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► **To cite this version:**

S.E. Derkachov, N.A. Kivel, A.S. Stepanenko, A.N. Vasil'Ev. On the calculation of $2 + \epsilon$ RG functions in the Gross-Neveu model from large- N expansions of critical exponents. Nuclear Physics B, 1994, 424, pp.619-627. hal-00164028

HAL Id: hal-00164028

<https://hal.science/hal-00164028v1>

Submitted on 19 Jul 2007

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On Calculation of $1/n$ Expansions of Critical Exponents in the Gross–Neveu Model with the Conformal Technique.

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Abstract

A proof of critical conformal invariance of Green's functions for a quite wide class of models possessing critical scale invariance is given. A simple method for establishing critical conformal invariance of a composite operator, which has a certain critical dimension, is also presented. The method is illustrated with the example of the Gross–Neveu model and the exponents η at order $1/n^3$, Δ and $1/\nu$ at order $1/n^2$ are calculated with the conformal bootstrap method.

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1 Introduction.

It is well known that the structure of renormalized field theory is determined by the renormalization group (RG) equations. One considerable problem then is the calculation of renormalization group functions. Usually one can compute only a few orders in perturbation theory. Hence, we do not have a complete knowledge of these functions. Other methods, such as the large N expansion, usually allow us to compute RG functions at leading order only because calculations become too complicated at higher orders.

To overcome these difficulties other methods have been developed. One such approach [6, 4] uses critical scale invariance at the fixed point of renormalization group for calculation of critical exponents, which are values of RG functions at fixed point. With this method it is possible to derive the critical exponents at order $1/N^2$. Using then the exact N dependence of RG functions at each order in perturbation theory one can reconstruct them at some high-loop approximation.

In order to derive the critical exponents at order $1/N^3$ a more powerful technique, the conformal bootstrap method, can be applied (see, for example, [5] and references given there). To use this method it is necessary that the model possesses conformal symmetry at the critical point. Up to now this method has been applied only to the calculation of the exponent η at order $1/N^3$ in the non-linear σ model [5] which has critical conformal symmetry. The aim of this paper is a proof of critical conformal invariance for a quite wide class of models. An example will be provided by the Gross–Neveu model.

2 Preliminaries and conventions.

The U_N -symmetric Gross–Neveu (GN) model describes a system of N d -dimensional Dirac spinors $\psi, \bar{\psi} \equiv \{\psi_a, \bar{\psi}_a, a = 1, \dots, N\}$, its non-renormalized action being

$$S = \int dx [\bar{\psi} \hat{\partial} \psi + g_0 (\bar{\psi} \psi)^2 / 2] \quad (1)$$

(in what follows we consider the theory in Euclidean space and functional distribution is denoted by $\exp S$ without the minus in the exponent). In statistical physics the coupling constant g_0 in (1) is a bare temperature. It is well known that the GN model has properties quite analogous to the non-linear σ -model. It is renormalizable in two dimensions and its renormalized action in $d = 2 + \epsilon$ dimension is

$$S = \int dx [Z_1 \bar{\psi} \hat{\partial} \psi + g M^{-\epsilon} Z_2 (\bar{\psi} \psi)^2 / 2], \quad (2)$$

where g is a dimensionless renormalized coupling constant (renormalized temperature), M is a renormalization mass and $Z_{1,2}$ are renormalization constants.

If g is small the fermions are massless due to a chiral symmetry. When g increases the system has a second order phase transition at some critical point g_* (fixed point of renormalization group). At this point the chiral symmetry is spontaneously broken and the fermions become massive. There is a critical scaling in the vicinity of g_* . Critical

dimensions $\Delta[F]$ of various quantities F (fields and parameters) can be calculated in the frameworks of $2 + \epsilon$, $4 - \epsilon$ and $1/n$ expansions with $n \equiv N \cdot \text{tr } \mathbf{1}$, $\text{tr } \mathbf{1}$ being a trace of the unit matrix in the space of d -dimensional spinors. This trace does not have (and does not require) an exact expression through d (often one takes $\text{tr } \mathbf{1} = 2, 2, 4$ for the integer dimensions $d = 2, 3, 4$).

The $2 + \epsilon$ expansion can be derived from the perturbation calculation of RG functions [1]. To generate a systematic $1/n$ expansion a scalar field σ is introduced and the GN action (1) is written in the following form:

$$S = \int dx [\bar{\psi} \hat{\partial} \psi - \sigma^2/2g_0 + \bar{\psi} \psi \sigma]. \quad (3)$$

The $4 - \epsilon$ expansion can be derived from the model suggested in [2]. This model differs from (3) by adding of terms like $(\partial\sigma)^2$ and σ^4 . In the framework of the $1/n$ expansion such terms play no role in the IR limit. Hence, the models [2] and (3) have the same $1/n$ expansion. But the model [2] is renormalizable in $d = 4$ and it allows us to obtain the $4 - \epsilon$ expansion of the same critical exponents with the standard RG technique.

