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STABILITY LOSS DELAY IN A CLASS OF SLOW AND FAST ECOLOGICAL MODELS

HAFIDA BOUDJELLABA * AND TEWFIK SARI †

Abstract. We study a class of three-dimensional systems of harvesting two species. The dynamics of the harvesting effort is assumed to be slow comparatively to the dynamics of the species. By using singular perturbation theory and the stability loss delay phenomenon, we give precise conditions which guarantee the existence of a semi global asymptotically stable equilibrium. As an application, a well-known model considered by Clark will be discussed.

Key words. Asymptotic stability; Delayed loss of stability; Singular perturbations; Canards; Biological models

AMS subject classifications. 34D15; 34E15; 92D25

1. Introduction. The main problem in the study of dynamical systems arising in applications is to determine the asymptotic behavior of the solutions. In mathematical ecology, this problem is related to the study of the persistence of the species. For instance, if there exists a globally asymptotically stable steady state of positive coordinates, then the system is persistent. On the other hand, in most applications the dynamics of different variables of the system are hierarchically scaled: for instance, in ecological models, often, the preys multiply much faster than the predators. Hence, the study and management of systems with various time scales were considered by many authors and remain a high point of interest from theoretical, computational and practical points of view [4, 5, 18, 19, 20, 22, 28].

In this paper, we investigate a class of slow and fast ecological models of particular interest for the applications and consider the question of the existence of a globally asymptotically stable persistent equilibrium. More precisely, we study the combined harvesting of two ecologically independent populations. Problems related to the exploitation of multispecies systems are more difficult than problems pertaining to the single-species model. For details and information, see Clark's book [7], or [8]. If each population is subject to logistic growth and E is the harvesting effort, we have the following equations (see [7], p. 311):

$$
\begin{aligned}
x' &= rx(1-x/K) - q_1 Ex, \\
y' &= sy(1-y/L) - q_2 Ey.
\end{aligned} \tag{1.1}
$$

In system (1.1), the parameters r and s are the intrinsic growth rates, q_1 and q_2 are the catchability coefficients and K and L are carrying capacities for populations x and y respectively. For net revenue, Clark used the following expression:

$$\pi(x,y,E) = (p_1 q_1 x + p_2 q_2 y - c)\, E,$$

where the prices p_1 and p_2 are constant and cE is the cost of fishing. Clark analyzed the system by finding the *bionomic equilibrium*, i.e. the equilibrium solutions $x' = 0$ and $y' = 0$ for equations (1.1) together with the condition $\pi(x,y,E) = 0$. He showed that, under some conditions, the bionomic equilibrium occurs, for a value E_∞ of

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the fishing effort, at a point (x_∞, y_∞) where both x_∞ and y_∞ are positive. Clark suggested to extend system (1.1) by adding a dynamic reaction of the effort of fishing E of form $E' = \varepsilon\pi(x, y, E)$ where $\varepsilon > 0$. He obtained the system

$$\begin{array}{rcl} x' & = & rx(1 - x/K) - q_1 Ex, \\ y' & = & sy(1 - y/L) - q_2 Ey, \\ E' & = & \varepsilon\,(p_1 q_1 x + p_2 q_2 y - c)\,E. \end{array} \tag{1.2}$$

Then Clark stated, without proof, that in system (1.2) the equilibrium $(x_\infty, y_\infty, E_\infty)$ is approached asymptotically (see [7], p. 312). Our aim is to prove this result in the case where ε is a small parameter.

More precisely, we study a general model given by the following equations:

$$\begin{array}{rcl} x' & = & xM(x, E), \\ y' & = & yN(y, E), \\ E' & = & \varepsilon EP(x, y), \end{array} \tag{1.3}$$

where M, N and P are continuous functions. For the simplicity of the presentation, we assume that system (1.3) has a unique solution with prescribed initial conditions. We study this system in the positive octant of $\mathbb{R}^3$. We assume that ε is small , which means that reaction E is much slower than reactions x and y. Let us denote by τ the time in system (1.3). In terms of the slow time $t = \varepsilon\tau$, system (1.3) becomes

$$\begin{array}{rcl} \varepsilon\dot{x} & = & xM(x, E), \\ \varepsilon\dot{y} & = & yN(y, E), \\ \dot{E} & = & EP(x, y). \end{array} \tag{1.4}$$

Throughout the paper, the dot designates the derivatives with respect to time t and the prime designates the derivatives with respect to time τ. System (1.4) is a slow and fast vector field and its study belongs to singular perturbation theory or Tikhonov's theory (see Section A.4, see also [21, 25, 26]). This system exhibits the phenomenon of *delayed loss of stability* or *canard solution* (see Section A.8). Our main aim is to give conditions on the functions M, N and P such that system (1.4) has an equilibrium $(x_\infty, y_\infty, E_\infty)$, with $x_\infty > 0$, $y_\infty > 0$ and $E_\infty > 0$, which is *practically semi-globally asymptotically stable* in the first octant, as $\varepsilon \to 0$, that is, for any solution $(x(t, \varepsilon), y(t, \varepsilon), E(t, \varepsilon))$ of system (1.4) we have

$$\lim_{t\to+\infty,\varepsilon\to 0} (x(t, \varepsilon), y(t, \varepsilon), E(t, \varepsilon)) = (x_\infty, y_\infty, E_\infty), \tag{1.5}$$

the limit being uniform with respect to the initial condition in any compact subset of the positive octant. The notion of practical semi-global asymptotic stability in systems depending on parameters is discussed in Section A.2. This notion, which is very important for applications, appeared first in control theory and is related to the problem of stabilization [6].

The paper is organized as follows. In Section 2, we apply Tikhonov's theory to system (1.4). In Section 3, we study the delayed loss of stability phenomenon in this system. In Section 4, the main results of the paper (theorems 4.6, 4.7 and 4.8) on the asymptotic stability of the equilibrium point $(x_\infty, y_\infty, E_\infty)$ of system (1.4) are stated. In Section 5, the results are illustrated on Clark's model (1.2) and by numerical simulations. In Appendix A, we give the main tools of singular perturbation theory which are disregarded in the literature. In this appendix, some stability results are obtained by using the decomposition of the dynamics of the system in a slow and a fast part.

2. Fast Dynamics and Slow Dynamics.

2.1. The fast equation. System (1.4) is a slow and fast system of the form (A.2) introduced in Section A.4, with $u = E$ as the slow variable and $v = (x, y)$ as the fast variable. For this system, the fast equation (A.4) is written as

$$\begin{aligned} x' &= xM(x, E) \\ y' &= yN(y, E) \end{aligned} \tag{2.1}$$

where E is a constant parameter. We assume that equations $x' = xM(x, E)$ and $y' = yN(y, E)$ have unique solutions with prescribed initial conditions, for every $E \geq 0$. The following assumptions are made.

(A1) There exists a positive continuous decreasing function $E \mapsto \xi(E)$ defined on $[0, a]$, $a > 0$, such that $\xi(a) = 0$ and for all $E \in [0, a]$ we have $M(\xi(E), E) = 0$ and $M(x, E) > 0$ for $x < \xi(E)$ and $M(x, E) < 0$ for $x > \xi(E)$.

(A2) There exists a positive continuous decreasing function $E \mapsto \eta(E)$ defined on $[0, b]$, $b > a$, such that $\eta(b) = 0$ and for all $E \in [0, b]$ we have $N(\eta(E), E) = 0$ and $N(y, E) > 0$ for $y < \eta(E)$, and $N(y, E) < 0$ for $y > \eta(E)$.

From assumptions **(A1)** and **(A2)**, we deduce the following properties: when $E \geq b$, $(0, 0)$ is the only equilibrium of (2.1); when $a \leq E < b$, (2.1) has two equilibria, $(0, 0)$ and $(0, \eta(E))$; when $0 \leq E < a$, (2.1) has four equilibria, $(0, 0)$, $(\xi(E), 0)$, $(0, \eta(E))$ and $(\xi(E), \eta(E))$. The stability of these equilibria is summarized in the following lemma:

LEMMA 2.1. *When $E \geq b$, $(0, 0)$ is a stable node. When $b > E \geq a$, $(0, 0)$ is a saddle point and $(0, \eta(E))$ is a stable node. When $a > E \geq 0$, $(0, 0)$ is an unstable node, $(\xi(E), 0)$ and $(0, \eta(E))$ are saddle points and $(\xi(E), \eta(E))$ is a stable node.*

Proof. The behavior of equilibria results easily from our assumptions, since the flow (2.1) is the product of two one-dimensional flows. □

The ecological interpretation of this lemma is as follows: when two populations are exploited jointly, with a harvesting effort $E \geq b$, then both of them are driven to extinction. If the harvesting effort is such that $a \leq E < b$, then one of the populations is driven to extinction, whereas the other population continues to support the fishery. If the harvesting effort is such that $E < a$, then both populations continue to support the fishery. The fast dynamics are illustrated in FIG. 2.1.

