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# A NON-HOMOGENEOUS ORBIT CLOSURE OF A DIAGONAL SUBGROUP

FRANÇOIS MAUCOURANT

ABSTRACT. Let  $G = \mathbf{SL}(n, \mathbf{R})$  with  $n \geq 6$ . We construct examples of lattices  $\Gamma \subset G$ , subgroups  $A$  of the diagonal group  $D$  and points  $x \in G/\Gamma$  such that the closure of the orbit  $Ax$  is not homogeneous and such that the action of  $A$  does not factor through the action of a one-parameter non-unipotent group. This contradicts a conjecture of Margulis.

## 1. INTRODUCTION

**1.1. Topological rigidity and related questions.** Let  $G$  be a real Lie group,  $\Gamma$  a lattice in  $G$ , meaning a discrete subgroup of finite covolume, and  $A$  a closed connected subgroup. We are interested in the action of  $A$  on  $G/\Gamma$  by left multiplication; we will restrict ourselves to the topological properties of these actions, referring the reader to [6] and [3] for references and recent developments on related measure theoretical problems.

Two linked questions arise when one studies continuous actions of topological groups: what are the closed invariant sets, and what are the orbit closures?

In the homogeneous action setting we are considering, there is a class of closed sets that admit a simple description: a closed subset  $X \subset G$  is said to be *homogeneous* if there exists a closed connected subgroup  $H \subset G$  such that  $X = Hx$  for some (and hence every)  $x \in X$ . Let us say that the action of  $A$  on  $G/\Gamma$  is *topologically rigid* if for any  $x \in G/\Gamma$ , the closure  $\overline{Ax}$  of the orbit  $Ax$  is homogeneous.

The most basic example of a topologically rigid action is when  $G = \mathbf{R}^n$ ,  $\Gamma = \mathbf{Z}^n$ ,  $A$  any vector subspace of  $G$ . It turns out that the behavior of elements of  $A$  for the adjoint action on the Lie algebra  $\mathfrak{g}$  of  $G$  plays an important role for our problem. Recall that an element  $g \in G$  is said to be **Ad**-unipotent if **Ad**( $g$ ) is unipotent, and **Ad**-split over  $\mathbf{R}$  if **Ad**( $g$ ) is diagonalizable over  $\mathbf{R}$ . If the closed, connected subgroup  $A$  of  $G$  is generated by **Ad**-unipotent elements, a celebrated theorem of Ratner [13] asserts that the action of  $A$  is always topologically rigid, settling a conjecture due to Raghunathan.

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When  $A$  is generated by elements which are **Ad**-split over  $\mathbf{R}$ , much less is known. Consider the model case of  $G = \mathbf{SL}(n, \mathbf{R})$  and  $A$  the group of diagonal matrices with nonnegative entries. If  $n = 2$ , it is easy to produce non-homogeneous orbit closures (see e.g. [7]); more generally, a similar phenomenon can be observed when  $A$  is a one-parameter subgroup of the diagonal group (see [6], 4.1). However, for  $A$  the full diagonal group, if  $n \geq 3$ , to the best of our knowledge, the only nontrivial example of a nonhomogeneous  $A$ -orbit closure is due to Rees, later generalized in [7]. In an unpublished preprint, Rees exhibited a lattice  $\Gamma$  of  $G = \mathbf{SL}(3, \mathbf{R})$  and a point  $x \in G/\Gamma$  such that for the full diagonal group  $A$ , the orbit closure  $\overline{Ax}$  is not homogeneous. Her construction was based on the following property of the lattice: there exists a  $\gamma \in \Gamma \cap A$  such that the centralizer  $C_G(\gamma)$  of  $\gamma$  is isomorphic to  $\mathbf{SL}(2, \mathbf{R}) \times \mathbf{R}^*$ , and such that  $C_G(\gamma) \cap \Gamma$  is, in this product decomposition and up to finite index,  $\Gamma_0 \times \langle \gamma \rangle$ , where  $\Gamma_0$  is a lattice in  $\mathbf{SL}(2, \mathbf{R})$  (see [4], [7]). Thus in this case the action of  $A$  on  $C_G(\gamma)/C_G(\gamma) \cap \Gamma$  factors to the action of a 1-parameter non-unipotent subgroup on  $\mathbf{SL}(2, \mathbf{R})/\Gamma_0$ , which, as we saw, has many non-homogeneous orbits.

Rees' example shows that factor actions of 1-parameter non-**Ad**-unipotent groups are obstructions to the topological rigidity of the action of diagonal subgroups. The following conjecture of Margulis [8, conjecture 1.1] (see also [6, 4.4.11]) essentially states that these are the only ones:

**Conjecture 1.** *Let  $G$  be a connected Lie group,  $\Gamma$  a lattice in  $G$ , and  $A$  a closed, connected subgroup of  $G$  generated by **Ad**-split over  $\mathbf{R}$  elements. Then for any  $x \in G/\Gamma$ , one of the following holds :*

- (a)  $\overline{Ax}$  is homogeneous, or
- (b) *There exists a closed connected subgroup  $F$  of  $G$  and a continuous epimorphism  $\phi$  of  $F$  onto a Lie group  $L$  such that*
  - $A \subset F$ ,
  - $Fx$  is closed in  $G/\Gamma$ ,
  - $\phi(F_x)$  is closed in  $L$ , where  $F_x$  denotes the stabilizer  $\{g \in F | gx = x\}$ ,
  - $\phi(A)$  is a one-parameter subgroup of  $L$  containing no nontrivial **Ad** <sub>$L$</sub> -unipotent elements.

A first step toward this conjecture has been done by Lindenstrauss and Weiss [7], who proved that in the case  $G = \mathbf{SL}(n, \mathbf{R})$  and  $A$  the full diagonal group, if the closure of a  $A$ -orbit contains a compact  $A$ -orbit that satisfy some irrationality conditions, then this closure is homogeneous. See also [15]. Recently, using an approach based on measure theory, Einsiedler, Katok and Lindenstrauss proved that if moreover  $\Gamma = \mathbf{SL}(n, \mathbf{Z})$ , then the set of bounded  $A$ -orbits has Hausdorff dimension  $n - 1$  [3, Theorem 10.2].