In what follows we shall consider $1/n$ expansions in arbitrary dimension of space d and the following notations will be used:

$$N \text{tr } \mathbf{1} \equiv n, \quad d \equiv 2\mu, \quad B_z \equiv \psi(z) + \psi(\mu - z), \quad (4)$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the gamma-function $\Gamma(z)$. In the framework of the $1/n$ expansion for the model (3), the canonical dimensions of the fields $\varphi \equiv \psi, \sigma$ and temperature $\tau \equiv g_* - g$ are

$$d_\psi = d_{\bar{\psi}} = \mu - 1/2, \quad d_\sigma = 1, \quad d_\tau = 2(\mu - 1). \quad (5)$$

The notation $d[F] \equiv d_F$ means canonical dimension of quantity F . Let us parametrize the critical dimensions $\Delta_F = d_F + \gamma_F^*$ (canonical + anomalous) as follows:

$$\Delta_\psi = d_\psi + \eta/2, \quad \Delta_\sigma = d_\sigma - \eta - 2\Delta, \quad \Delta_\tau \equiv 1/\nu \equiv 2\lambda \quad (\lambda_0 = \mu - 1), \quad (6)$$

where η and ν are the usual notations for the critical exponents, the quantity 2Δ is a critical dimension of the vertex $\bar{\psi} \psi \sigma$ in the model (3). For this model a connection between the quantity Δ_τ and the critical dimension $\Delta[\sigma^2]$ of a composite operator $F = \sigma^2$ is expressed by the following relation

$$\Delta_\tau \equiv 1/\nu \equiv 2\lambda = 2\mu - \Delta[\sigma^2]. \quad (7)$$

Note that in the framework of the $2 + \epsilon$ scheme for the purely fermion model (2) (without the field σ) the quantity Δ_σ can be calculated directly from the relation $\Delta_\sigma = \Delta[\bar{\psi} \psi] = 2\mu - \Delta_m$ through the dimension of the composite operator $F = \bar{\psi} \psi$ or through the dimension of the fermion mass m for a massive generalization of the model (2) (this mass is generated by the substitution $\hat{\partial} \rightarrow \hat{\partial} + m_0$ with $m_0 = mZ_m$).

For the $1/n$ expansion of any exponent z its coefficients in powers of $1/n$ are denoted by z_k : $z = z_0 + z_1/n + z_2/n^2 + \dots$. In what follows the results for the three independent exponents η, Δ, λ in (6) are presented.

3 Main results for the Gross–Neveu model.

The first coefficients $\eta_1, \Delta_1, \lambda_1$ for the special case of dimension $d = 2\mu = 3$ have been calculated in [3]; in our notations these results have the form:

$$\eta_1 = 8/3\pi^2, \quad \Delta_1/\eta_1 = 3/2, \quad \lambda_1/\eta_1 = -2. \quad (8)$$

The same coefficients and η_2 for arbitrary dimension $d = 2\mu$ have been calculated in [4]. Their results are

$$\left. \begin{aligned} \eta_1 &= -\frac{2\Gamma(2\mu-1)}{\Gamma(\mu)\Gamma(\mu+1)\Gamma(\mu-1)\Gamma(1-\mu)}, \\ \Delta_1/\eta_1 &= \frac{\mu}{2(\mu-1)}, \quad \lambda_1/\eta_1 = -(2\mu-1), \\ \eta_2/\eta_1^2 &= (B_{1-\mu} - B_1)\frac{(2\mu-1)}{(\mu-1)} + \frac{1}{2\mu} - \frac{\mu}{2(\mu-1)^2}. \end{aligned} \right\} \quad (9)$$

We calculated the following coefficients Δ_2, λ_2 and η_3 :

$$\left. \begin{aligned} \Delta_2/\eta_1^2 &= \frac{\mu}{2(\mu-1)} \left[(B_{1-\mu} - B_1)\frac{(2\mu-1)}{(\mu-1)} - 3\mu B'_1 - \right. \\ &\quad \left. -2\mu - 1 - \frac{6}{(\mu-1)} - \frac{4}{(\mu-1)^2} \right], \end{aligned} \right\} \quad (10a)$$

$$\left. \begin{aligned} \lambda_2/\eta_1^2 &= \frac{\mu}{2(\mu-1)} \left[\frac{8}{\eta_1(2-\mu)^2} + [B'_{2-\mu} - (B_{2-\mu} - B_1)^2]\frac{4\mu(3-2\mu)}{(2-\mu)} + \right. \\ &\quad + (B_{2-\mu} - B_1) \left(-8\mu^2 + 2 - \frac{8}{(\mu-1)} - \frac{10}{(2-\mu)} + \frac{4}{(2-\mu)^2} - \frac{2}{\mu} \right) + \\ &\quad + \mu B'_1 \left(6\mu - 35 + \frac{22}{(2-\mu)} \right) + 8\mu^2 - 16\mu - 2 + \\ &\quad \left. + \frac{14}{(\mu-1)} - \frac{5}{(\mu-1)^2} + \frac{42}{(2-\mu)} - \frac{4}{(2-\mu)^2} + \frac{4}{\mu} - \frac{1}{\mu^2} \right] \end{aligned} \right\} \quad (10b)$$