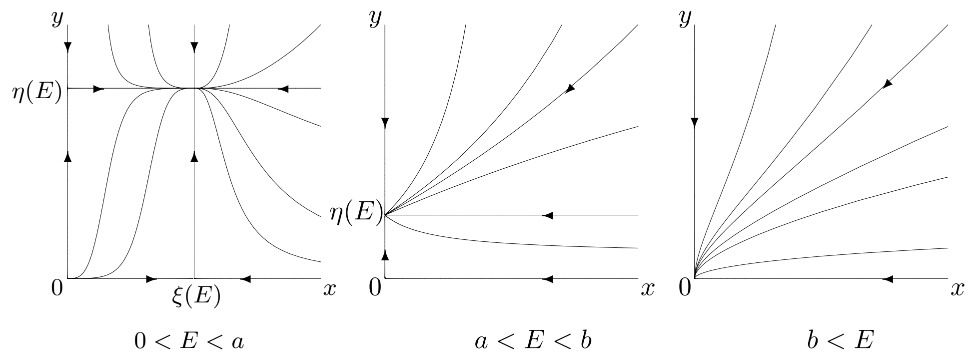

FIG. 2.1. *The fast dynamics of system (2.1). The equilibrium $(0, 0)$ is attracting for $E > b$. The equilibrium $(0, \eta(E))$ is attracting for $a < E < b$. The equilibrium $(\xi(E), \eta(E))$ is attracting for $0 < E < a$.*

2.2. The slow equation. After a fast transition close to the trajectories of the fast equation (2.1), the solutions of (1.4) are approximated by those of the slow equation. A slow equation is defined on each component of the slow manifold. The slow manifold which is the set of equilibra of the fast equations consists of several components (see FIG. 2.2, left):

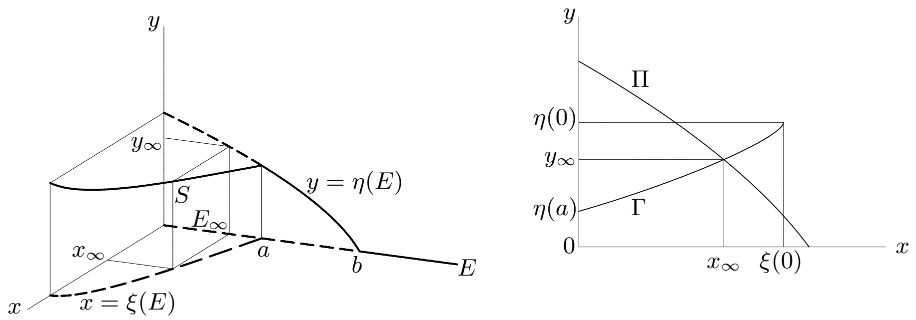

FIG. 2.2. *On the left, the slow manifold of system (1.4): attracting parts of the slow manifold are indicated by a bold line, non attracting parts of the slow manifold are indicated by a dashed line. On the right the relative positions of* Γ *and* Π.

i) The E axis which is the set of equilibra $(0,0)$. This component is attracting for $E > b$. On this slow curve, the slow equation is given by

$$\dot{E} = EP(0,0), \qquad 0 < E < +\infty. \tag{2.2}$$

ii) The curve $(x = 0,\ y = \eta(E))$, $0 < E < b$. This component is attracting for $a < E < b$. On this slow curve, the slow equation is given by

$$\dot{E} = EP(0,\eta(E)), \qquad 0 < E < b. \tag{2.3}$$

iii) The curve $(x = \xi(E),\ y = \eta(E))$, $0 < E < a$. This component is attracting. On this slow curve, the slow equation is given by

$$\dot{E} = EP(\xi(E),\eta(E)), \qquad 0 < E < a. \tag{2.4}$$

iv) The curve $(x = \xi(E),\ y = 0)$, $0 < E < a$. This component is not attracting. On this slow curve, the slow equation is given by

$$\dot{E} = EP(\xi(E),0), \qquad 0 < E < a. \tag{2.5}$$

We assume that the solution with prescribed initial conditions is unique for all equations (2.3), (2.4) and (2.5).

We add the following assumption (see FIG. 2.2, right):

(A3) The subset $\Pi = \{(x,y) : P(x,y) = 0\}$ is the graph of a decreasing function. We have $P(x,y) < 0$ under Π, $P(x,y) > 0$ above Π, $P(\xi(0),\eta(0)) > 0$ and $P(0,\eta(a)) < 0$.

2.3. Equilibria of the system. By **(A3)** curves $\Gamma = \{(\xi(E),\eta(E)) : 0 < E < a\}$ and Π intersect (see FIG. 2.2, right) at a unique point denoted by (x_∞, y_∞). Let E_∞ be defined by $\xi(E_\infty) = x_\infty$ and $\eta(E_\infty) = y_\infty$. We have $0 < E_\infty < a$ and

$$S = (x_\infty, y_\infty, E_\infty) \tag{2.6}$$

is an equilibrium point of (1.4). In fact (1.4) has at least three other equilibria,

$$(0,0,0), \qquad (0,\eta(0),0), \qquad (\xi(0),0,0), \tag{2.7}$$

which lie in the invariant xy-plane. In the case where $P(0,\eta(0)) > 0$, there exists a value $c \in]0,a[$ such that $P(0,\eta(c)) = 0$. Thus

$$(0,\eta(c),c) \tag{2.8}$$

is an equilibrium of (1.4) which lies in the invariant yE-plane. This equilibrium is unstable in the x-direction. In the case where $P(\xi(0),0) > 0$, there exists a value $d \in]0,a[$ such that $P(\xi(d),0) = 0$. Thus (1.4) has another equilibrium

$$(\xi(d),0,d) \tag{2.9}$$

which lies in the invariant xE-plane. This equilibrium is unstable in the y-direction.

2.4. Application of Tikhonov's theory. For all $E \in [0,a[$, the equilibrium $(\xi(E),\eta(E))$ of (2.1) is Globally Asymptotically Stable (GAS) in the positive quadrant of $\mathbb{R}^2$. Moreover, (2.4) has an asymptotically stable equilibrium point $E = E_\infty$. Hence, Theorem A.7 applies and predicts that in the region $0 \le E < a$, the solutions of (1.4) jump quickly near the slow curve $(x,y) = (\xi(E),\eta(E))$ and then move near this slow curve towards the equilibrium point S (see Proposition 4.1).

For all $E \in [a,b[$, the equilibrium $(0,\eta(E))$ of (2.1) is GAS in the positive quadrant of $\mathbb{R}^2$. Hence, Theorem A.6 applies and predicts that in the region $a \le E < b$, the solutions of (1.4) jump quickly near the slow curve $(x,y) = (0,\eta(E))$ and then move near this slow curve with decreasing E, until E reaches the value $E = a$ at which this slow curve loses its stability (see Fig. 2.3, left). More precisely, we have the following result:

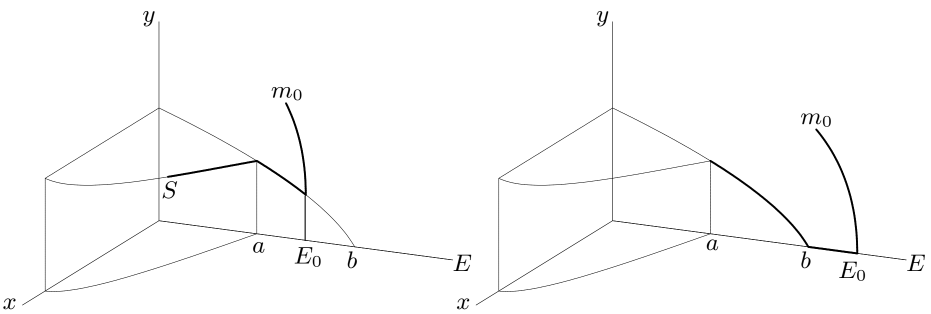

Fig. 2.3. *The asymptotic behavior of the solution of (1.4) with initial condition $m_0 = (x_0,y_0,E_0)$ when $a < E_0 < b$ (on the left) and $E_0 > b$ (on the right). The expected asymptotic behavior, when E crosses the value a (on the left) or the value b (on the right) is not correct. The correct behavior, showing the delayed loss of stability, is described in* Fig. *3.2, 3.3, 3.4 or 3.5.*