**1.2. Statement of the results.** In this article we exhibit some counterexamples to the above conjecture when  $G = \mathbf{SL}(n, \mathbf{R})$  for  $n \geq 6$  and  $A$  is some strict subgroup of the diagonal group of matrices with nonnegative entries. Let  $D$  be

the diagonal subgroup of  $G$ ; note that  $D$  has dimension  $n - 1$ . Our main result is:

**Theorem 1.** *Assume  $n \geq 6$ .*

- (1) *There exists a  $(n - 3)$  dimensional closed and connected subgroup  $A$  of  $D$ , and a point  $x \in \mathbf{SL}(n, \mathbf{R})/\mathbf{SL}(n, \mathbf{Z})$  such that the closure of the  $A$ -orbit of  $x$  satisfies neither condition (a) nor condition (b) of the conjecture.*
- (2) *There exists a lattice  $\Gamma$  of  $\mathbf{SL}(n, \mathbf{R})$ , a  $(n - 2)$  dimensional closed and connected subgroup  $A$  of  $D$  and a point  $x \in \mathbf{SL}(n, \mathbf{R})/\Gamma$  such that the closure of the  $A$ -orbit of  $x$  satisfies neither condition (a) nor condition (b) of the conjecture.*

It will be clear from the proofs that these examples however satisfy a third condition:

- (c) *There exists a closed connected subgroup  $F$  of  $G$  and two continuous epimorphisms  $\phi_1, \phi_2$  of  $F$  onto Lie groups  $L_1, L_2$  such that*
  - $A \subset F$ ,
  - $Fx$  is closed in  $G/\Gamma$ ,
  - For  $i = 1, 2$ ,  $\phi_i(Fx)$  is closed in  $L_i$ ,
  - $(\phi_1, \phi_2) : F \rightarrow L_1 \times L_2$  is surjective
  - $(\phi_1, \phi_2) : A \rightarrow \phi_1(A) \times \phi_2(A)$  is not surjective.

Construction of these examples is the subject of Section 2, whereas the proof that they satisfy the required properties is postponed to Section 3.

**1.3. Toral endomorphisms.** To conclude this introduction, we would like to mention that the idea behind this construction can be also used to yield examples of 'non-homogeneous' orbits for diagonal toral endomorphisms.

Let  $1 < p_1 < \dots < p_q$ , with  $q \geq 2$ , be integers generating a multiplicative non-lacunary semigroup of  $\mathbf{Z}$  (that is, the  $\mathbf{Q}$ -subspace  $\bigoplus_{1 \leq i \leq q} \mathbf{Q} \log(p_i)$  has dimension at least 2). We consider the abelian semigroup  $\Omega$  of endomorphisms of the torus  $T^n = \mathbf{R}^n/\mathbf{Z}^n$  generated by the maps  $z \mapsto p_i z \bmod \mathbf{Z}^n$ ,  $1 \leq i \leq q$ .

In the one-dimensional situation, described by Furstenberg [5], every  $\Omega$ -orbit is finite or dense. If  $n \geq 2$ , Berend [1] showed that minimal sets are the finite orbits of rational points, but there are others obvious closed  $\Omega$ -invariant sets, namely the orbits of rational affine subspaces. Meiri and Peres [10] showed that closed invariant sets have integer Hausdorff dimension.

Note that the study of the orbit of a point lying in a proper rational affine subspace reduces to the study of finitely many orbits in lower dimensional tori, although some care must be taken about the pre-periodic part of the rational affine subspace (for example, if  $q = n = 2$ , and if  $\alpha \in T^1$  is irrational with non-dense  $p_1$ -orbit, the orbit closure of the point  $(\alpha, 1/p_2) \in T^2$  is the union of a horizontal circle and a finite number of strict closed infinite subsets of some horizontal circles).

With this last example in mind, Question 5.2 of [10] can be re-formulated: is a proper closed invariant set necessarily a subset of a finite union of rational affine tori? Or, equivalently, if a point is outside any rational affine subspace, does it necessarily have a dense orbit? It turns out that this is not the case at least for  $n \geq 2q$ , as the following example shows.

**Theorem 2.** *Let  $N$  be an integer greater than  $q \frac{\log p_q}{\log p_1}$ , and let  $z$  be the point in the  $2q$ -dimensional torus  $T^{2q}$  defined by the coordinates modulo 1:*

$$z = (z_1, \dots, z_{2q}) = \left( \sum_{k \geq 1} p_1^{-N^{2k}}, \dots, \sum_{k \geq 1} p_q^{-N^{2k}}, \sum_{k \geq 1} p_1^{-N^{2k+1}}, \dots, \sum_{k \geq 1} p_q^{-N^{2k+1}} \right).$$

*Then the point  $z \in T^{2q}$  is not contained in any rational affine subspace, but its orbit  $\Omega z$  is not dense.*

The proof of Theorem 2 will be the subject of Section 4.

## 2. SKETCH OF PROOF OF THEOREM 1

**2.1. The direct product setup.** We now describe how these examples are built. Choose two integers  $n_1 \geq 3$ ,  $n_2 \geq 3$ , such that  $n_1 + n_2 = n$ . For  $i = 1, 2$ , let  $\Gamma_i$  be a lattice in  $G_i = \mathbf{SL}(n_i, \mathbf{R})$ .

Let  $g_i$  be an element of  $G_i$  such that  $g_i \Gamma_i g_i^{-1}$  intersects the diagonal subgroup  $D_i$  of  $\mathbf{SL}(n_i, \mathbf{R})$  in a lattice, in other words  $g_i \Gamma_i$  has a compact  $D_i$ -orbit; such elements exist, see [11]. In fact, we will need an additional assumption on  $g_i$ , namely that the tori  $g_i^{-1} D_i g_i$  are *irreducible over  $\mathbf{Q}$* . The precise definition of this property and the proof of the existence of such a  $g_i$ , a consequence of a theorem of Prasad and Rapinchuk [12, Theorem 1], will be the subject of Section 3.1.