$$\left. \begin{aligned} \eta_3/\eta_1^3 &= \frac{1}{4} \left[(B'_1 - B'_{1-\mu})\frac{2(2\mu-1)^2}{(\mu-1)^2} - I(\mu)B'_1\frac{6\mu^2}{(\mu-1)} + \right. \\ &\quad + B'_1 \left(8 - \mu + \frac{16}{(\mu-1)} + \frac{9}{(\mu-1)^2} \right) - \\ &\quad - B'_1(B_{1-\mu} - B_1)\frac{12\mu^2}{(\mu-1)} + (B_{1-\mu} - B_1)^2\frac{6(2\mu-1)^2}{(\mu-1)^2} + \\ &\quad + (B_{1-\mu} - B_1) \left(-4\mu - 6 + \frac{6}{\mu} - \frac{28}{(\mu-1)} - \frac{40}{(\mu-1)^2} - \frac{14}{(\mu-1)^3} \right) + \\ &\quad \left. -4\mu - 6 + \frac{2}{\mu^2} - \frac{9}{(\mu-1)} + \frac{3}{(\mu-1)^2} + \frac{12}{(\mu-1)^3} + \frac{5}{(\mu-1)^4} \right], \end{aligned} \right\} \quad (10c)$$

where the notations (4) have been used and $I(\mu)$ is a function which cannot be expressed explicitly through $\Gamma(z)$ and its derivatives. A definition of this function will be given below. The expression for η_3/η_1^3 in the non-linear σ -model [5], which is analogous to (10c), also contains this function. We write below this expression for the σ -model in the notations (10c), correcting a misprint in [5]:

$$\left. \begin{aligned}
& \eta_3/\eta_1^3 = -B'_1 I(\mu) \frac{3\mu^2(\mu-1)(4\mu-5)}{2(2-\mu)^2} + \frac{2\mu^2(\mu-1)(2\mu-3)^2}{3(2-\mu)^3} \times \\
& \times \left[3(B_{2-\mu} - B_1)(B'_{2-\mu} - B'_1) - B''_{2-\mu} + B''_1 - (B_{2-\mu} - B_1)^3 \right] + \\
& + \frac{1}{2} \left\{ 8\mu^2 + 26\mu + 70 - \frac{177}{(2-\mu)} + \frac{67}{(2-\mu)^2} + \frac{58}{(2-\mu)^3} - \right. \\
& - \frac{16}{(2-\mu)^4} + \frac{9}{(\mu-1)} + \frac{1}{(\mu-1)^2} + \frac{1}{\mu^2} + (B_{2-\mu} - B_2) \times \\
& \times \left[4\mu^2 + 14\mu + 66 - \frac{187}{(2-\mu)} + \frac{102}{(2-\mu)^2} + \frac{16}{(2-\mu)^3} + \frac{2}{(\mu-1)} + \frac{3}{\mu} \right] + \\
& + (B_{2-\mu} - B_2)^2 \left[20 - \frac{50}{(2-\mu)} + \frac{32}{(2-\mu)^2} \right] - \\
& - B'_1 \left[7\mu^2 - 10\mu - 45 + \frac{127}{(2-\mu)} - \frac{64}{(2-\mu)^2} - \frac{48}{(2-\mu)^3} + \frac{32}{(2-\mu)^4} \right] + \\
& + B'_{2-\mu} \left[8\mu^2 + 8\mu + 14 - \frac{30}{(2-\mu)} \right] - \\
& \left. - (B_{2-\mu} - B_2) B'_1 \left[-2\mu^2 - 13\mu - 45 + \frac{136}{(2-\mu)} - \frac{108}{(2-\mu)^2} + \frac{32}{(2-\mu)^3} \right] \right\}, \quad (11)
\end{aligned} \right\}$$

in the notations (4). In the denominator of the second term of the expression for η_3 in [5] (eq.(22) of [5]) $(2-\mu)^2$ appeared instead of $(2-\mu)^3$. The value of η_3 for the integer dimension $d = 2\mu = 3$ is not affected by the misprint but it is essential for comparing ϵ -expansions. The expression (11) is more complicated than (10c) and this is a general rule: calculations in the GN model are simpler because the number of diagrams is smaller due to the non-existence of odd fermion cycles.

The function $I(\mu)$ is expressed through the self-energy massless diagram showed in Fig.1 on which a line with index a means a propagator like $|x-y|^{-2a}$. According to its dimension the diagram is a simple line with index $\mu-1+\Delta$ multiplied by the coefficient $\Pi(\mu, \Delta)$ which is the “value” of the diagram. The quantity $I(\mu)$ is defined from $\Pi(\mu, \Delta)$ by the relation $I(\mu) = d \ln \Pi(\mu, \Delta) / d\Delta|_{\Delta=0}$ (for $\Delta = 0$ the diagram is calculated exactly: $\Pi(\mu, 0) = 3\pi^{2\mu} B'_{\mu-1} \Gamma(\mu-1) \Gamma(2-\mu) / \Gamma(2\mu-2)$). In [5] the value of $I(\mu)$ for the dimension $d = 2\mu = 3$ and the first terms of ϵ -expansions around $d = 2$ and $d = 4$ are given: $I(3/2) = 3\psi''(1/2)/2\pi^2 + 2 \ln 2$, $I(1+\epsilon) = -2/3\epsilon + O(\epsilon)$, $I(2-\epsilon) = O(\epsilon)$.

The result (9) was obtained in [4] with the technique of self-consistent equations for propagators suggested in [6]. The result (11) at order $1/n^3$, which was the unique one until quite recently, was obtained in [5] with the conformal bootstrap method. In this method not only propagators but 3-point vertices also are dressed and this essentially

simplifies calculations at high orders in $1/n$. We obtained the results (10) with the same method, first proving conformal invariance. Technical details of the calculations will be published elsewhere [7].