Lemma 2.2. *Let $(x(t,\varepsilon),y(t,\varepsilon),E(t,\varepsilon))$ be the solution of (1.4) with initial condition $x(0,\varepsilon) > 0$, $y(0,\varepsilon) > 0$ and $E(0,\varepsilon) = E_0$ such that $a \le E_0 < b$. Let $\overline{E}_1(t)$ be the solution of (2.3) with initial condition $\overline{E}_1(0) = E_0$. Let $t_a > 0$ be the instant of time for which $\overline{E}_1(t_a) = a$. Then, for any $\delta > 0$, we have $\lim_{\varepsilon\to 0} E(t,\varepsilon) = \overline{E}_1(t)$, uniformly for $t \in [0,t_a-\delta]$ and $\lim_{\varepsilon\to 0} x(t,\varepsilon) = 0$, $\lim_{\varepsilon\to 0} y(t,\varepsilon) = \eta(\overline{E}_1(t))$, uniformly for $t \in [\delta,t_a-\delta]$.*

Proof. The solution $\overline{E}_1(t)$ of (2.3) is decreasing towards the value $E = a$, since $P(0,\eta(E)) < 0$ for all $a < E < b$ (see assumption **(A3)**). When E crosses this value, the slow curve $(x, y) = (0, \eta(E))$ becomes unstable. To conclude, we apply Theorem A.6 because its hypothesis are now satisfied for $t \in [0, T]$ where $T = t_a - \delta$. □

For all $E \geq b$, the equilibrium $(0, 0)$ of (2.1) is GAS in the positive quadrant of $\mathbb{R}^2$. Hence, Theorem A.6 applies and predicts that in the region $E > b$, the solutions of (1.4) jump quickly near the slow curve $(x, y) = (0, 0)$ and then move near this axis with decreasing E, until E reaches the value $E = b$ at which this slow curve loses its stability (see FIG. 2.3, right). More precisely, we have the following result:

LEMMA 2.3. *Let $(x(t, \varepsilon), y(t, \varepsilon), E(t, \varepsilon))$ be the solution of (1.4) with initial condition $x(0, \varepsilon) > 0$, $y(0, \varepsilon) > 0$ and $E(0, \varepsilon) = E_0 \geq b$. Let $\overline{E}_0(t)$ be the solution of (2.2) with initial condition $\overline{E}_0(0) = E_0$. Let $t_b > 0$ be the instant of time for which $\overline{E}_0(t_b) = b$. Then, for any $\delta > 0$, we have $\lim_{\varepsilon \to 0} E(t, \varepsilon) = \overline{E}_0(t)$, uniformly for $t \in [0, t_b - \delta]$ and $\lim_{\varepsilon \to 0} x(t, \varepsilon) = 0$, $\lim_{\varepsilon \to 0} y(t, \varepsilon) = 0$, uniformly for $t \in [\delta, t_b - \delta]$.*

Proof. The solution $\overline{E}_0(t)$ of (2.2) is decreasing towards the value $E = b$ since $P(0, 0) < 0$. When E crosses this value, the slow curve $(x, y) = (0, 0)$ becomes unstable. Thus the hypothesis of Theorem A.6 are satisfied for $t \in [0, T]$ where $T = t_b - \delta$. The result follows from this theorem. □

3. Delayed loss of stability. One might believe then, that the solution of (1.4) with initial condition $E(0, \varepsilon) = E_0 > b$ described in Lemma 2.3 will move, for $t > t_b$, near the attracting slow curve $(x, y) = (0, \eta(E))$, with decreasing E, until E reaches the value $E = a$ (see FIG. 2.3, right). Similarly, one might believe then, that the solution of (1.4) with initial condition $E(0, \varepsilon) = E_0 > b$ described in Lemma 2.2 will move, for $t > t_a$, near the attracting slow curve $(x, y) = (\xi(E), \eta(E))$, towards the equilibrium point S (see FIG. 2.3, left).

In fact, due to the delayed loss of stability phenomenon, this behavior is not the right one and a solution which jumps quickly near the slow curve $(x, y) = (0, 0)$ with $E_0 > b$, will move near this axis, with decreasing E, until E reaches a value $E_1 < b$ (see FIG. 3.3, 3.4 or 3.5, right). Similarly, a solution which jumps quickly near the slow curve $(x, y) = (0, \eta(E))$, with $E_0 \in]a, b[$ will move near this slow curve, with decreasing E, until E reaches a value $E_1 < a$, (see FIG. 3.2, right). Now, we need to compute E_1 with respect to E_0. The mapping $E_0 \mapsto E_1$ is called the *entrance-exit function* along the slow curve.

3.1. Entrance-exit functions. For c defined as in (2.8) when $P(0, \eta(0)) > 0$ and $c = 0$ when $P(0, \eta(0)) \leq 0$, we have $P(0, \eta(u)) < 0$ for all $u \in]c, b]$, so the function

$$E \in]c, b[\mapsto f(E) \in [0, +\infty[, \qquad f(E) = \int_a^E \frac{M(0, u)}{uP(0, \eta(u))} du,$$

is well defined. The function f reaches its minimum at a (see FIG. 3.1, left). It is decreasing from $+\infty$ to 0 on $]c, a]$ and increasing on $[a, b[$.

DEFINITION 3.1. *The function $E \in [a, b[\mapsto F(E) \in]c, a]$, defined by $F = f_+^{-1} \circ f_-$, where f_- and f_+ are the restrictions of f on $[a, b[$ and $]c, a]$ respectively, is called the entrance-exit function along the slow curve $(x, y) = (0, \eta(E))$.*

Since $f(E) = f(F(E))$ for all $E \in [a, b[$ we have

$$\int_E^{F(E)} \frac{M(0, u)}{uP(0, \eta(u))} du = 0. \tag{3.1}$$

Now to define the entrance-exit function near the slow curve $(x, y) = (0, 0)$, let g

the function

$$E \in]0,+\infty[\mapsto g(E) \in [0,+\infty[, \qquad g(E) = \int_b^E \frac{N(0,u)}{uP(0,0)}du,$$

that has a minimum at b (see FIG. 3.1, right). This function is decreasing from $+\infty$ to 0 on $]0,b]$ and increasing on $[b,+\infty[$. It defines a mapping $E \in [b,\infty[\mapsto G(E) \in]0,b]$, by $G = g_+^{-1} \circ g_-$, where g_- and g_+ are the restrictions of g on $[b,+\infty[$ and $]0,b]$ respectively. Let h the function defined by

$$E \in]0,+\infty[\mapsto f(E) \in [0,+\infty[, \qquad h(E) = \int_a^E \frac{M(0,u)}{uP(0,0)}du.$$

This function has a minimum at a (see FIG. 3.1, right). It is decreasing from $+\infty$ to 0 on $]0,a]$ and increasing on $[a,+\infty[$. It defines a mapping $E \in [a,\infty[\mapsto H(E) \in]0,a]$, by $H = h_+^{-1} \circ h_-$, where h_- and h_+ are the restrictions of h on $[a,+\infty[$ and $]0,a]$ respectively. Let $a_* = G^{-1}(a)$. If $E \in [b,a_*]$ then (see FIG. 3.1, right) $H(E) < a \leq G(E)$. If $E > a_*$ then, we either have $H(E) < G(E)$ or $H(E) > G(E)$. However, in (1.2), the property $H(E) < G(E)$ is true for all $E > a_*$ (see Lemma 5.2).