Let  $\pi_i : G_i \rightarrow G_i/\Gamma_i$  be the canonical quotient map. Define for  $i = 1, 2$ :

$$y_i = \pi_i \left( \left[ \begin{array}{cccccc} 1 & 0 & \dots & 0 & 1 \\ 0 & 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \end{array} \right] g_i \right).$$

The  $D_i$ -orbit of  $y_i$  is dense, by the following argument. It is easily seen that the closure of  $D_i y_i$  contains the compact  $D_i$ -orbit  $\mathcal{T}_i = \pi_i(D_i g_i)$ . The  $\mathbf{Q}$ -irreducibility of  $\mathcal{T}_i$  is sufficient to show that the assumptions of the theorem of Lindenstrauss and Weiss [7, Theorem 1.1] are satisfied (Lemma 3.1); thus, by this theorem, we obtain that there exists a group  $H_i < G_i$  such that  $H_i y_i = \overline{D_i y_i}$ . Again because

of  $\mathbf{Q}$ -irreducibility, the group  $H_i$  is necessarily the full group, i.e.  $H_i = G_i$  (proof of Lemma 3.2)<sup>1</sup>.

Let  $A_1$  be the  $(n - 3)$  dimensional subgroup of  $G_1 \times G_2$  given by:

$$(1) \quad A_1 = \left\{ (diag(a_1, \dots, a_{n_1}), diag(b_1, \dots, b_{n_2})) : \prod_{i=1}^{n_1} a_i = \prod_{j=1}^{n_2} b_j = \frac{a_1 b_1}{a_{n_1} b_{n_2}} = 1, a_i > 0, b_j > 0 \right\}.$$

Then the  $A_1$ -orbit of  $(y_1, y_2)$  is not dense in  $G_1 \times G_2 / \Gamma_1 \times \Gamma_2$  (Lemma 3.3), but  $G_1 \times G_2$  is the smallest closed connected subgroup  $F$  of  $G_1 \times G_2$  such that  $\overline{A_1(y_1, y_2)} \subset F(y_1, y_2)$  (Lemma 3.7).

This yields a counterexample to Conjecture 1 which can be summarized as follows:

**Proposition 1.** *For  $i = 1, 2$ , let  $n_i \geq 3$  and  $\Gamma_i$  be a lattice in  $G_i = \mathbf{SL}(n_i, \mathbf{R})$ . For  $A_1, y_1, y_2$  depicted as above, the  $A_1$ -orbit of  $(y_1, y_2)$  in  $G_1 \times G_2 / \Gamma_1 \times \Gamma_2$  satisfies neither condition (a) nor condition (b) of Conjecture 1.*

**2.2. Theorem 1, part (1).** In order to obtain the first part of Theorem 1, choose  $\Gamma_i = \mathbf{SL}(n_i, \mathbf{Z})$ ,  $\Gamma = \mathbf{SL}(n, \mathbf{Z})$  and consider the embedding of  $G_1 \times G_2$  in  $G$ , where matrices are written in blocks:

$$(2) \quad \Psi : (M_{n_1, n_1}, N_{n_2, n_2}) \mapsto \begin{bmatrix} M_{n_1, n_1} & 0_{n_1, n_2} \\ 0_{n_2, n_1} & N_{n_2, n_2} \end{bmatrix}.$$

This embedding gives rise to an embedding  $\overline{\Psi}$  of  $G_1 \times G_2 / \Gamma_1 \times \Gamma_2$  into  $G / \Gamma$ . Let  $y_1, y_2$  be two points as above, let  $x = \overline{\Psi}(y_1, y_2)$  and take  $A = \Psi(A_1)$ . We claim that this point  $x$  and this group  $A$  satisfy Theorem 1, part (1). In fact, since the image of  $\overline{\Psi}$  is a closed connected  $A$ -invariant subset of  $\mathbf{SL}(n, \mathbf{R}) / \mathbf{SL}(n, \mathbf{Z})$ , everything takes place in this direct product.

**2.3. Theorem 1, part (2).** The second part of Theorem 1 is obtained as follows. Let  $\sigma$  be the nontrivial field automorphism of the quadratic extension  $\mathbf{Q}(\sqrt[4]{2}) / \mathbf{Q}(\sqrt{2})$ . Consider for any  $m \geq 1$ :

$$\mathbf{SU}(m, \mathbf{Z}[\sqrt[4]{2}], \sigma) = \left\{ M \in \mathbf{SL}(m, \mathbf{Z}[\sqrt[4]{2}]) : ({}^t M^\sigma) M = I_m \right\}.$$

Then  $\mathbf{SU}(m, \mathbf{Z}[\sqrt[4]{2}], \sigma)$  is a lattice in  $\mathbf{SL}(m, \mathbf{R})$ , as will be proved in Section 3.5 (see [4, Appendix] for  $m = 3$ ). Define for  $i = 1, 2$ ,  $\Gamma_i = \mathbf{SU}(n_i, \mathbf{Z}[\sqrt[4]{2}], \sigma)$ , and  $\Gamma = \mathbf{SU}(n, \mathbf{Z}[\sqrt[4]{2}], \sigma)$ . Now consider the map:

$$\varphi : G_1 \times G_2 \times \mathbf{R} \rightarrow G,$$

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<sup>1</sup>The reader only interested in the case  $n = 6$  and  $\Gamma = \mathbf{SL}(6, \mathbf{Z})$  might note that when  $\Gamma_1 = \Gamma_2 = \mathbf{SL}(3, \mathbf{Z})$ , [7, Corollary 1.4] can be used directly in the proof of Lemma 3.2; then the notion of  $\mathbf{Q}$ -irreducibility becomes unnecessary, and the entire Section 3.1 can be skipped.

$$(X, Y, t) \mapsto \begin{bmatrix} e^{n_2 t} X & 0 \\ 0 & e^{-n_1 t} Y \end{bmatrix}.$$