Here we restrict ourself to a discussion of the results (10) and their consequences for RG-functions in the model (3) (section 6). Now we give a proof of the critical conformal invariance for a quite wide class of models which includes the GN model.

4 General proof of critical conformal invariance.

Let us consider an arbitrary multiplicatively renormalizable local model. The simplest dimensional regularization and MS scheme are supposed. Let us also suppose that the model has a renormalization group fixed point at which a critical scaling occurs. Hence, the corresponding massless model has a critical scale symmetry. The questions which we ask ourself are 1) would the model under some assumptions have a conformal symmetry of Green's functions $\langle\varphi\varphi\dots\rangle$ for simple fields only and 2) would it have conformal symmetry of Green's function $\langle F\varphi\varphi\dots\rangle$ with one insertion of a some composite operator F (we assume the conformal symmetry with the same critical dimensions as in the scale symmetry).

Let $S(\varphi) = \int dx \mathcal{L}(x)$ be the renormalized action of the massless model, φ being a set of fields (needed indices and summations over them are implied). Let us also assume that $\mathcal{L} = \mathcal{L}_0 + \mathcal{V}$, where \mathcal{L}_0 is the quadratic part of the lagrangian, \mathcal{V} the interaction containing higher powers of φ . The quantity

$$W(A) = \ln \int D\varphi \exp[S(\varphi) + \varphi A] \quad (12)$$

is a generating functional of renormalized connected Green's functions for fields φ and the quantity

$$\langle\langle F \rangle\rangle \equiv \frac{\int D\varphi F \exp[S(\varphi) + \varphi A]}{\int D\varphi \exp[S(\varphi) + \varphi A]} \quad (13)$$

with arbitrary composite operator $F \equiv F(\varphi)$ (local or non-local) is a generating functional of connected Green's functions $\langle F\varphi\varphi\dots\rangle$ with one operator F insertion and any number of fields φ . The source $A(x)$ in (12) and (13) is a functional argument.

Let us consider an arbitrary group of transformations on the fields $\varphi(x) \rightarrow \varphi'(x)$ which has an infinitesimal form like:

$$\varphi'(x) = \varphi(x) + \omega_\alpha T^\alpha \varphi(x) , \quad (14)$$

where ω_α are infinitesimal parameters, T^α are the corresponding generators acting on the fields φ , α is a numbering index. Summation over repeated indices is implied. We are interested in the scale and the special conformal transformations for which we have

$$T^\alpha \equiv T = \delta_\varphi + x\partial \quad (\text{scale}) \quad (15a)$$

$$T^\alpha \equiv T^k = x^2\partial_k - 2x_k x_s \partial_s - 2\delta_\varphi x_k + 2\Sigma_{ks} x_s \quad (\text{conformal}) \quad (15b)$$

where δ_φ is the free given dimension of φ , Σ_{ks} is the spin part of the rotation generator (it is a known matrix acting on the φ field indices). With each transformation group one can associate the corresponding Noether currents J_i^α : $J_i = x_k \theta_i^k + \delta_\varphi \Phi^i \varphi$ for the scale transformations and $J_i^k = (x^2 \delta_{ks} - 2x_k x_s) \theta_i^s - 2x_k \delta_\varphi \Phi^i \varphi + 2x_s \Phi^i \Sigma_{ks} \varphi$ for the special conformal ones. Here we use the following notations $\Phi^i \equiv \partial \mathcal{L} / \partial (\partial_i \varphi)$ and $\theta_i^k \equiv \Phi^i \partial_k \varphi - \delta_{ik} \mathcal{L}$ (the stress-energy tensor). Note that in these notations we have $\delta S(\varphi) / \delta \varphi = \partial \mathcal{L} / \delta \varphi - \partial_i \Phi^i$. We do not distinguish between upper and lower indices, their position is for convenience only.

The following Ward identity is true for any group of transformations even if the system does not have the corresponding symmetry

$$\partial_i J_i^\alpha(x) + [\delta S(\varphi) / \delta \varphi(x)] \cdot T^\alpha \varphi(x) + N^\alpha(x) = 0, \quad (16)$$

where N^α is a “breaking” operator. If the symmetry is exact then $N^\alpha = 0$. From (16) we have

$$\delta^\alpha S \equiv \int dx [\delta S(\varphi) / \delta \varphi(x)] T^\alpha \varphi(x) = - \int dx N^\alpha(x). \quad (17)$$

On other hand from the following Schwinger–Dyson equation

$$0 = \int D\varphi \delta\{T^\alpha \varphi(x) \cdot \exp[S(\varphi) + A\varphi]\} / \delta \varphi(x), \quad (18)$$

one can derive the following relation $\langle\langle \delta^\alpha S + \int dx A(x) T^\alpha \varphi(x) \rangle\rangle = 0$ (here the notation $\delta^\alpha \equiv \int dx T^\alpha \varphi(x) \delta / \delta \varphi(x)$ is used) integrating over x and neglecting terms like $\delta(0)$ (due to the dimensional regularization in the massless theory). From this relation and (17) substituting $\varphi \rightarrow \delta / \delta A$ it follows

$$\mathcal{D}_A^\alpha W(A) = \int dx \langle\langle N^\alpha \rangle\rangle, \quad (19)$$

where $\mathcal{D}_A^\alpha \equiv \int dx A(x) T^\alpha [\delta / \delta A(x)]$. If $N^\alpha = 0$ equation (16) implies that the corresponding symmetry of Green’s functions $\langle \varphi \dots \rangle$, but in the general case of non-zero N^α the symmetry is broken. All the above relations are satisfied for any free chosen dimension δ in (15). The exact form of the breaking operators $N^\alpha \equiv \{N \text{ (scale)}, N^k \text{ (conformal)}\}$ are influenced by the choice of δ . In what follows we assume that δ in (15) are the canonical dimensions $\delta_\varphi = d_\varphi$. Then under some assumptions on \mathcal{L} the following relation between N and N^k holds:

$$N^k(x) = -2x_k N(x). \quad (20)$$

The assumptions are the following:

1. \mathcal{L} is local and invariant under translations and rotations.
2. The quadratic part of \mathcal{L}_0 is invariant under the scale and the conformal transformations with canonical dimensions d_φ .
3. The interaction \mathcal{V} contains explicitly the fields φ but not their derivatives $\partial \varphi$.

With these assumptions, only the interaction \mathcal{V} contributes to N^α . It is then easy to find $N^\alpha \equiv \{N, N^k\}$ exactly, looking for contributions of \mathcal{V} to the l.h.s.. We have $N = -\mathcal{V}d + d_\varphi[\partial\mathcal{V}/\partial\varphi]\varphi$ and $N^k = -2x_k N + 2x_s[\partial\mathcal{V}/\partial\varphi]\Sigma_{ks}\varphi$. The latter contribution is a variation of \mathcal{V} under the purely spin rotations and it has to disappear with our assumptions (locality + rotation symmetry + absence of derivatives $\partial\varphi$ in \mathcal{V}). Hence, the equation (20) is satisfied. For the scale operator we have $N \sim \mathcal{V}$ if \mathcal{V} consists of one term only and it is a linear combination of the terms in the general case. This operator has the same symmetry as \mathcal{L} and canonical dimension d .

The relation (20) is a basis for further constructions. In order to analyze consequences of the relation (19) in the critical region the r.h.s. of (19) should be expressed in terms of renormalized composite operators $[F]_R$ because the non-renormalized ones have no meaning at $g = g_*$ due to the singularities of renormalization constants $Z(g)$ at $g = g_*$. For the scale operator $N^\alpha \equiv N$ it is convenient to use as elements of the basis the following renormalized ones (which correspond to each independent component of the field φ)

$$\varphi(x)\delta S(\varphi)/\delta\varphi(x) = [\varphi(x)\delta S_{\text{bas}}(\varphi)/\delta\varphi(x)]_R . \quad (21)$$

S_{bas} is a ‘‘basis’’ action which is given by the renormalized one S after substitution $Z \rightarrow 1$, the notation $[M]_R$ for any monomial M means the appropriate renormalized one. Note that just the r.h.s. gives a meaning to the operator in (21) at $g = g_*$ because the constants Z contained in the l.h.s. do not exist at $g = g_*$. The Schwinger equation which follows from (18) after the substitution $T^\alpha \rightarrow 1$ yields

$$\langle\langle \varphi(x)\delta S(\varphi)/\delta\varphi(x) \rangle\rangle = -A(x)\delta W(A)/\delta A(x) , \quad (22)$$

which proves UV-finiteness of the operator (21) and vanishing of its anomalous dimension. Note that the equation (21) is a consequence of the MS scheme.

In general the expansion of the scale operator N in terms of renormalized ones contains three kinds of contributions (we denote them by A, B, C respectively):

$$N = \sum_\varphi a_\varphi \varphi \delta S / \delta\varphi + \sum b[F]_R + \sum c \partial[F]_R \quad (23)$$

where a, b, c are numeric coefficients depending on the coupling constant g . The first sum (A) contains all the operators of the type (21), the other two are the remaining independent operators with (C) and without (B) external derivatives. There is one contribution of type (21) (A), one $[\varphi^4]_R$ of type B and one $\partial^2[\varphi^2]_R$ of type C in the simple φ^4 -model. After substitution of (23) into equation (19) for the scale operator, contributions of type C vanish due to the integration over x . Contributions of type A lead to a change of the field dimensions $d_\varphi \rightarrow d_\varphi + a_\varphi$ due to (22) and contributions of type B correspond to the breaking of scale invariance with the dimensions $d_\varphi + a_\varphi$. Since we have assumed that the model has critical scale invariance with the dimensions $d_\varphi + \gamma_\varphi^*$ then we have for the coefficients in (23)

$$a_\varphi(g_*) = \gamma_\varphi^* , \quad b(g_*) = 0 . \quad (24)$$

The quantities $c(g_*)$ are not constrained.

Let us consider now the equation (19) for the conformal case. The renormalization is not influenced by the multiplication by $-2x_k$. Hence, the expansion of the conformal operator $N^\alpha \equiv N^k$ in the r.h.s. of (19) in terms of the renormalized ones is given by (23) multiplied by $-2x_k$. Then from (15b) and (24) it follows that after the substitution into (19) the contributions of types A and B lead to the change $d_\varphi \rightarrow \Delta_\varphi$ of the canonical dimensions into the critical ones. Hence, in the absence of type C contributions the critical conformal invariance would be a consequence of the scale invariance.