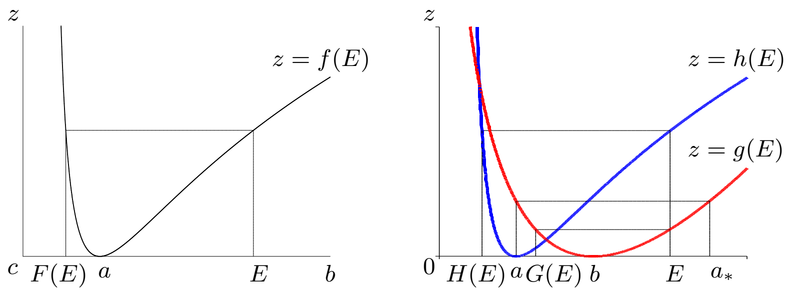

FIG. 3.1. *On the left: the function f defining the entrance-exit function $E \mapsto F(E)$ along the slow curve $(x,y) = (0,\eta(E))$. On the right: the functions g (in red) and h (in blue) defining the mappings $E \mapsto G(E)$ and $E \mapsto H(E)$ respectively.*

DEFINITION 3.2. *The function $E \in [b,\infty[\mapsto K(E) = \max(G(E),H(E)) \in]0,b]$ is called the entrance-exit function along the slow curve $(x,y) = (0,0)$.*

If $E \in [b,a_*]$ or $E > a_*$ and $H(E) < G(E)$ then $K(E) = G(E)$ and

$$\int_E^{G(E)} \frac{N(0,u)}{uP(0,0)}du = 0. \tag{3.2}$$

If $E > a_*$ and $H(E) > G(E)$ then $K(E) = H(E)$ and

$$\int_E^{H(E)} \frac{M(0,u)}{uP(0,0)}du = 0. \tag{3.3}$$

3.2. Behavior in the vicinity of the slow curve $(0,\eta(E))$. Let (x_0,y_0,E_0) be an initial condition such that $a < E_0 < b$. By Lemma 2.2, the corresponding trajectory of (1.4) will move quickly towards the equilibrium $(0,\eta(E))$ and then remains close to this equilibrium with E decreasing and until E reaches the value a where the equilibrium $(0,\eta(E))$ loses its stability. The departure of the trajectory from the neighborhood of the equilibrium takes place not immediately after a but rather after a time during which the effort E changes by a finite amount. There is a delayed loss of stability. More precisely, let $x_0 < 1$ positive and not too big. We consider the trajectory $\gamma(t,\varepsilon)$ of (1.4) that starts at (x_0, y_0, E_0) with $a < E_0 < b$. If ε is small enough, according to Lemma 2.2, the trajectory remains between the planes $x = x_0$ and $x = 0$ and goes towards the equilibrium $(0, \eta(E))$ while E is decreasing as far as $E > a$. Denote now, the next intersection of this trajectory and the plane $x = x_0$ by $(x_0, y(t_1,\varepsilon), E(t_1,\varepsilon))$ where $t_1 = t_1(x_0, y_0, E_0, \varepsilon)$ depends on the initial condition (x_0, y_0, E_0) and on ε. The limit

$$E_1 = \lim_{\varepsilon\to 0} E(t_1(x_0, y_0, E_0, \varepsilon), \varepsilon) \tag{3.4}$$

exists and depends only on E_0. This limit, as well as, a precise behavior of the trajectory are given by the following theorem:

THEOREM 3.3. *Let $E_0 \in]a, b[$. Then, we have $E_1 = F(E_0)$ and for small $\varepsilon > 0$, the trajectory leaves the neighborhood of point $(0, \eta(E_1), E_1)$ and jumps to the neighborhood of point $(\xi(E_1), \eta(E_1), E_1)$ close to the unstable separatrix $y = \eta(E_1)$ of the saddle point $(0, \eta(E_1))$ of the fast dynamics (see* FIG. *3.2).*

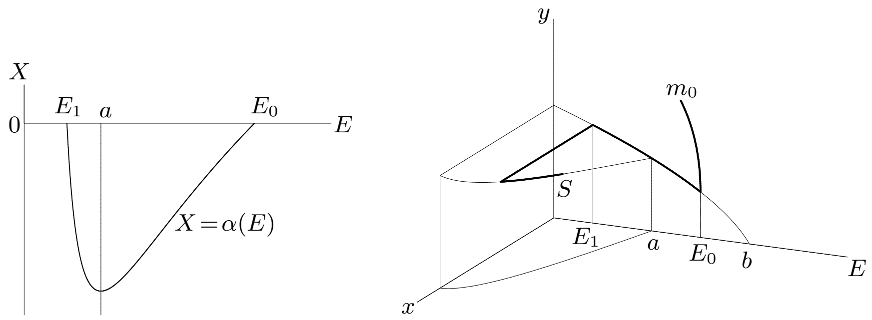

FIG. 3.2. *On the right: the asymptotic behavior of the solution of (1.4) with initial condition $m_0 = (x_0, y_0, E_0)$, when $a < E_0 < b$, showing the delayed loss of stability when E crosses the value a. On the left: the asymptotic behavior in the coordinates (X, E) of (3.5). The function α is defined by (3.8).*

Proof. The change of variable $X = \varepsilon \ln x$ maps the strip $0 < x < 1$ into the half space $X < 0$. This change of variable transforms (1.4) into

$$\begin{array}{rcl} \dot{X} & = & M(\exp(X/\varepsilon), E), \\ \varepsilon\dot{y} & = & yN(y, E) \\ \dot{E} & = & EP(\exp(X/\varepsilon), y) \end{array} \tag{3.5}$$

The initial condition becomes $(\varepsilon \ln x_0, y_0, E_0)$. System (3.5) is a slow and fast system, similar to (A.2) of Section A.4, with $u = (X, E)$ as the slow variable and $v = y$ as the fast variable. For this system, the fast equation (A.4) is written as

$$y' = yN(y, E) \tag{3.6}$$

The equilibrium $y = \eta(E)$ of (3.6) is attracting for all $E \in [0, b[$. We have $\lim_{\varepsilon\to 0} \exp X/\varepsilon = 0$ since $X < 0$. Thus, on the slow surface $y = \eta(E)$, the slow equation is

$$\dot{X} = M(0, E), \qquad \dot{E} = EP(0, \eta(E)). \tag{3.7}$$

According to Theorem A.6, y goes very quickly towards the stable equilibrium $y = \eta(E)$. Then a slow transition develops near the surface $y = \eta(E)$. This slow transition is approximated by the solution of (3.7) with initial condition $X(0) = 0$, $E(0) = E_0$. This solution is given by $E = \overline{E}(t)$ and

$$X = \alpha(E) := \int_{E_0}^{E} \frac{M(0,u)}{uP(0,\eta(u))} du, \tag{3.8}$$

where $\overline{E}(t)$ is the solution of (2.3) such that $\overline{E}(0) = E_0$. Thus, according to (3.1), we have again (see FIG. 3.2, left) $X = 0$ for $E_1 = F(E_0)$. Returning to the original variables, we see that the trajectory $\gamma(t,\varepsilon)$ crosses again the plane $x = x_0$ when E is asymptotically equal to $E_1 = F(E_0)$. Then (see FIG. 3.2, right), a fast transition brings the trajectory from the neighborhood of point $(0, \eta(E_1), E_1)$ to the neighborhood of point $(\xi(E_1), \eta(E_1), E_1)$ close to the unstable separatrix $y = \eta(E_1)$ of the saddle point $(0, \eta(E_1))$ of the fast dynamics. □

3.3. Behavior in the vicinity of the slow curve $(0,0)$**.** By analogy to the previous section, let (x_0, y_0, E_0) be an initial condition such that $E_0 > b$. By Lemma 2.3, the corresponding trajectory of system (1.4) will go quickly towards the origin $(x, y) = (0, 0)$ and then remains close to this equilibrium while E is decreasing and until E reaches the value b where the origin loses its stability. The departure of the trajectory from the neighborhood of the equilibrium takes place not immediately after b. There is a delayed loss of stability and the trajectory will leave the neighborhood of the equilibrium $(0, 0)$ when E is asymptotically equal to a value E_1. More precisely, we have the following result:

PROPOSITION 3.4. *For small $\varepsilon > 0$, the solution remains near the slow curve $(x, y) = (0, 0)$ as long as $E_1 < E < E_0$ where $E_1 = K(E_0)$.*
a) If $E_0 \in [b, a_*]$ then, the solution leaves the neighborhood of point $(0, 0, E_1)$ and jumps to the neighborhood of point $(0, \eta(E_1), E_1)$ close to the separatrix $x = 0$ of the saddle point $(0, 0)$ of the fast dynamics (see FIG. *3.3).*
b) If $E_0 > a_*$ and $H(E_0) < G(E_0)$ then, the solution leaves the neighborhood of point $(0, 0, E_1)$ and jumps to the neighborhood of point $(0, \eta(E_1), E_1)$ close to the orbit $x = 0$ of the unstable node $(0, 0)$ of the fast dynamics (see FIG. *3.4).*
c) If $E_0 > a_*$ and $H(E_0) > G(E_0)$ then, the solution leaves the neighborhood of point $(0, 0, E_1)$ and jumps to the neighborhood of point $(\xi(E_1), 0, E_1)$ close to the orbit $y = 0$ of the unstable node $(0, 0)$ of the fast dynamics (see FIG. *3.5).*

