1,\ A_n=\pm B_{2n-1}} &= \\ = -\epsilon^{\pm} L^{(n-1)}_{\mathcal{O}}(A_1,\cdots,A_{n-1}|B_2,\cdots,B_{2n-1}) \end{aligned} \tag{59}$$

($\epsilon = +$ or $-$ respectively for the operators Φ_{2k} and their descendents or Φ_{2k+1} and their descendents) is sufficient for the locality of the operator whose form factors are given by

$$\begin{aligned} &f_{\mathcal{O}}(\beta_1,\beta_2,\cdots,\beta_{2n})_{-\cdots-+\cdots+} = \\ &= c^n\prod_{i<j}\zeta(\beta_i-\beta_j)\prod_{i=1}^{n}\prod_{j=n+1}^{2n}\frac{1}{\sinh\nu(\beta_j-\beta_i-\pi i)}\exp(-\frac{1}{2}(\nu(n-1)-n)\sum\beta_j) \\ &\times \widehat{f}_{\mathcal{O}}(\beta_1,\beta_2,\cdots,\beta_{2n})_{-\cdots-+\cdots+} \end{aligned}$$

where

$$\begin{aligned} &\widehat{f}_{\mathcal{O}}(\beta_1,\beta_2,\cdots,\beta_{2n})_{-\cdots-+\cdots+} \quad = \\ &= \frac{1}{(2\pi i)^n}\int dA_1\cdots\int dA_n\prod_{i=1}^{n}\prod_{j=1}^{2n}\psi(A_i,B_j)\prod_{i<j}(A_i^2-A_j^2)\ L^{(n)}_{\mathcal{O}}(A_1,\cdots,A_n|B_1,\cdots,B_{2n})\prod_{i=1}^{n}a_i^{-i} \end{aligned}$$

The calculations which we are going to make are well known even for more general case [3]. However, we want to repeat them for our particular situation for the completeness of the exposition.

In the case of diagonal scattering the only non-trivial requirement for the form factors is the following:

$$2\pi i\ res_{\beta_{2n}=\beta_1+\pi i} f_{\mathcal{O}}(\beta_1,\beta_2,\cdots,\beta_{2n-1},\cdots\beta_{2n})_{-\cdots-+\cdots+} = \\ = f_{\mathcal{O}}(\beta_2,\cdots,\cdots,\beta_{2n-1})_{-\cdots-+\cdots+}\left(\prod_{j=2}^{2n-1} S(\beta_j-\beta_1)-\epsilon\right) \tag{60}$$

where the S-matrix in our case is

$$S(\beta_2-\beta_1)=\prod_{j=1}^{\nu-1}\frac{\sinh\frac{1}{2}(\beta_2-\beta_1+\frac{\pi i}{\nu}j)}{\sinh\frac{1}{2}(\beta_2-\beta_1-\frac{\pi i}{\nu}j)}=\frac{\psi(-B_1,B_2)}{\psi(B_1,B_2)}$$

Using the identity [3]

$$\exp(-\frac{1}{2}(\nu-1)(\beta_1+\beta_j))\frac{\zeta(\beta_1-\beta_j)\zeta(\beta_j-\beta_1-\pi i)}{(\sinh\nu(\beta_j-\beta_1))^2}=\frac{1}{\psi(B_1,B_j)}$$

one finds that the relation (60) is equivalent to

$$\widehat{f}_{\mathcal{O}}(\beta_1,\beta_2,\cdots,\beta_{2n-1},\beta_{2n})_{-\cdots-+\cdots+}\Big|_{\beta_{2n}=\beta_1+\pi i} = \\ =\frac{1}{2B_1}\widehat{f}_{\mathcal{O}}(\beta_2,\cdots,\beta_{2n-1})_{-\cdots-+\cdots+}\left(\prod_{j=2}^{2n-1}\psi(-B_1,B_j)-\epsilon\prod_{j=2}^{2n-1}\psi(B_1,B_j)\right)$$

if the constant c is taken as $c=2\nu(\zeta(-\pi i))^{-1}$. Explicitly the LHS of this equation is