But the in general case type C contributions are present in (23) and can lead to a breaking of conformal symmetry. It is clear that after integration over x in (19), only the type C contributions containing only one external derivative like $\partial_k \mathcal{O}_k$ are dangerous because they are multiplied by one factor x_k .

In the presence of such contributions the conformal relation (19) in the critical region yields

$$\mathcal{D}_A^k W(A) = c \int dx \langle\langle \mathcal{O}_k(x) \rangle\rangle , \quad (25)$$

where \mathcal{O}_k is a “dangerous” operator which does not depend explicitly on x , c is a scalar numeric coefficient. Now in the definition (15) of generators T^k contained in \mathcal{D}_A^k , the dimensions δ_φ should be taken as critical ones Δ_φ .

The expression in l.h.s. of (25) is UV-finite and has identical canonical and critical dimensions (the functional W is dimensionless and the operation \mathcal{D}_A^k decreases the dimension by the one). The coefficient c is dimensionless at criticality because in the massless model it is proportional to some power of μ^ϵ with the renormalization scale μ . Hence, if $c \neq 0$ the critical dimension $\Delta[\mathcal{O}_k]$ of the composite operator \mathcal{O}_k is obtained from (25) as follows:

$$\Delta[\mathcal{O}_k] = d - 1 . \quad (26)$$

Now let us consider an arbitrary renormalized composite operator $F(x)$, which is canonically conformal in the lowest order of the ϵ expansion, with critical dimension Δ_F . Then in analogous way we can derive the following relation

$$[\mathcal{D}_A^k + T_F^k] \langle\langle F(x) \rangle\rangle = c \langle\langle \mathcal{O}_k(x) \rangle\rangle , \quad (27)$$

where T_F^k is a generator like (15b) with a suitable dimension and spin rotation generator, \mathcal{O}_k is an analogue to (25) “breaking” operator which does not explicitly depend on x . If F is non-scalar and is characterized by some indices then \mathcal{O}_k has the same. If $c \neq 0$ an analogue of (26) is now given by the following relation

$$\Delta[\mathcal{O}_k] = \Delta[F] - 1 . \quad (28)$$

Note that the canonical conformal invariance of the operator at $\epsilon = 0$ is an essential restriction. For example, for the φ^4 -model with $\epsilon = 4 - d$, scalar operators containing only φ and $\partial^2 \varphi$ are canonically conformal while operators like $\partial^4 \varphi, \partial^2 \varphi^2$ are not.

A practical use of the formulae (25)–(28) is the following: possible candidates to be \mathcal{O}_k are defined by the symmetry and the canonical dimension $d_F^* \equiv d_F|_{\epsilon=0} = \Delta_F|_{\epsilon=0}$. If from the corresponding renormalized monomials one can construct a linear combination which

satisfy the criterion (26) or (28) then the combination can be \mathcal{O}_k , i.e. the conformal symmetry can be broken (however the final answer is given by an exact calculation of the coefficient c in the ϵ expansion). But if there are no renormalized operators with the required dimension $\Delta[\mathcal{O}_k]$, then the r.h.s. of the corresponding equations (25) or (27) have to vanish due to $c(g_*) = 0$. In this case the critical conformal symmetry is a consequence of scale invariance.

5 Examples of models possessing critical conformal invariance.

Below we present some models for which the conformal symmetry of the Green's function of the simple fields is guaranteed (all the models satisfy the assumptions).

- 1.** For models with one scalar field φ the operator \mathcal{O}_k in (25) is necessarily the gradient of another scalar $\partial_k F$. Such operator does not contribute after integration over x in (25).
- 2.** Let us consider the complex φ^4 -model with the fields φ, φ^+ and the interaction $\mathcal{V} \sim (\varphi^+ \varphi)^2$. Gauge current $J_k^{\text{cal}} = [\varphi^+ \partial_k \varphi - \partial_k \varphi^+ \cdot \varphi]_R$ only is a candidate to be \mathcal{O}_k in (25). It has no anomalous dimension [8], is conserved, and is multiplicatively renormalizable (nothing can mix with it). Hence, J_k^{cal} satisfies the criterion (26). But the dangerous term $\partial_k J_k^{\text{cal}}$ in r.h.s. of (23) is forbidden by charge symmetry, because it is odd under the charge transformation $\varphi \leftrightarrow \varphi^+$ and it cannot mix with the operator $N \sim \mathcal{V} \sim (\varphi^+ \varphi)^2$ which is charge-even. Note that the quantity $\partial_k J_k^{\text{cal}}$ is a difference of two operators like (21) for φ and φ^+ . Its presence in r.h.s. of (23) would imply the inequality $\gamma^*[\varphi] \neq \gamma^*[\varphi^+]$ due to (24), while conformal symmetry would be conserved in a wide sense. But the inequality $\gamma^*[\varphi] \neq \gamma^*[\varphi^+]$ is forbidden by the charge symmetry of the model.
- 3.** The same remark is valid for $O(n)$ -symmetrical models with a n -component field $\varphi \equiv \{\varphi_a\}$, for instance the $O(n)$ - φ^4 -model.
- 4.** The charge symmetry arguments are valid for other models possessing it, in particular, for models like the Gross-Neveu model (2). For this model the basis of the renormalized monomials in r.h.s. of (23) consists of three operators $[(\bar{\psi}\psi)^2]_R$, $[\bar{\psi}\hat{\partial}\psi]_R$ and $[\partial\bar{\psi} \cdot \gamma\psi]_R$. Only the gauge current $J_k^{\text{cal}} = [\bar{\psi}\gamma_k\psi]_R$ gives use to dangerous $\partial_k J_k^{\text{cal}}$. The contribution of $\partial_k J_k^{\text{cal}}$ is prohibited by the parity of the breaking operator $N \sim \mathcal{V} \sim (\bar{\psi}\psi)^2$ again. The inequality $\Delta[\psi] \neq \Delta[\bar{\psi}]$ in the presence of conformal symmetry in the wide sense with unknown $\Delta[\psi]$ and $\Delta[\bar{\psi}]$ is prohibited also by the exact form of the massless propagator $\langle\psi\bar{\psi}\rangle \sim \hat{x}/|x|^a$ which is a conformal invariant only if $\Delta[\psi] = \Delta[\bar{\psi}]$. The last argument allows us to eliminate the dangerous operator $\partial_k J_k^{\text{cal}}$ without the reference to charge symmetry. This is important for calculations in arbitrary dimension d because the matrix of charge conjunction C , which is contained in the definition of the charge transformation $\{\psi, \bar{\psi}\} \rightarrow \{\psi' = C\bar{\psi}, \bar{\psi}' = -\psi C^{-1}\}$ for the spinors, has no (similar to γ_5) unique natural definition in arbitrary dimension d .