Proof. Let $x_0 < 1$ and $y_0 < 1$ be positive and not too big. By Lemma 2.3, for small $\varepsilon > 0$ the solution remains between the planes $x = x_0$, $y = y_0$, $x = 0$ and $y = 0$. The change of variables $X = \varepsilon \ln x$, $Y = \varepsilon \ln y$ maps the open set $0 < x < 1$, $0 < y < 1$ into the octant $X < 0$, $Y < 0$. This change of variables transforms (1.4) into

$$\begin{aligned} \dot{X} &= M(\exp(X/\varepsilon), E), \\ \dot{Y} &= N(\exp(Y/\varepsilon), E), \\ \dot{E} &= EP(\exp(X/\varepsilon), \exp(Y/\varepsilon)). \end{aligned} \tag{3.9}$$

The initial condition (x_0, y_0, E_0) becomes $(\varepsilon \ln x_0, \varepsilon \ln y_0, E_0)$. Since $X < 0$ and $Y < 0$, (3.9) is a regular perturbation of

$$\dot{X} = M(0, E), \qquad \dot{Y} = N(0, E), \qquad \dot{E} = EP(0, 0). \tag{3.10}$$

The solution of (3.10) with initial condition $X(0) = 0$, $Y(0) = 0$, $E(0) = E_0$ is

$$X = \beta(E) := \int_{E_0}^{E} \frac{M(0,u)}{uP(0,0)} du, \qquad Y = \gamma(E) := \int_{E_0}^{E} \frac{N(0,u)}{uP(0,0)} du, \tag{3.11}$$

where $E = \overline{E}(t) := E_0 e^{P(0,0)t}$ is the solution of (2.2) such that $\overline{E}(0) = E_0$. Let $E_1 = K(E_0)$ and $t_1 = \frac{1}{P(0,0)} \ln \frac{E_1}{E_0}$ (t_1 satisfies $\overline{E}(t_1) = E_1$). Hence $E(t_1, \varepsilon) = E_1 + o(1)$. When E is asymptotically equal to E_1, i.e. when t is asymptotically equal to t_1, the solution jumps far from the neighborhood of the E-axis as shown below.

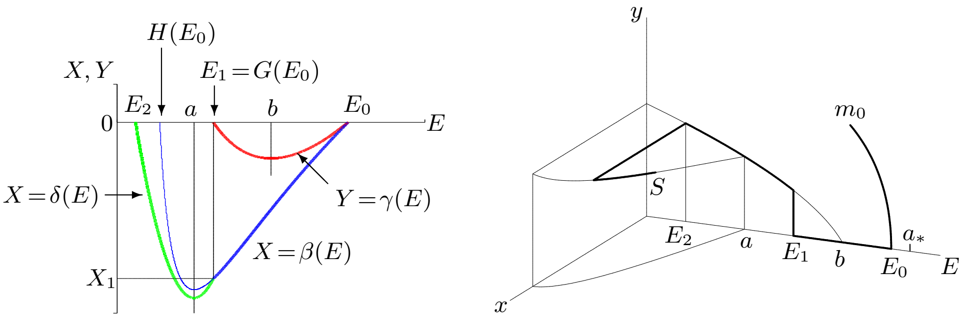

FIG. 3.3. *On the right: the asymptotic behavior of the solution of (1.4) with initial condition $m_0 = (x_0, y_0, E_0)$, when $b < E_0 < a_*$, showing the delayed loss of stability when E crosses values b and a. On the left: the asymptotic behavior in the coordinates (X, Y, E) of (3.9). The functions β (in blue), γ (in red) and δ (in green) are defined by (3.11) and (3.15) respectively.*

Cases a and b) If $E_0 \in [b, a_*]$ (see FIG. 3.3, left), or $E_0 > a_*$ and $H(E_0) < G(E_0)$ (see FIG. 3.4, left), then $E_1 = G(E_0)$. Thus, according to (3.2) and $h(G(E_0)) < h(E_0)$, we have

$$X_1 = \int_{E_0}^{E_1} \frac{M(0,u)}{uP(0,0)} du < 0, \qquad Y_1 = \int_{E_0}^{E_1} \frac{N(0,u)}{uP(0,0)} du = 0. \tag{3.12}$$

Since $X(t_1, \varepsilon) = X_1 + o(1)$ and $Y(t_1, \varepsilon) = o(1)$, the solution reaches again the plane $y = y_0$ asymptotically at time t_1, and $x(t_1, \varepsilon) = \exp((X_1 + o(1))/\varepsilon)$ is exponentially small. Thus, asymptotically at time t_1, the solution jumps (see FIG. 3.3 or 3.4, right) from the neighborhood of point $(0, 0, E_1)$ to the neighborhood of point $(0, \eta(E_1), E_1)$ close to the orbit $x = 0$ of equilibrium point $(0, 0)$ of the fast dynamics. According to Lemma 2.1, this equilibrium is a saddle point if $E_0 \in [b, a_*]$. It is an unstable node if $E_0 > a_*$.

Case c) If $E_0 > a_*$ and $H(E_0) > G(E_0)$ (see FIG. 3.5, left) then $E_1 = H(E_0)$. Thus, according to (3.3) and $g(H(E_0)) < g(E_0)$, we have

$$X_1 = \int_{E_0}^{E_1} \frac{M(0,u)}{uP(0,0)} du = 0, \qquad Y_1 = \int_{E_0}^{E_1} \frac{N(0,u)}{uP(0,0)} du < 0. \tag{3.13}$$

Since $X(t_1, \varepsilon) = o(1)$ and $Y(t_1, \varepsilon) = Y_1 + o(1)$, the solution reaches again the plane $x = x_0$ asymptotically at value E_1 and $y(t_1, \varepsilon) = \exp((Y_1 + o(1))/\varepsilon)$ is exponentially small. Thus, asymptotically at time t_1, the solution jumps (see FIG. 3.5, right) from the neighborhood of point $(0, 0, E_1)$ to the neighborhood of point $(\xi(E_1), 0, E_1)$ close to the orbit $y = 0$ of the unstable node $(0, 0)$ of the fast dynamics. □

As stated in Proposition 3.4, the jump of the solution, far from the slow curve $(0, 0)$, happens when E is asymptotically equal to $E_1 = K(E_1)$. After this jump, the asymptotic behavior of the solution is given by Theorems 3.5 and 3.6 below.

THEOREM 3.5. *Let $E_0 \in [b, a_*]$. For small $\epsilon > 0$, the solution remains near the slow curve $(x, y) = (0, \eta(E))$ as long as $E_2 < E < E_1$ where $E_1 = G(E_0)$ and*

$E_2 \in]c, a]$ *is given by*

$$\int_{E_0}^{E_1} \frac{M(0,E)}{EP(0,0)} dE + \int_{E_1}^{E_2} \frac{M(0,E)}{EP(0,\eta(E))} dE = 0. \tag{3.14}$$

Afterwards, it jumps from the neighborhood of point $(0, \eta(E_2), E_2)$ *to the neighborhood of point* $(\xi(E_2), \eta(E_2), E_2)$ *close to the unstable separatrix* $y = \eta(E_2)$ *of the saddle point* $(0, \eta(E_2))$ *of the fast dynamics (see* FIG. *3.3).*
This result is also true in the case where $E_0 > a_*$, $H(E_0) < G(E_0)$ *and* $P(0, \eta(0)) < 0$ *(see* FIG. *3.4). In this case* $E_2 \in]0, E_1[$.