Define  $M$  to be the image of  $\varphi$ . This time,  $\varphi$  factors into a finite covering  $\bar{\varphi}$  of homogeneous spaces:

$$\bar{\varphi} : G_1 \times G_2 \times \mathbf{R}/\Gamma_1 \times \Gamma_2 \times (\log \alpha)\mathbf{Z} \rightarrow M/M \cap \Gamma \subset G/\Gamma,$$

where  $\alpha = (3 + 2\sqrt{2}) + \sqrt[4]{2}(2 + 2\sqrt{2})$  satisfies  $\alpha^{-1} = \sigma(\alpha)$ . Consider the points  $y_i$  constructed above, and let  $x = \bar{\varphi}(y_1, y_2, 0)$ . Choose:

$$A = \left\{ \text{diag}(a_1, \dots, a_n) \mid \prod_{i=1}^n a_i = \frac{a_1 a_{n_1+1}}{a_{n_1} a_n} = 1, a_i > 0 \right\} \subset \mathbf{SL}(n, \mathbf{R}).$$

We claim that this lattice  $\Gamma$ , this point  $x$  and this group  $A$  satisfy Theorem 1, part (2). What happens here is that the  $A$ -orbit of  $x$  is a circle bundle over an  $A_1$ -orbit (up to the finite cover  $\bar{\varphi}$ ), like in Rees' example.

### 3. PROOF OF THEOREM 1

**3.1.  $\mathbf{Q}$ -irreducible tori.** Fix  $i \in \{1, 2\}$ . Recall that  $\Gamma_i$  is a lattice in  $G_i = \mathbf{SL}(n_i, \mathbf{R})$ . Since  $n_i \geq 3$ , by Margulis's arithmeticity Theorem [16, Theorem 6.1.2], there exists a semisimple algebraic  $\mathbf{Q}$ -group  $\mathbf{H}_i$  and a surjective homomorphism  $\theta$  from the connected component of identity of the real points of this group  $\mathbf{H}_i^0(\mathbf{R})$  to  $\mathbf{SL}(n_i, \mathbf{R})$ , with compact kernel, such that  $\theta(\mathbf{H}_i(\mathbf{Z}) \cap \mathbf{H}_i^0(\mathbf{R}))$  is commensurable with  $\Gamma_i$ .

Following Prasad and Rapinchuk, we say that a  $\mathbf{Q}$ -torus  $\mathbf{T} \subset \mathbf{H}_i$  is  $\mathbf{Q}$ -irreducible if it does not contain any proper subtorus defined over  $\mathbf{Q}$ . By [12, Theorem 1, (ii)], there exists a maximal  $\mathbf{Q}$ -anisotropic  $\mathbf{Q}$ -torus  $\mathbf{T}_i \subset \mathbf{H}_i$ , which is  $\mathbf{Q}$ -irreducible. Because any two maximal  $\mathbf{R}$ -tori of  $\mathbf{SL}(n_i, \mathbf{R})$  are  $\mathbf{R}$ -conjugate, there exists  $g_i \in G_i$  such that  $\theta(\mathbf{T}_i^0(\mathbf{R})) = g_i^{-1} D_i g_i$ . The subgroup  $\mathbf{T}_i(\mathbf{Z})$  is a cocompact lattice in  $\mathbf{T}_i(\mathbf{R})$  since  $\mathbf{T}_i$  is  $\mathbf{Q}$ -anisotropic [2, Theorem 8.4 and Definition 10.5]. Because  $\theta(\mathbf{H}_i(\mathbf{Z}) \cap \mathbf{H}_i^0(\mathbf{R}))$  and  $\Gamma_i$  are commensurable and  $\theta$  has compact kernel, it follows that both  $\Gamma_i \cap g_i^{-1} D_i g_i$  and  $\theta(\mathbf{T}_i^0(\mathbf{Z})) \cap \Gamma_i \cap g_i^{-1} D_i g_i$  are also cocompact lattices in  $g_i^{-1} D_i g_i$ . The resulting topological torus  $\pi_i(D_i g_i) \subset G_i/\Gamma_i$  will be denoted  $\mathcal{T}_i$ . Write  $z_i = \pi_i(g_i)$ , so that  $\mathcal{T}_i = D_i z_i$ .

For every  $1 \leq k < l \leq n_i$ , define as in [7]:

$$N_{k,l}^{(i)} = \left\{ \text{diag}(a_1, \dots, a_{n_i}) : \prod_{s=1}^{n_i} a_s = 1, a_k = a_l, a_s > 0 \right\} \subset D_i,$$

Of interest to us amongst the consequences of  $\mathbf{Q}$ -irreducibility is the fact that an element of  $\Gamma_i \cap g_i^{-1} D_i g_i$  lying in a wall of a Weyl chamber is necessarily trivial. This is expressed in the following form:

**Lemma 3.1.** *For every  $1 \leq k < l \leq n_i$ , and any closed connected subgroup  $L$  of positive dimension of  $N_{k,l}^{(i)}$ , the  $L$ -orbit of  $z_i$  is not compact.*

*Proof.* Assume the contrary, that is  $Lz_i$  is compact. This implies that  $g_i^{-1}Lg_i \cap \Gamma_i$  is a uniform lattice in  $g_i^{-1}Lg_i$ , so  $g_i^{-1}Lg_i \cap \theta(\mathbf{H}_i(\mathbf{Z}))$  is also a uniform lattice. Since  $L$  is nontrivial, there exists an element  $\gamma \in \mathbf{H}_i(\mathbf{Z}) \cap \mathbf{H}_i^0(\mathbf{R})$  of infinite order, such that  $g_i\theta(\gamma)g_i^{-1}$  is in  $L$ . Note that since  $\theta$  has compact kernel,  $\mathbf{T}_i(\mathbf{Z})$  is a lattice in  $\theta^{-1}(\theta(\mathbf{T}_i^0(\mathbf{R})))$  and is then a subgroup of finite index in  $\mathbf{H}_i(\mathbf{Z}) \cap \mathbf{H}_i^0(\mathbf{R}) \cap \theta^{-1}(\theta(\mathbf{T}_i^0(\mathbf{R})))$ , so there exists  $n > 0$  such that  $\gamma^n$  belongs to  $\mathbf{T}_i(\mathbf{Z})$ . Consider the representation:

$$\begin{aligned} \rho : \mathbf{H}_i^0(\mathbf{R}) &\rightarrow \mathbf{GL}(\mathfrak{sl}(n_i, \mathbf{R})), \\ x &\mapsto \mathbf{Ad}(g_i\theta(x)g_i^{-1}). \end{aligned}$$