It is clear that the above proof of the critical conformal symmetry of Green's functions for fields $\varphi = \bar{\psi}, \psi$ in the Gross-Neveu model (2) generalizes directly to other models with four-fermion interaction without derivatives.

5. For an application of the conformal bootstrap method in $1/n$ calculations of critical dimensions it is necessary to generalize the proof of the critical conformal invariance to models like (3) with fields $\varphi = \psi, \bar{\psi}, \sigma$. Moreover, we have to do that in ϵ scheme, for which the standard theory of composite operator renormalization can be applied because we have used it in our proof. Note that the existence of conformal symmetry is an objective property and is not influenced by calculational schemes. The model (3) in the $2 + \epsilon$ scheme is not suitable for our purpose because it is not multiplicatively renormalizable in the above scheme due to appearance of the second interaction $\sim (\bar{\psi}\psi)^2$ as a counterterm. Hence, the field σ apparently has no well defined critical dimension due to the mixing with the operator $\bar{\psi}\psi$. In the framework of the $2 + \epsilon$ renormalized model (3) the quantity Δ_σ calculated in $1/n$ is an eigenvalue of a 2×2 matrix of the critical dimensions of the mixing operators σ and $\bar{\psi}\psi$ which complicates the analysis. It is simpler to consider a two-coupling generalization of the model (3) adding the second independent interaction $\sim (\bar{\psi}\psi)^2$ which does not change the theory in the fermion sector. The two-coupling model is multiplicatively renormalizable and, hence, the field σ has now a well defined dimension Δ_σ . Just this dimension is calculated in $1/n$ expansion. The proof of the conformal invariance of Green's functions for the fields $\sigma, \psi, \bar{\psi}$ is valid for the two-coupling model because the quadratic part $\sim \sigma^2$ of the field σ in the action is canonically conformal and the interactions do not contain the derivatives. It follows that the divergence of the gauge current is still an unique candidate to be a dangerous operator in r.h.s. of (23). But it is prohibited as we have seen.

We need to prove also the conformal invariance of the composite operator $F = \sigma^2$ in order to calculate the exponent $1/\nu$ in (7) with the conformal bootstrap method. This property follows from the relations (27), (28) and the fact that the operator $[\sigma^2]_R$ has a non-trivial anomalous dimension. Indeed, from the analysis of canonical dimensions it follows that the gauge current J_k^{cal} only is a candidate to be the operator \mathcal{O}_k in (27). But it has no anomalous dimension, hence, it does not satisfy the criterion (28). Thus, $[\sigma^2]_R$ is a conformal operator. In the same way it can be shown that the operator $[\varphi^2]_R$ in the massless $O(n)$ - φ^4 -model ($4 - \epsilon$ scheme) is critically conformal. For the same model two of the three operators with $d_F^* = d_F|_{\epsilon=0} = 4$ are conformal, but the third one $(\partial^2[\varphi^2]_R)$ is not.

The scheme presented above gives us a quite simple way to analyze the problem of critical conformal invariance in any concrete model. One significant conclusion is that the fermion fields are not dangerous for conformal symmetry because the form of a massless spinor propagator, as for scalars but contrary to vectors, is solely defined by the standard symmetries without the conformal one. Note that vector fields are not dangerous by themselves (for instance, in the U_N -symmetrical Thirring model analogous to (3) the term $\sigma^2 \equiv \sigma_i \sigma_i$ is canonically conformal). However the presence of vector fields is usually correlated with the existence of a local gauge symmetry. In models like massless QED in an arbitrary gauge even the critical scale symmetry is broken due to the presence of the gauge parameter. In the transverse gauge $\partial_i A_i = 0$ the critical scale symmetry occurs, but holds not the conformal symmetry because the constraint $\partial_i A_i = 0$ is not preserved by conformal transformations. Note that we consider usual conformal transformations,

and various attempts to generalize them [9] are not included here because there are no real calculation applications of such methods.