$$\frac{1}{(2\pi i)^n}\int dA_1\cdots\int dA_n\prod_{i=1}^{n}\frac{a_i-b_1}{A_i^2-B_1^2}\prod_{j=2}^{2n-1}\psi(A_i,B_j) \\ \times\prod_{i<j}(A_i^2-A_j^2)\ L_{\mathcal{O}}^{(n)}(A_1,\cdots,A_n|B_1,\cdots,B_{2n-1},-B_1)\prod_{i=1}^{n}a_i^{-i} \tag{61}$$

where we have used the identity

$$\psi(A,B)\psi(A,-B)=\frac{a-b}{A^2-B^2}$$

Let us consider the integral over A_n. If the contour is such that $|A_n|>|B_1|$ we can replace in this integral a_n^{-n} by $b_1^{-n+1}(a_n-b_1)^{-1}$. Indeed

$$\frac{b_1^{-n+1}}{(a_n-b_1)}=a_n^{-n}+\sum_{j=1}^{n-1}a_n^{-j}b_1^{-n+j}+\sum_{j\geq n+1}a_n^{-j}b_1^{-n+j}$$

the sum $\sum\limits_{j=1}^{n-1}$ can be omitted due to anti-symmetry with respect to a_j, $j=1,\cdots,n-1$, the sum $\sum\limits_{j\geq n+1}$ can be omitted because the integrand decreases faster than A_n^{-2} as $A_n\to\infty$. Thus the integral over A_n becomes

$$b_1^{-n+1}\frac{1}{2\pi i}\int\limits_{A_n>B_1}dA_n\frac{1}{A_n^2-B_1^2}\prod_{j=2}^{2n-1}\psi(A_n,B_j)\prod_{i<n}(A_i^2-A_n^2)\ L_{\mathcal{O}}^{(n)}(A_1,\cdots,A_n|B_1,\cdots,B_{2n-1},-B_1)= \\ =\frac{1}{2}b_1^{-n+1}B_1^{-1}\prod_{i<n}(A_i^2-B_1^2)\left(\prod_{j=2}^{2n-1}\psi(-B_1,B_j)L_{\mathcal{O}}^{(n)}(A_1,\cdots,A_{n-1},-B_1|B_1,\cdots,B_{2n-1},-B_1)-\right. \\ \left.-\prod_{j=2}^{2n-1}\psi(B_1,B_j)L_{\mathcal{O}}^{(n)}(A_1,\cdots,A_{n-1},B_1|B_1,\cdots,B_{2n-1},-B_1)\right)$$

Let us substitute this expression into (61):

$$
\begin{aligned}
&\frac{1}{2B_1}\frac{1}{(2\pi i)^{n-1}}\int dA_1\cdots\int dA_{n-1}\prod_{i=1}^{n-1}\prod_{j=2}^{2n-1}\psi(A_i,B_j)\prod_{i<j}(A_i^2-A_j^2)b_1^{-n+1}\prod_{i=1}^{n-1}a_i^{-i}(a_i-b_1)\\
&\times\left(\prod_{j=2}^{2n-1}\psi(-B_1,B_j)L_{\mathcal{O}}^{(n)}(A_1,\cdots,A_{n-1},-B_1|B_1,\cdots,B_{2n-1},-B_1)-\right.\\
&\left.-\prod_{j=2}^{2n-1}\psi(B_1,B_j)L_{\mathcal{O}}^{(n)}(A_1,\cdots,A_{n-1},B_1|B_1,\cdots,B_{2n-1},-B_1)\right)=\\
=\ &\frac{1}{2B_1}\frac{1}{(2\pi i)^{n-1}}\int dA_1\cdots\int dA_{n-1}\prod_{i=1}^{n-1}\prod_{j=2}^{2n-1}\psi(A_i,B_j)\prod_{i<j}(A_i^2-A_j^2)\prod_{i=1}^{n-1}a_i^{-i}\\
&\times\left(\prod_{j=2}^{2n-1}\psi(-B_1,B_j)L_{\mathcal{O}}^{(n)}(A_1,\cdots,A_{n-1},-B_1|B_1,\cdots,B_{2n-1},-B_1)-\right.\\
&\left.-\prod_{j=2}^{2n-1}\psi(B_1,B_j)L_{\mathcal{O}}^{(n)}(A_1,\cdots,A_{n-1},B_1|B_1,\cdots,B_{2n-1},-B_1)\right)
\end{aligned}
\tag{62}
$$

where we have replaced $b_1^{-1}a_i^{-i}(a_i-b_1)$ by $-a_i^{-i}$ for the following reason: for a_1 this expression is $b_1^{-1}-a_1^{-1}$, the integrand with b_1^{-1} is regular at zero, for a_i with $i>1$ we use anti-symmetry. The final formula (62) shows that the equation (59) is sufficient for locality.