Recall that  $\chi(\text{diag}(a_1, \dots, a_{n_i})) = a_k/a_l$  is a weight of  $\mathbf{Ad}$  with respect to  $D_i$ , so  $\chi$  is a weight of  $\rho$  with respect to  $\mathbf{T}_i$ . By [12, Proposition 1, (iii)], the  $\mathbf{Q}$ -irreducibility of  $\mathbf{T}_i$  implies that  $\chi(\gamma^n) \neq 1$ , but this contradicts the fact that  $\theta(\gamma^n) \in g_i^{-1}N_{k,l}^{(i)}g_i$ .  $\square$

**3.2. Contraction and expansion.** For real  $s$ , denote by  $a_i(s)$  the following  $n_i \times n_i$ -matrix:

$$a_i(s) = \text{diag}(e^{s/2}, \underbrace{1, \dots, 1}_{n_i-2 \text{ times}}, e^{-s/2}),$$

and write simply  $N_i$  for  $N_{1,n_i}^{(i)}$ . Write also:

$$h_i(t) = \begin{bmatrix} 1 & 0 & \dots & 0 & t \\ 0 & 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix}.$$

Then the following commutation relation holds:

$$a_i(s)h_i(t) = h_i(e^st)a_i(s),$$

that is the direction  $h_i$  is expanded for positive  $s$ ; note that both  $h_i$  and  $a_i$  commute with elements of  $N_i$ . It is easy to check from Equation (1) that

$$A_1 = \{(a_1(s)d_1, a_2(-s)d_2) : s \in \mathbf{R}, d_i \in N_i, i = 1, 2\}.$$

Recall that  $y_i = h_i(1)z_i$ .

**Lemma 3.2.** (1) *If  $s \leq 0$ , for any  $d \in N_i$  the point  $a_i(s)dy_i$  lies in the compact set  $K_i = h_i([0, 1])\mathcal{T}_i$ .*

(2) *The  $D_i$ -orbit of  $y_i$  is dense in  $G_i/\Gamma_i$ .*

(3) *The set  $\{a_i(s)dy_i : s \geq 0, d \in N_i\}$  is dense in  $G_i/\Gamma_i$ .*

*Proof.* The first statement is clear from the commutation relation. It also implies that  $D_i y_i$  contains the compact torus  $\mathcal{T}_i$  in its closure.



To prove the second point, we rely heavily on the paper of Lindenstrauss and Weiss. [7, Theorem 1.1] applies here, since the hypotheses of their Theorem is precisely the conclusion of Lemma 3.1 for  $L = N_{k,l}^{(i)}$ . So the following holds: there exists a reductive subgroup  $H_i$ , containing  $D_i$ , such that  $\overline{D_i y_i} = H_i y_i$ , and  $H_i \cap \Gamma_i$  is a lattice in  $H_i$ . Write  $L = D_i \cap C_{G_i}(H_i)$ .

Since  $D_i y_i$  is not closed,  $H_i \neq D_i$ , so there exists a nontrivial root relatively to  $D_i$  for the Adjoint representation of  $H_i$  on its Lie algebra, which is a subalgebra of  $\mathfrak{sl}(n_i, \mathbf{R})$ . Thus there exist  $k, l$  such that  $L \subset N_{k,l}^{(i)}$ . By [7, step 4.1 of Lemma 4.2],  $L z_i$  is compact, so by Lemma 3.1,  $L$  is trivial. By [7, Proposition 3.1],  $H_i$  is the connected component of the identity of  $C_{G_i}(L)$ , so  $H_i = G_i$ , as desired.

The third claim follows from the first and second claim together with the fact that  $K_i$  has empty interior.  $\square$

### 3.3. Topological properties of the $A_1$ -orbit.

**Lemma 3.3.** *The  $A_1$ -orbit of  $(y_1, y_2)$  is not dense in  $G_1 \times G_2 / \Gamma_1 \times \Gamma_2$ .*

*Proof.* Consider the open set  $U = K_1^c \times K_2^c$ . We claim that the  $A_1$ -orbit of  $(y_1, y_2)$  does not intersect  $U$ . Indeed, if  $(a_1(s)d_1, a_2(-s)d_2) \in A_1$  with  $s \in \mathbf{R}$  and  $d_i \in N_i$ , the previous Lemma implies that if  $s \geq 0$ ,  $a_2(-s)d_2 y_2 \in K_2$ , and if  $s \leq 0$ ,  $a_1(s)d_1 y_1 \in K_1$ .  $\square$

The following elementary result will be useful:

**Lemma 3.4.** *Let  $p_i : G_1 \times G_2 \rightarrow G_i$  be the first (resp. second) coordinate morphism. If  $F \subset G_1 \times G_2$  is a subgroup such that  $p_i(F) = G_i$  for  $i = 1, 2$ , and  $A_1 \subset F$ , then  $F = G_1 \times G_2$ .*

*Proof.* Let  $F_1 = \text{Ker}(p_1) \cap F$ . Since  $F_1$  is normal in  $F$ ,  $p_2(F_1)$  is normal in  $p_2(F) = G_2$ . Note that  $N_2 \subset p_2(A_1 \cap \text{Ker}(p_1)) \subset p_2(F_1)$  is not finite, and that  $G_2$  is almost simple, consequently the normal subgroup  $p_2(F_1)$  of  $G_2$  is equal to  $G_2$ . Let  $(a, b) \in G_1 \times G_2$ , by assumption there exists  $f \in F$  such that  $p_1(f) = a$ . Let  $f_1 \in F_1$  be such that  $p_2(f_1) = b p_2(f)^{-1}$ , then  $(a, b) = f_1 f \in F$ .  $\square$

We will have to apply several times the two following well-known Lemmas:

**Lemma 3.5.** *Let  $L$  be a Lie group,  $\Lambda \subset L$  a lattice,  $M, N$  two closed, connected subgroups of  $L$ , such that for some  $w \in L/\Lambda$ ,  $Mw$  and  $Nw$  are closed. Then  $(M \cap N)w$  is closed.*

*Proof.* This is a weaker form of [14, Lemma 2.2].  $\square$

**Lemma 3.6.** *Let  $L$  be a connected Lie group,  $\Lambda \subset L$  a discrete subgroup,  $M, N$  two subgroups of  $L$ , such that  $M$  is closed and connected, and  $N$  is a countable union of closed sets. For any  $w \in L/\Lambda$ , if  $Mw \subset Nw$ , then  $M \subset N$ .*

*Proof.* Up to changing  $\Lambda$  by one of its conjugate in  $L$ , one can assume that  $w = \Lambda \in L/\Lambda$ . By assumption,  $M\Lambda \subset N\Lambda$  so  $M \subset N\Lambda \subset L$ . Recall that  $M$

is closed, that  $\Lambda$  is countable, and that  $N$  is a countable union of closed sets, so Baire's category Theorem applies, and there exists  $\lambda \in \Lambda$  and an open set  $U$  of  $M$  such that  $U \subset N\lambda$ , so  $UU^{-1} \subset N$ . Since  $M$  is a connected subgroup,  $UU^{-1}$  generates  $M$ , so  $M \subset N$ .  $\square$

The following lemma will be useful both for proving that the closure of  $A_1(y_1, y_2)$  is not homogeneous, and for proving it does not fiber over a 1-parameter group orbit.

**Lemma 3.7.** *Let  $F$  be a closed connected subgroup of  $G_1 \times G_2$  such that  $F(y_1, y_2)$  contains the closure of  $A_1(y_1, y_2)$ . Then  $F = G_1 \times G_2$ .*

*Proof.* By Lemma 3.2, the set of first coordinates of the set

$$\{(a(s)d_1y_1, a(-s)d_2y_2) : s \geq 0, d_i \in N_i\},$$

is dense in  $G_1/\Gamma_1$  and the second coordinates lies in the compact set  $K_2$ , so the closure of  $A_1(y_1, y_2)$  contains points of arbitrary first coordinate with their second coordinate in  $K_2$ . Consequently, the set of first coordinates of  $F(y_1, y_2)$  is the whole  $G_1/\Gamma_1$ , and similarly for the set of second coordinates. For  $i = 1, 2$ , Lemma 3.6 now applies to  $L = M = G_i$ ,  $\Lambda = \Gamma_i$ ,  $N = p_i(F)$ , which is a countable union of closed sets because  $G_1 \times G_2$  is  $\sigma$ -compact, and  $w = y_i$ , and so  $p_i(F) = G_i$ .

In order to apply Lemma 3.4 and finish the proof, we have to show that  $A_1 \subset F$ . Again, this follows from a direct application of Lemma 3.6 to  $L = G_1 \times G_2$ ,  $\Lambda = \Gamma_1 \times \Gamma_2$ ,  $M = A_1$ ,  $N = F$ ,  $w = (y_1, y_2)$ .  $\square$

**3.4. Proof of Theorem 1, part (1).** We now proceed to proving Theorem 1, part (1). The proof of Proposition 1 is similar and is omitted.

Recall that in this case, we fixed  $A = \overline{\Psi}(A_1)$  and  $x = \overline{\Psi}(y_1, y_2)$ .

Assume  $\overline{Ax}$  is homogeneous, that is  $\overline{Ax} = Fx$  for a closed connected subgroup  $F$  of  $G$ . Since  $Ax \subset \overline{\Psi}(G_1 \times G_2/\Gamma_1 \times \Gamma_2)$ , which is closed in  $G/\Gamma$ , Lemma 3.6 imply that  $F \subset \overline{\Psi}(G_1 \times G_2)$ . By Lemma 3.7,  $F = \Psi(G_1 \times G_2)$ , so  $Fx = G/\Gamma$  and  $Ax$  is dense in  $\overline{\Psi}(G_1 \times G_2)$ , which is a contradiction.

Now assume  $\overline{Ax}$  fibers over the orbit of a one-parameter subgroup. Let  $F$  be a closed connected subgroup,  $L$  a Lie group and  $\phi : F \rightarrow L$  a continuous epimorphism satisfying the (b) of the conjecture. Let  $F' = F \cap \Psi(G_1 \times G_2)$ , we have  $A \subset F'$ . By Lemma 3.5,  $F'x$  is closed in  $F'x \cap \overline{\Psi}(G_1 \times G_2)$ , so is closed in  $G/\Gamma$ . By Lemma 3.7,  $F' = \Psi(G_1 \times G_2)$  necessarily. Let  $H = \text{Ker}(\phi \circ \Psi) \subset G_1 \times G_2$ , so  $A_1/(A_1 \cap H)$  is a one-parameter group by assumption (b).

The subgroup  $H$  is a normal subgroup of the semisimple group  $G_1 \times G_2$ , which has only four kind of normal subgroups : finite,  $G_1 \times G_2$ ,  $G_1 \times \text{finite}$  and  $\text{finite} \times G_2$ . None of these possible normal subgroups have the property that they intersect  $A_1$  in a codimension 1 subgroup, so this is a contradiction.