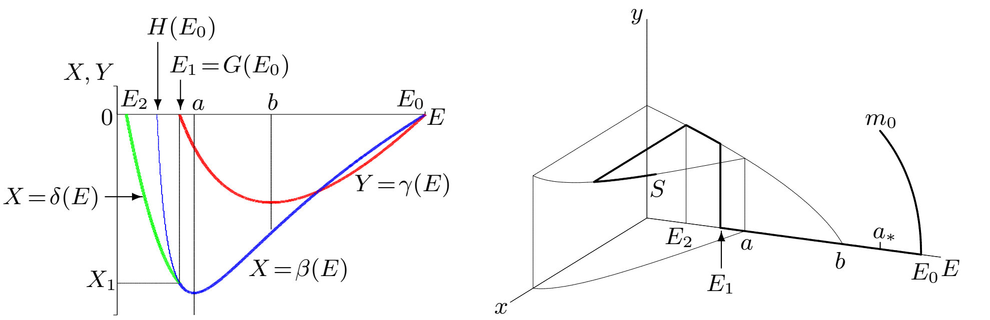

FIG. 3.4. *On the right: the asymptotic behavior of the solution of (1.4) with initial condition* $m_0 = (x_0, y_0, E_0)$*, when* $E_0 > a_*$ *and* $H(E_0) < G(E_0)$*, showing the delayed loss of stability when* E *crosses values* b *and* a*. On the left: the asymptotic behavior in the coordinates* (X, Y, E) *of (3.9). The functions* β *(in blue),* γ *(in red) and* δ *(in green) are defined by (3.11) and (3.15) respectively.*

Proof. The asymptotic behavior of the solution for $t \in [0, t_1]$ is described in cases *(a)* and *(b)* of Proposition 3.4. For $t > t_1$ we use, the same change of variable as in the proof of Theorem 3.3, $X = \varepsilon \ln x$ which maps the strip $0 < x < 1$ into the half space $X < 0$. This change of variable transforms (1.4) into (3.5) with conditions $X(t_1, \varepsilon) = X_1 + o(1)$, $Y(t_1, \varepsilon) > 0$, and $E(t_1, \varepsilon) = E_1 + o(1)$, where X_1 is defined by (3.12). According to Theorem A.6, y goes very quickly towards the stable equilibrium $y = \eta(E)$. Then, a slow transition develops near the surface $y = \eta(E)$. This slow transition is approximated by the solution of (3.7) with initial condition $X(t_1) = X_1$, $E(t_1) = E_1$. This solution is given by

$$X = \delta(E) := X_1 + \int_{E_1}^{E} \frac{M(0,u)}{uP(0,\eta(u))} du, \tag{3.15}$$

where $E = \overline{E}(t)$ is the solution of (2.3) such that $\overline{E}(t_1) = E_1$. In case $E_0 \in [b, a_*]$, the integral in (3.15) is well defined because $E_1 \in [a, b]$ and $P(0, \eta(u)) < 0$ for all $u \in]c, b]$. Thus, E_2 defined by (3.14) satisfies $E_2 \in]c, a[$ (see FIG. 3.3, left). We have again $X = 0$ at value E_2. Returning to the original variables, we notice that the trajectory $\gamma(t, \varepsilon)$ crosses again the plane $x = x_0$ when E is asymptotically equal to E_2. Then (see FIG. 3.3, right), a fast transition brings the trajectory from the neighborhood of point $(0, \eta(E_2), E_2)$ to the neighborhood of point $(\xi(E_2), \eta(E_2), E_2)$, close to the unstable separatrix $y = \eta(E_2)$ of the saddle point $(0, \eta(E_2))$ of the fast dynamics.
In the case where $E_0 > a_*$, $H(E_0) < G(E_0)$ and $P(0, \eta(0)) < 0$ the proof is the same as in case $E_0 \in [b, a_*]$, adapted to FIG. 3.4. Now $E_1 \in]0, a[$ and $c = 0$. The integral

in (3.15) is well defined since $P(0,\eta(u)) < 0$ for all $u \in [0,a]$. Hence E_2, given by (3.14), satisfies $E_2 \in]0,E_1[$ (see FIG. 3.4, left). □

Note that the mapping $E_1 \to E_2$ given by formula (3.14) is not equal to the entrance-exit function $E_1 \mapsto F(E_1)$ of the slow curve $(x,y) = (0,\eta(E))$ as it was the case in Theorem 3.3. Indeed, in Theorem 3.5, the solution is exponentially close to the plane $x = 0$ before it arrives near the slow curve $(x,y) = (0,\eta(E))$. Recall that in Theorem 3.3 the solution arrived from a point (x_0,y_0,E_0) which was not very close to the plane $x = 0$.

THEOREM 3.6. *Let $E_0 > a_*$. If $H(E_0) > G(E_0)$ and $P(\xi(0),0) < 0$ then for small $\varepsilon > 0$, the solution remains near the slow curve $(x,y) = (\xi(E),0)$ as long as $E_2 < E < E_1$ where $E_1 = H(E_0)$ and $E_2 \in]0,E_1[$ is given by*

$$\int_{E_0}^{E_1} \frac{N(0,E)}{EP(0,0)}dE + \int_{E_1}^{E_2} \frac{M(0,E)}{EP(\xi(E),0)}dE = 0. \tag{3.16}$$

Afterwards, it jumps from the neighborhood of point $(\xi(E_2),0,E_2)$ to the neighborhood of point $(\xi(E_2),\eta(E_2),E_2)$ close to the unstable separatrix $x = \xi(E_2)$ of the saddle point $(\xi(E_2),0)$ of the fast dynamics (see FIG. 3.5).
If $H(E_0) = G(E_0)$, $P(0,\eta(0)) < 0$ and $P(\xi(0),0) < 0$ then for small $\varepsilon > 0$, the solution jumps from the neighborhood of point $(0,0,E_1)$, where $E_1 = H(E_0)$, to the neighborhood of point $(\xi(E_1),\eta(E_1),E_1)$ close to one of the trajectories of the fast dynamics connecting the unstable node $(0,0)$ to the stable node $(\xi(E_2),\eta(E_2))$ (see FIG. 2.1, left).

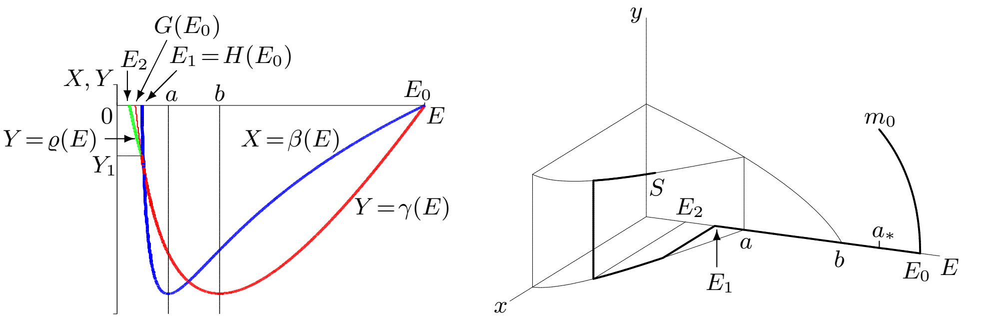

FIG. 3.5. *On the right: the asymptotic behavior of the solution of (1.4) with initial condition $m_0 = (x_0,y_0,E_0)$, in the case where $E_0 > a_*$ and $H(E_0) > G(E_0)$, showing the delayed loss of stability when E crosses values b and a. On the left: the asymptotic behavior in the coordinates (X,Y,E) of (3.9). The functions β (in blue), γ (in red) and ρ (in green) are defined by (3.11) and (3.20) respectively.*

Proof. The asymptotic behavior of the solution for $t \in [0,t_1]$ is described in case *(c)* of Proposition 3.4. For $t > t_1$ we use the change of variable $Y = \varepsilon \ln x$ which maps the strip $0 < y < 1$ into the half space $Y < 0$. This change of variable transforms (1.4) into

$$\begin{array}{rcl} \varepsilon\dot{x} &=& xM(x,E), \\ \dot{Y} &=& N(\exp(Y/\varepsilon),E), \\ \dot{E} &=& EP(x,\exp(Y/\varepsilon)) \end{array} \tag{3.17}$$

with conditions $X(t_1,\varepsilon) > 0$, $Y(t_1,\varepsilon) = Y_1 + o(1)$, and $E(t_1,\varepsilon) = E_1 + o(1)$, where Y_1 is defined by (3.13). System (3.17) is a slow and fast system as (A.2) of Section A.4, with $u = (Y, E)$ as the slow variable and $v = x$ as the fast variable. For this system, the fast equation (A.4) is written as

$$x' = xM(x, E) \tag{3.18}$$

The equilibrium $x = \xi(E)$ of (3.6) is attracting for all $E \in [0, a[$. We have $\lim_{\varepsilon \to 0} \exp Y/\varepsilon = 0$ since $Y < 0$. And on the slow surface $x = \xi(E)$, the slow equation is

$$\dot{Y} = N(0, E), \qquad \dot{E} = EP(\xi(E), 0). \tag{3.19}$$

According to Theorem A.6, x goes very quickly towards the stable equilibrium $x = \xi(E)$. Then a slow transition develops near the surface $x = \xi(E)$. This slow transition is approximated by the solution of (3.19) with initial condition $Y(t_1) = Y_1$, $E(t_1) = E_1$. This solution is given by