7 Appendix B

The reflectionless case is a rather degenerate one, so, the deformed Riemann bilinear identity [4] does not exist in complete form. However for our needs we use only the consequence of the deformed Riemann bilinear identity which allows a simple proof in the reflectionless case. We give this proof here for the sake of completeness. We have used the following fact:

$$
\int dA_1\int dA_2\prod_{i=1}^{2}\prod_{j=1}^{2n}\psi(A_i,B_j)C(A_1,A_2)a_1^ka_2^l=0\quad\forall k,l \tag{63}
$$

where we had the expression for $C(A_1,A_2)$:

$$
C(A_1,A_2)=\frac{1}{A_1A_2}\left\{\frac{A_1-A_2}{A_1+A_2}(P(A_1)P(A_2)-P(-A_1)P(-A_2))+(P(-A_1)P(A_2)-P(A_1)P(-A_2))\right\} \tag{64}
$$

Let us introduce the functions

$$
F(A)=\prod_{j=1}^{2n}\psi(A,B_j)P(A),\qquad G(A)=\prod_{j=1}^{2n}\psi(A,B_j)P(-A)
$$

Recall that the function $\psi(A,B)$ satisfies the difference equation

$$
\psi(Aq,B)=\left(\frac{B-A}{B+qA}\right)\ \psi(A,B), \tag{65}
$$

which implies that

$$
F(Aq)=G(A)
$$

The integral (63) can be rewritten as follows:

$$
\int\frac{dA_1}{A_1}\int\frac{dA_2}{A_2}\left\{\frac{A_1-A_2}{A_1+A_2}(F(A_1)F(A_2)-F(qA_1)F(qA_2))+(F(qA_1)F(A_2)-F(A_1)F(qA_2))\right\}a_1^ka_2^l
$$

Changing variables $A_i \to qA_i$ where needed one easily finds that this integral equals zero. Recall that $a_i = A_i^{2\nu}$ and $q^{2\nu} = 1$, so $a_1^k a_2^l$ do not change under these changes of variables.

Similar proof for the case of generic coupling constant is more complicated because one crosses singularities when changing variables and moving contours.

Now let us prove the Proposition 1. We want to find equivalent expressions for $C(A_1, A_2)$. Let us rewrite the expression (64) in the integral form:

$$C(A_1, A_2) = C'(A_1, A_2) - C'(A_2, A_1)$$

where

$$C'(A_1, A_2) = \frac{1}{2A_2}\left(\frac{P(A_1)P(A_2)}{A_2 + A_1} + \frac{P(A_1)P(-A_2)}{A_2 - A_1} - \frac{P(-A_1)P(A_2)}{A_2 - A_1} - \frac{P(-A_2)P(-A_1)}{A_2 + A_1}\right) =$$
$$= A_1 \int\limits_{|D_2|>|D_1|} dD_2 \int\limits_{|D_1|>|A_1|,|A_2|} dD_1 \frac{P(D_1)P(D_2)}{(D_1 + D_2)(D_1^2 - A_1^2)(D_2^2 - A_2^2)}$$

Let us modify this expression by adding "exact forms" in variables A_1. It is convenient to use the formula

$$Q(A_1)P(A_1) - qQ(qA_1)P(-A_1) =$$
$$= \int\limits_{|D_1|>|A_1|} dD_1 \frac{P(D_1)}{(D_1^2 - A_1^2)}(Q(D_1)(D_1 + A_1) - qQ(-qD_1)(D_1 - A_1))$$

Suppose that the polynomial $Q(A)$ solves the equation

$$Q(D_1) + qQ(-qD_1) = \int\limits_{|D_2|>|D_1|} dD_2 \frac{P(D_2)}{(D_1 + D_2)(D_2^2 - A_2^2)} \tag{66}$$