**3.5. The arithmetic lattice.** Here we prove that  $\mathbf{SU}(n, \mathbf{Z}[\sqrt[4]{2}], \sigma)$  is a lattice in  $\mathbf{SL}(n, \mathbf{R})$ . Let  $P, Q$  be the polynomials with coefficients in  $\mathbf{Q}(\sqrt{2})$  such that for any  $X, Y \in M_n(\mathbf{C})$

$$\det(X + \sqrt[4]{2}Y) = P(X, Y) + \sqrt[4]{2}Q(X, Y).$$

For an integral domain  $\mathbf{A} \subset \mathbf{C}$ , consider the set of pairs of matrices:

$$\mathbf{G}(\mathbf{A}) = \{(X, Y) \in M_n(\mathbf{A})^2 : {}^tXX - \sqrt{2}{}^tYY = I_n, {}^tXY - {}^tYX = 0, \\ P(X, Y) = 1, Q(X, Y) = 0\},$$

which implies that  $({}^tX - \sqrt[4]{2}{}^tY)(X + \sqrt[4]{2}Y) = I_n$  and  $\det(X + \sqrt[4]{2}Y) = 1$  for all  $(X, Y) \in \mathbf{G}(\mathbf{A})$ . Endow  $\mathbf{G}(\mathbf{A})$  with the multiplication given by

$$(X, Y)(X', Y') = (XX' + \sqrt{2}YY', XY' + YX'),$$

which is such that the map  $\phi : \mathbf{G}(\mathbf{A}) \rightarrow \mathbf{SL}(n, \mathbf{C})$ ,  $(X, Y) \mapsto X + \sqrt[4]{2}Y$  is a morphism. With this structure,  $\mathbf{G}$  is an algebraic group, which is clearly defined over  $\mathbf{Q}(\sqrt{2})$ . Let  $\tau$  be the nontrivial field automorphism of  $\mathbf{Q}(\sqrt{2})/\mathbf{Q}$ , it can be checked that the map  $\phi$  is an isomorphism between  $\mathbf{G}(\mathbf{R})$  and  $\mathbf{SL}(n, \mathbf{R})$ , and that moreover  $\phi' : \mathbf{G}^\tau(\mathbf{R}) \rightarrow \mathbf{SL}(n, \mathbf{C})$ ,  $(X, Y) \mapsto X + i\sqrt[4]{2}Y$  is an isomorphism onto  $\mathbf{SU}(n)$ . Let  $\mathbf{H} = \text{Res}_{\mathbf{Q}(\sqrt{2})/\mathbf{Q}} \mathbf{G} = \mathbf{G} \times \mathbf{G}^\tau$ . Then  $\mathbf{H}$  is defined over  $\mathbf{Q}$  (see for example [16, 6.1.3], for definition and properties of the restriction of scalars functor). It follows from a Theorem of Borel and Harish-Chandra [16, Theorem 3.1.7] that  $\mathbf{H}(\mathbf{Z})$  is a lattice in  $\mathbf{H}(\mathbf{R})$ . Since  $\mathbf{SU}(n)$  is compact, it follows that the projection of  $\mathbf{H}(\mathbf{Z})$  onto the first factor of  $\mathbf{G}(\mathbf{R}) \times \mathbf{G}^\tau(\mathbf{R})$  is again a lattice. Using the isomorphism between  $\mathbf{G}(\mathbf{R})$  and  $\mathbf{SL}(n, \mathbf{R})$ , this projection can be identified with

$$\mathbf{G}(\mathbf{Z}[\sqrt{2}]) = \mathbf{SU}(n, \mathbf{Z}[\sqrt{2}] + \sqrt[4]{2}\mathbf{Z}[\sqrt{2}], \sigma) = \mathbf{SU}(n, \mathbf{Z}[\sqrt[4]{2}], \sigma).$$

**3.6. Proof of Theorem 1, part (2).** Note that, as stated implicitly in Section 2.3,

$$\varphi(\Gamma_1 \times \Gamma_2 \times (\log \alpha)\mathbf{Z}) \subset \Gamma \cap M,$$

so  $\Gamma \cap M$  is a lattice in  $M$ , and  $M/(M \cap \Gamma)$  is a closed,  $A$ -invariant subset of  $G/\Gamma$ . Notice also that the map  $\Psi$  defined by Equation (2) defines an embedding  $\overline{\Psi} : G_1 \times G_2/\Gamma_1 \times \Gamma_2 \rightarrow G/\Gamma$ .

Assume  $\overline{Ax}$  is homogeneous, that is  $\overline{Ax} = Fx$  for a closed connected subgroup  $F$  of  $G$ . Since  $Ax \subset M/(M \cap \Gamma)$ , which is closed in  $G/\Gamma$ , Lemma 3.6 applied twice gives that  $A \subset F \subset M$ . Let  $F' = F \cap \Psi(G_1 \times G_2)$ , again by Lemma 3.5,  $F'x$  is a closed subset of  $\text{Im}(\overline{\Psi})$ . Since  $A_1 \subset F'$ ,  $\Psi(A_1)x \subset F'x$  and Lemma 3.7 implies that  $F' = \Psi(G_1 \times G_2)$ . Since  $A$  contains  $\varphi(e, e, t)$  for all  $t \in \mathbf{R}$ , we have  $M = AF' \subset F$  so  $F = M$  necessarily.

By Lemma 3.3, the  $A_1$ -orbit of  $(y_1, y_2)$  is not dense; the topological transitivity of the action of  $A_1$  on  $G_1 \times G_2/\Gamma_1 \times \Gamma_2$  implies that moreover the closure of this orbit has empty interior. Thus, the  $A_1 \times \mathbf{R}$ -orbit of  $(y_1, y_2, 0)$  is also nowhere

dense in  $G_1 \times G_2 \times \mathbf{R}/\Gamma_1 \times \Gamma_2 \times (\log \alpha)\mathbf{Z}$ . The map  $\bar{\varphi}$  being a finite covering, the  $A$ -orbit of  $x$  is nowhere dense. This is a contradiction with  $F = M$ .

Now assume  $\overline{Ax}$  fibers over the orbit of a one-parameter non-**Ad**-unipotent subgroup. Let  $F$  be a closed connected subgroup,  $L$  a Lie group and  $\phi : F \rightarrow L$  a continuous epimorphism satisfying the (b) of the conjecture. Let  $F' = F \cap \Psi(G_1 \times G_2)$  and  $F'' = F \cap M$ , we have  $A_1 \subset F'$  and  $A \subset F''$ . Similarly,  $F'x$  and  $F''x$  are closed in  $G/\Gamma$ . Again, by Lemma 3.7,  $F' = \Psi(G_1 \times G_2)$  necessarily, and like before,  $AF' \subset F'' \subset M$  so  $F'' = M$ .