$$Y = \varrho(E) = Y_1 + \int_{E_1}^{E} \frac{N(0, u)}{uP(\xi(u), 0)} du, \tag{3.20}$$

where $E = \overline{E}(t)$ is the solution of (2.5) such that $\overline{E}(t_1) = E_1$. The integral in (3.20) is well defined because $E_1 \in]0, a[$ and $P(\xi(u), 0) < 0$ for all $u \in]0, a]$. Thus, E_2 given by (3.16) satisfies $E_2 \in]0, E_1[$ (see Fig. 3.3, left). We have again $Y = 0$ at value E_2. Returning to the original variables, we see that the trajectory $\gamma(t, \varepsilon)$ crosses again the plane $y = y_0$ when E is asymptotically equal to E_2. Then (see Fig. 3.5, right), a fast transition brings the trajectory from the neighborhood of point $(\xi(E_2), 0, E_2)$ to the neighborhood of point $(\xi(E_2), \eta(E_2), E_2)$, close to the unstable separatrix $x = \xi(E_2)$ of the saddle point $(\xi(E_2), 0)$ of the fast dynamics.
In the case where $H(E_0) = G(E_0)$, $E_2 = E_1$ in (3.16). The fast transition from $(0, 0, E_1)$ to $(\xi(E_1), \eta(E_1), E_1)$ would hold close to one of the orbits of the fast dynamics. We exclude equilibrium points on $(\xi(E), 0, E)$ and on $(0, \eta(E), E)$ since $P(0, \eta(0)) < 0$ and $P(\xi(0), 0) < 0$. Thus, the trajectory does not stay near these slow curves and the jump happens when E is asymptotically equal to E_1. □

In the case where $H(E_0) = G(E_0)$, our analysis does not predict the orbit along which the fast transition from $(0, 0, E_1)$ to $(\xi(E_1), \eta(E_1), E_1)$ would hold. By continuous dependance, when $E(0, \varepsilon)$ crosses a value E_0 such that $H(E_0) = G(E_0)$, all orbits of the fast dynamics arise as transition orbits, since the transition holds close to $x = 0$ when $H(E_0) < G(E_0)$, and close to $y = 0$ when $H(E_0) < G(E_0)$. The geometric singular perturbation theory (see remark following Theorem A.8) could provide tools to address this question. However, in (1.2), the case $H(E_0) = G(E_0)$ does not appear (see Lemma 5.2).

Remark. In the case where $P(0, \eta(0)) > 0$, (2.8) is an equilibrium of (1.4). Let $c_* > b$ be defined by $g(c_*) = g(c)$. Assume that $H(c_*) < G(c_*)$. Then, according to Theorem 3.4, the solution of (1.4) with initial condition $x(0, \varepsilon) > 0$, $y(0, \varepsilon) > 0$, $E(0, \varepsilon) = c_*$ will jump quickly near the slow curve $(x, y) = (0, 0)$ and remains close to this curve, as long as $c < E < c_*$. Then, the solution will jump quickly from the neighborhood of $(0, 0, c)$ to the neighborhood of the equilibrium $(0, \eta(c), c)$, so it could stay near this equilibrium for a long time. On the other hand, in the case where $P(\xi(0), 0) > 0$, (2.9) is an equilibrium of (1.4). Let $d_* > b$ be defined by $h(d_*) = h(d)$. Assume that $H(d_*) > G(d_*)$. Then, according to Proposition 3.4, the solution of (1.4) with initial condition $x(0, \varepsilon) > 0$, $y(0, \varepsilon) > 0$, $E(0, \varepsilon) = d_*$ will jump quickly near the slow curve $(x, y) = (0, 0)$ and remains close to this curve, as long as $d < E < d_*$. Then, the solution will jump quickly from the neighborhood of

$(0, 0, d)$ to the neighborhood of the equilibrium $(\xi(d), 0, d)$, so it could remain near this equilibrium for a long time. To achieve a full description of the solution, as in Theorem 3.5 or 3.6, we must exclude the existence of equilibrium (2.8) or (2.9).

4. Persistence. In this section $\gamma(t, \epsilon) = (x(t, \varepsilon), y(t, \varepsilon), E(t, \varepsilon))$ is the solution of (1.4) with initial condition $x(0, \varepsilon) > 0$, $y(0, \varepsilon) > 0$ and $E(0, \varepsilon) = E_0 > 0$. Our aim is to show that the limit (1.5) is uniform with respect to the initial condition in any compact subset of the positive octant.

4.1. Asymptotic behavior for all $t \geq 0$**.** We begin with the asymptotic behavior of the solutions when $E_0 < a$.

PROPOSITION 4.1. *Let* $E_0 \in]0, a[$*. Let* $\overline{E}(t)$ *be the solution of (2.4) with initial condition* $\overline{E}(0) = E_0$*. Let* $\overline{x}(t) = \xi(\overline{E}(t))$ *and* $\overline{y}(t) = \eta(\overline{E}(t))$*. Then for any* $\delta > 0$*, we have* $\lim_{\varepsilon \to 0}(x(t, \varepsilon), y(t, \varepsilon)) = (\overline{x}(t), \overline{y}(t))$*, uniformly for* $t \in [\delta, +\infty[$ *and* $\lim_{\varepsilon \to 0} E(t, \varepsilon) = \overline{E}(t)$*, uniformly for* $t \in [0, +\infty[$*.*

Proof. For all $E \in [0, a]$, the solution $(\xi(E), \eta(E))$ of (2.1) is GAS. Moreover, by assumption **(A3)**, the solutions of (2.4) are converging towards $E = E_\infty$, which is an asymptotically stable equilibrium. The result follows from Theorem A.7. □

Assume now that $E_0 \in [a, b]$. From Theorem 3.3, we see that the solution $\gamma(t, \varepsilon)$ reaches the neighborhood of point $(\xi(E_1), \eta(E_1), E_1)$. Then, as shown in Proposition 4.1, it is approximated by a solution of (2.4). More precisely, we have the following result which completes, for $t > t_a$, the one given by Lemma 2.2.

PROPOSITION 4.2. *let* $E_0 \in [a, b]$*. Let* $\overline{E}_1(t)$ *be the solution of (2.3) with initial condition* $\overline{E}_1(0) = E_0$*. Let* $E_1 = F(E_0)$*. Let* $t_1 > 0$ *be the instant of time for which* $\overline{E}(t_1) = E_1$*. Let* $\overline{E}_2(t)$ *be the solution of (2.4) with initial condition* $\overline{E}_2(t_1) = E_1$*. Let*

$$\overline{E}(t) = \begin{cases} \overline{E}_1(t) & \text{for} \quad t \in [0, t_1], \\ \overline{E}_2(t) & \text{for} \quad t \in [t_1, +\infty[, \end{cases} \tag{4.1}$$

and

$$\overline{x}(t) = \begin{cases} 0 & \text{for} \quad t \in]0, t_1[, \\ \xi(\overline{E}_2(t)) & \text{for} \quad t \in]t_1, +\infty[\end{cases} \qquad \overline{y}(t) = \eta(\overline{E}(t)) \text{ for } t \in]0, +\infty[.$$

Then, for any $\delta > 0$ *we have* $\lim_{\varepsilon \to 0}(x(t, \varepsilon), y(t, \varepsilon)) = (\overline{x}(t), \overline{y}(t))$*, uniformly for* $t \in [\delta, t_1 - \delta] \cup [t_1 + \delta, +\infty[$ *and* $\lim_{\varepsilon \to 0} E(t, \varepsilon) = \overline{E}(t)$*, uniformly for* $t \in [0, +\infty[$*.*

Proof. The limit behavior of the solution for $0 \leq t < t_1$ is given by Lemma 2.2 and Theorem 3.3 and by Proposition 4.1 for $t > t_1$ (see FIG 3.2). □

Assume that $E_0 > b$. From Theorems 3.5 and 3.6 we see that the solution $\gamma(t, \varepsilon)$ reaches the neighborhood of point $(\xi(E_2), \eta(E_2), E_2)$. Then , as shown in Proposition 4.1, it is approximated by a solution of (2.4). In the next propositions, we set three results, in the case where $H(E_0) < G(E_0)$, in the case where $H(E_0) > G(E_0)$ and in the exceptional case where $H(E_0) = G(E_0)$. These results complete, for $t > t_b$, the one given in Lemma 2.3.