(obviously the RHS of this equation is a polynomial) then we can rewrite the expression for $C'(A_1, A_2)$ in the following equivalent form

$$C'(A_1, A_2) = \int\limits_{|D_1|>|A_1|} dD_1 D_1 \frac{P(D_1)}{(D_1^2 - A_1^2)}(Q(D_1) - qQ(-qD_1))$$

Now we have to solve the equation (66). It is simple:

$$Q(D_1) = \frac{1}{D_1} \int\limits_{|D_2|>|D_1|} dD_2 \eta\left(\frac{D_1}{D_2}\right) \frac{P(D_2)}{(D_2^2 - A_2^2)}$$

where

$$\eta(x) = \sum_{k=0}^{\infty} \frac{1}{1 + q^{2k+1}} x^{2k} - \sum_{k=1}^{\infty} \frac{1}{1 - q^{2k}} x^{2k-1}$$

Hence

$$C'(A_1, A_2) = \int\limits_{|D_2|>|D_1|} dD_2 \int\limits_{|D_1|>|A_1|} dD_1 P(D_1)P(D_2)\tau_e\left(\frac{D_1}{D_2}\right) \frac{1}{(D_1^2 - A_1^2)(D_2^2 - A_2^2)}$$

where

$$\tau_e(x) = \sum_{k=1}^{\infty} \frac{1 - q^{2k-1}}{1 + q^{2k-1}} x^{2k-1} - \sum_{k=1}^{\infty} \frac{1 + q^{2k}}{1 - q^{2k}} x^{2k}$$

The expression for $C_e(A_1, A_2)$ given in Proposition 1 follows from these formulae. The expression for $C_o(A_1, A_2)$ can be obtained in a similar way eliminating the even degrees of A_1 from $C'(A_1, A_2)$.

References

[1] F.A. Smirnov, Nucl. Phys. B453[FS] (1995) p.807

[2] O. Babelon, D. Bernard, F.A. Smirnov, *Quantization of Solitons and the Restricted sine-Gordon Model,* hep-th/9603010 (to be published in Comm. Math. Phys.)

[3] F.A.Smirnov, *Form Factors in Completely Integrable Models of Quantum Field Theory.* Adv. Series in Math. Phys. 14, World Scientific, Singapore (1992)

[4] F.A. Smirnov, Lett.Math.Phys. 36 (1996) p.267

[5] A.B. Zamolodchikov, Adv. Studies Pure Math. 19 (1989) 641.

[6] B. Feigin, D. Fuchs, in *Representations of Lie Groups and Related Topics,* eds. A.M. Vershik and D.P. Zhelodenko, Gordon and Breach (1990) p.465

[7] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, Nucl. Phys. B241 (1984) p.333.

[8] B.A. Kupershmidt, P. Mathieu, Phys. Lett. B227 (1989) 245.

[9] B. Feigin, E. Frenkel *Integrals of Motion and Quantum Groups,* Lect. Notes in Math. v.1620 (1995)

[10] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, in *Nonlinear Integrable Systems.* World Scientific, Singapore (1983)

[11] F.A. Smirnov, *Particle-Field Duality in Sine-Gordon Theory,* to be published in Proceedings of Buckow Conference (1995)

[12] V.G. Knizhnik, Comm. Math. Phys. 112 (1987) p.567

[13] L.A. Dickey, *Soliton Equations and Hamiltonian Systems* Adv. Series in Math. Phys. 12, World Scientific (1991)

[14] E.D. Belokolos, A.I. Bobenko, V.Z. Enol'skii, A.R. Its, V.B. Matveev *Algebra-geometric Approach to Non-linear Integrable Equations,* Springer series in non-linear dynamics (1994)

[15] S.Novikov, S.Manakov, L.Pitaevski, V.Zakharov, *Theory of Solitons.* Consultants Bureau, New York, (1984).

[16] G.B. Whitham, *Linear and Nonlinear Waves.* Wiley-Interscience, New York (1974)

[17] H. Flashka, M.G. Forest, D.W. McLaughlin, Comm. Pure Appl. Math. XXXIII (1980) p.739

[18] I.M. Krichever, Functional Anal. Appl. 22 (1988) p.200

[19] B.A. Dubrovin, S.P. Novikov, Russian Math. Surveys 44:6 (1989) p.35