6 Discussions.

The expressions (10) can be tested by independently calculating $2 + \epsilon$ and $4 - \epsilon$ expansions of the corresponding quantities. In the $2 + \epsilon$ scheme the quantities (6) are calculated from RG-functions of the model (3) with the following relations

$$\begin{aligned} \Delta_\psi &= d_\psi + \gamma_\psi^* , \quad \Delta_\sigma = \Delta[\bar{\psi}\psi] = d[\bar{\psi}\psi] + \gamma^*[\bar{\psi}\psi] , \\ \Delta_\tau &= 1/\nu = -\beta'(g_*) \quad (d_\psi = \mu - 1/2 , \quad d[\bar{\psi}\psi] = 2\mu - 1) , \end{aligned} \quad (29)$$

where $d_F \equiv d[F]$ are canonical dimensions of quantities F in the model (3) and γ^* are values of corresponding RG function $\gamma(g)$ in the fixed point $g_* \sim \epsilon$. Three loop calculation of the RG functions in the MS scheme in the model (3) for $d = 2 + 2\epsilon$ gives [10]:

$$\beta = 2\epsilon u - 2u^2(n-2) + 4u^3(n-2) + 2u^4(n-2)(n-7) + O(u^5) \quad (30a)$$

$$\gamma_\psi = u^2(n-1) - u^3(n-1)(n-2) + O(u^4) \quad (30b)$$

$$\gamma[\bar{\psi}\psi] = -2u(n-1) + 2u^2(n-1) + 2u^3(n-1)(2n-3) + O(u^4) \quad (30c)$$

where $u \equiv g/4\pi$ is a more convenient variable. From (30) it follows

$$\Delta_\psi = 1/2 + \epsilon + \epsilon^2 \frac{(n-1)}{(n-2)^2} + \epsilon^3 \frac{(n-1)(6-n)}{(n-2)^3} + O(\epsilon^4) \quad (31a)$$

$$\Delta_\sigma \equiv \Delta[\bar{\psi}\psi] = 1 + 2\epsilon - \epsilon \frac{(n-1)}{(n-2)} - 2\epsilon^2 \frac{(n-1)}{(n-2)^2} + 2\epsilon^3 \frac{n(n-1)}{(n-2)^3} + O(\epsilon^4) \quad (31b)$$

$$\Delta_\tau \equiv 1/\nu = 2\epsilon - \frac{4\epsilon^2}{(n-2)} - 4\epsilon^3 \frac{(n-3)}{(n-2)^2} + O(\epsilon^4) \quad (31c)$$

Expanding (31) in $1/n$ and (10) in ϵ we can see that both results are in agreement.

$4 - \epsilon$ expansions of the quantities (6) also correlates with RG functions of the model [2], but in $\Delta_F = d_F + \gamma_F^*$ the canonical dimensions of the latter model are implied: $d_\psi = (d-1)/2$, $d_\sigma = (d-2)/2$, $d_\tau = 2$ (the parameter $\tau \equiv g - g_*$ is the coefficient of σ^2 in the action [2]). A one loop calculation [2] gives ($d = 4 - 2\epsilon$):

$$\left. \begin{aligned} \gamma_\psi^* &= \frac{\epsilon}{n+6} + O(\epsilon^2) , \quad \gamma_\sigma^* = \frac{\epsilon n}{n+6} + O(\epsilon^2) , \\ \Delta_\tau \equiv 1/\nu &= 2 + \gamma_\tau^* = 2 - \frac{\epsilon}{3(n+6)} [5n+6 + \sqrt{n^2 + 132n + 36}] + O(\epsilon^2) . \end{aligned} \right\} \quad (32)$$

Expanding (32) in $1/n$ and (10) in $\epsilon = (4-d)/2$ we can see again the agreement between these results.

The quantities (10) can be expanded to any order in $1/n$, what gives us information about further terms in the expansions of β and γ_ψ functions. Using the method of [4] we could find all five-loop contribution to $\gamma_\psi(g)$, if we would know the ϵ expansion of the function $I(1 + \epsilon)$ at order ϵ^2 . However we know it only at order ϵ : $I(1 + \epsilon) = -2/3\epsilon + \epsilon/2 + O(\epsilon^2)$.

While preparing this manuscript we received the preprint [11] which contains the result for Δ_2 that is in agreement with (10a).

One of the author thanks Dr. R. Lacaze for providing ref.[11].

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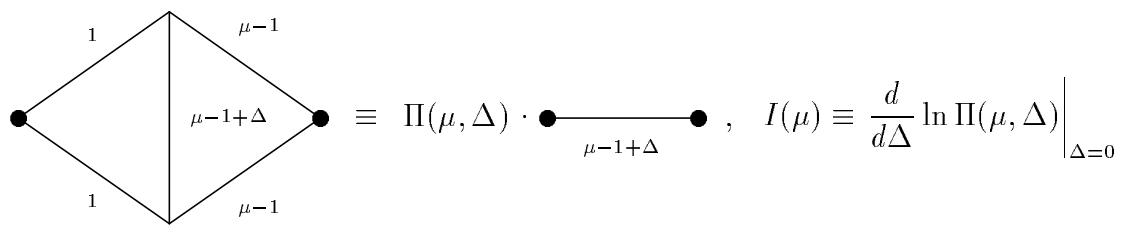


Fig.1