Let  $H = \text{Ker}(\phi \circ \varphi) \subset G_1 \times G_2 \times \mathbf{R}$ , so  $A_1 \times \mathbf{R}/(A_1 \times \mathbf{R} \cap H)$  is a one-parameter group. This time, possibilities for the closed normal subgroup  $H$  are: finite  $\times \Lambda$ ,  $G_1 \times G_2 \times \Lambda$ ,  $G_1 \times \text{finite} \times \Lambda$  and finite  $\times G_2 \times \Lambda$ , where  $\Lambda$  is a closed subgroup of  $\mathbf{R}$ . Of all these possibilities, only  $G_1 \times G_2 \times \Lambda$ , where  $\Lambda$  is discrete, has the required property that  $A_1 \times \mathbf{R}/(A_1 \times \mathbf{R} \cap H)$  is a one-parameter group. This proves that  $\Psi(G_1 \times G_2) \subset \text{Ker}(\phi)$ , so  $F \subset N_G(\Psi(G_1 \times G_2))$ . However, the normalizer of  $\Psi(G_1 \times G_2)$  in  $G$  is the group of block matrices having for connected component of the identity the group  $M$ . So by connectedness of  $F$ ,  $F \subset M$ , and since  $M = F'' \subset F$ , we have  $F = M$ . Thus  $L = F/\text{Ker}(\phi) = \mathbf{R}/\Lambda$  is abelian, and a fortiori every element of  $L$  is unipotent; this contradicts (b).

#### 4. PROOF OF THEOREM 2

The proof of Theorem 2 is divided in two independent lemmas.

**Lemma 4.1.** *The family  $(z_1, \dots, z_{2q}, 1)$  is linearly independent over  $\mathbf{Q}$ .*

*Proof.* Consider a linear combination:

$$\sum_{i=1}^q a_i z_i + b_i z_{i+q} = c.$$

We can assume that  $a_i, b_i$  and  $c$  are integers. Let  $k_0 \geq 1$ , write

$$(3) \quad \left( \prod_{i=1}^q p_i \right)^{N^{2k_0+1}} \left( \sum_{i=1}^q \sum_{k=1}^{k_0} a_i p_i^{-N^{2k}} + b_i p_i^{-N^{2k+1}} - c \right) = \\ - \left( \prod_{i=1}^q p_i \right)^{N^{2k_0+1}} \left( \sum_{i=1}^q \sum_{k \geq k_0+1} a_i p_i^{-N^{2k}} + b_i p_i^{-N^{2k+1}} \right).$$

It is clear the left hand side is an integer. Since  $1 < p_1 < \dots < p_q$ , the right hand side is less in absolute value than

$$\begin{aligned} p_q^{qN^{2k_0+1}} 2q \sup_i (|a_i|, |b_i|) \sum_{k \geq 0} \left( p_1^{-N^{2k_0+2}} \right)^{N^{2k}} \\ \leq 4q \sup_i (|a_i|, |b_i|) p_q^{qN^{2k_0+1}} p_1^{-N^{2k_0+2}} \\ \leq 4q \sup_i (|a_i|, |b_i|) \exp(N^{2k_0+1}(q \log p_q - N \log p_1)). \end{aligned}$$

Since  $N > q \frac{\log(p_q)}{\log(p_1)}$ , the last expression tends to zero. This proves the right-hand side of (3) is zero for large enough  $k_0$ , so for all large  $k$ ,

$$\sum_{i=1}^q a_i p_i^{-N^{2k}} + b_i p_i^{-N^{2k+1}} = 0.$$

The  $p_i$  being distincts, this implies that for  $i \in \{1, \dots, q\}$ ,  $a_i = b_i = 0$ .  $\square$

The following Lemma implies easily that the orbit of  $z$  under  $\Omega$  cannot be dense.

**Lemma 4.2.** *For all  $\epsilon > 0$ , there exists  $L > 0$ , such that for all  $n_1, \dots, n_q \geq 0$  with  $\sum_{i=1}^q n_i \geq L$ , there exists  $j \in \{1, \dots, 2q\}$  such that  $p_1^{n_1} \dots p_q^{n_q} z_j$  lies in the interval  $[0, \epsilon]$  modulo 1.*

*Proof.* Let  $s \in \{1, \dots, q\}$  such that for all  $r \in \{1, \dots, q\}$ ,  $p_s^{n_s} \geq p_r^{n_r}$ . Let  $k_0$  be the integer part of  $\log(n_s)/2 \log(N)$ , then either  $N^{2k_0} \leq n_s \leq N^{2k_0+1}$ , or  $N^{2k_0+1} \leq n_s \leq N^{2k_0+2}$ . In the first case, take  $j = s$ , then:

$$p_1^{n_1} \dots p_q^{n_q} z_j = p_1^{n_1} \dots p_q^{n_q} \sum_{k \geq 1} p_s^{-N^{2k}} = p_1^{n_1} \dots p_q^{n_q} \sum_{k \geq k_0+1} p_s^{-N^{2k}} \pmod{1}.$$

We have

$$\sum_{k \geq k_0+1} p_s^{-N^{2k}} \leq 2p_s^{-N^{2k_0+2}},$$

so, using the fact that for all  $r \in \{1, \dots, q\}$ ,  $p_r^{n_r} \leq p_s^{n_s} \leq p_s^{N^{2k_0+1}}$ , we obtain:

$$p_1^{n_1} \dots p_q^{n_q} \sum_{k \geq k_0+1} p_s^{-N^{2k}} \leq 2p_s^{qN^{2k_0+1} - N^{2k_0+2}} \leq 2p_s^{N^{2k_0+1}(q-N)},$$

but by hypothesis we have  $N > q \frac{\log(p_q)}{\log(p_1)} > q$ , so the preceding bound is small whenever  $k_0$  is large. Because of the definition of  $k_0$ , we have

$$k_0 \geq \frac{\log \frac{\sum_{i=1}^q n_i \log p_i}{q \log p_q}}{2 \log N} \geq \frac{\log \frac{L \log p_1}{q \log p_q}}{2 \log N},$$

so  $k_0$  is arbitrary large when  $L$  is large.

In the second case  $N^{2k_0+1} \leq n_s \leq N^{2k_0+2}$ , one can proceed similarly with  $j = s + q$ .

□

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