PROPOSITION 4.3. *Let* $E_0 > b$*. We assume that* $P(0, \eta(0)) < 0$ *and* $H(E_0) < G(E_0)$*. Let* $\overline{E}_0(t)$ *be the solution of (2.2) with initial condition* $\overline{E}_0(0) = E_0$*. Let* $E_1 = G(E_0)$*. Let* $t_1 > 0$ *be the instant of time for which* $\overline{E}_0(t_1) = E_1$*. Let* $\overline{E}_1(t)$ *be the solution of (2.3) with initial condition* $\overline{E}_1(0) = E_0$*. Let* E_2 *be defined by (3.14) and* $t_2 > t_1$ *be the instant of time for which* $\overline{E}_1(t_2) = E_2$*. Let* $\overline{E}_2(t)$ *be the solution of (2.4) with initial condition* $\overline{E}_2(t_2) = E_2$*. Let*

$$\overline{E}(t) = \begin{cases} \overline{E}_0(t) & \text{for} \quad t \in [0, t_1], \\ \overline{E}_1(t) & \text{for} \quad t \in [t_1, t_2], \\ \overline{E}_2(t) & \text{for} \quad t \in [t_2, +\infty[, \end{cases} \tag{4.2}$$

and

$$\overline{x}(t) = \begin{cases} 0 & \text{for} \quad t \in]0, t_2[, \\ \xi(\overline{E}(t)) & \text{for} \quad t \in]t_2, +\infty[, \end{cases} \qquad \overline{y}(t) = \begin{cases} 0 & \text{for} \quad t \in]0, t_1[, \\ \eta(\overline{E}(t)) & \text{for} \quad t \in]t_1, +\infty[, \end{cases}$$

Then, for any $\delta > 0$ *we have* $\lim_{\varepsilon\to 0}(x(t,\varepsilon), y(t,\varepsilon)) = (\overline{x}(t), \overline{y}(t))$, *uniformly for* $t \in [\delta, t_1 - \delta] \cup [t_1 + \delta, t_2 - \delta] \cup [t_2 + \delta, +\infty[$ *and* $\lim_{\varepsilon\to 0} E(t,\varepsilon) = \overline{E}(t)$, *uniformly for* $t \in [0, +\infty[$.

Proof. The limit behavior of the solution for $0 \le t < t_2$ is given by Lemma 2.3 and Theorem 3.5. The limit behavior of the solution for $t_2 \le t$ is given in Proposition 4.1 (see FIG. 3.3 or 3.4). □

PROPOSITION 4.4. *Let* $E_0 > a_*$. *We assume that* $P(\xi(0), 0) < 0$ *and* $H(E_0) > G(E_0)$. *Let* $\overline{E}_0(t)$ *be the solution of (2.2) with initial condition* $\overline{E}_0(0) = E_0$. *Let* $E_1 = H(E_0)$. *Let* $t_1 > 0$ *be the instant of time for which* $\overline{E}_0(t_1) = E_1$. *Let* $\overline{E}_1(t)$ *be the solution of (2.5) with initial condition* $\overline{E}_1(0) = E_0$. *Let* E_2 *be defined by (3.16) and* $t_2 > t_1$ *be the instant of time for which* $\overline{E}_1(t_2) = E_2$. *Let* $\overline{E}_2(t)$ *be the solution of (2.4) with initial condition* $\overline{E}_2(t_2) = E_2$. *Let* $\overline{E}(t)$ *be defined by (4.2). Let*

$$\overline{x}(t) = \begin{cases} 0 & \text{for} \quad t \in]0, t_1[, \\ \xi(\overline{E}(t)) & \text{for} \quad t \in]t_1, +\infty[, \end{cases} \qquad \overline{y}(t) = \begin{cases} 0 & \text{for} \quad t \in]0, t_2[, \\ \eta(\overline{E}(t)) & \text{for} \quad t \in]t_2, +\infty[, \end{cases}$$

Then, for any $\delta > 0$ *we have* $\lim_{\varepsilon\to 0}(x(t,\varepsilon), y(t,\varepsilon)) = (\overline{x}(t), \overline{y}(t))$, *uniformly for* $t \in [\delta, t_1 - \delta] \cup [t_1 + \delta, t_2 - \delta] \cup [t_2 + \delta, +\infty[$ *and* $\lim_{\varepsilon\to 0} E(t,\varepsilon) = \overline{E}(t)$, *uniformly for* $t \in [0, +\infty[$.

Proof. The limit behavior of the solution for $0 \le t < t_2$ is given by Lemma 2.3 and Theorem 3.6. The limit behavior of the solution for $t_2 \le t$ is given in Proposition 4.1 (see FIG. 3.5). □

PROPOSITION 4.5. *Let* $E_0 > a_*$. *We assume that* $P(\xi(0), 0) < 0$, $P(0, \eta(0)) < 0$ *and* $H(E_0) = G(E_0)$. *Let* $\overline{E}_1(t)$ *be the solution of (2.2) with initial condition* $\overline{E}_1(0) = E_0$. *Let* $E_1 = H(E_0)$. *Let* $t_1 > 0$ *be the instant of time for which* $\overline{E}(t_1) = E_1$. *Let* $\overline{E}_2(t)$ *be the solution of (2.4) with initial condition* $\overline{E}_2(t_1) = E_1$. *Let* $\overline{E}(t)$ *be defined by (4.1). and*

$$\overline{x}(t) = \begin{cases} 0 & \text{for} \quad t \in]0, t_1[, \\ \xi(\overline{E}(t)) & \text{for} \quad t \in]t_1, +\infty[, \end{cases} \qquad \overline{y}(t) = \begin{cases} 0 & \text{for} \quad t \in]0, t_1[, \\ \eta(\overline{E}(t)) & \text{for} \quad t \in]t_1, +\infty[, \end{cases}$$

Then, for any $\delta > 0$ *we have* $\lim_{\varepsilon\to 0}(x(t,\varepsilon), y(t,\varepsilon)) = (\overline{x}(t), \overline{y}(t))$, *uniformly for* $t \in [\delta, t_1 - \delta] \cup [t_1 + \delta, +\infty[$ *and* $\lim_{\varepsilon\to 0} E(t,\varepsilon) = \overline{E}(t)$, *uniformly for* $t \in [0, +\infty[$.

Proof. The limit behavior of the solution for $0 \le t < t_1$ is given by Lemma 2.2 and Theorem 3.6 and by Proposition 4.1 for $t > t_1$. □

4.2. Stability results. We make the following assumption:
(A4) We have $P(0, \eta(0)) < 0$ and $P(\xi(0), 0) < 0$.
For example, this last assumption is satisfied in FIG. 2.2, right. It guarantees that the conclusions of Propositions 4.3, 4.4 and 4.5 hold. Thus, as a consequence of the results given in Section 4.1, after some fast and slow transitions, every trajectory of (1.4) arrives near the slow curve $(x, y) = (\xi(E), \eta(E))$ and approaches the equilibrium S as $t \to \infty$ and $\varepsilon \to 0$. More precisely, we prove the following result:

THEOREM 4.6. *Let assumptions* **(A1)** *to* **(A4)** *be satisfied. Then the equilibrium* S *is practically semi-globally asymptotically stable in the positive octant, as* $\varepsilon \to 0$, *i.e. for any solution* $\gamma(t,\varepsilon) = (x(t,\varepsilon), y(t,\varepsilon), E(t,\varepsilon))$ *of (1.4) we have* $\lim\limits_{t\to+\infty,\varepsilon\to 0} \gamma(t,\varepsilon) =$

S, the limit being uniform with respect to the initial condition in any compact subset of the positive octant.

Proof. Let $t \mapsto e(t,E_0) = \overline{E}(t)$ where $\overline{E}(t)$ is the function defined by Proposition 4.1 when $E_0 < a$, by Proposition 4.2 when $a < E_0 < b$ and by Proposition 4.3, 4.4 or 4.5 when $E_0 > b$ respectively. For all $E_0 > 0$, we have $\lim_{t\to\infty} e(t,E_0) = E_\infty$. Let $r > 0$ and let $T(E_0) = \inf\{t > 0 : |e(t,E_0) - E_\infty| \leq r/2\}$. Let K be a compact subset of the positive octant and let $(x_0,y_0,E_0) \in K$. Let $T = \max\{T(E_0) : (x_0,y_0,E_0) \in K\}$. There exists $\varepsilon_0 > 0$ such that for all $t > T$ and all $\varepsilon \in]0,\varepsilon_0]$, we have $\|\gamma(t,\varepsilon) - S\| \leq r$. Thus S is PSGAS as $\varepsilon \to 0$. □

In order to reach an asymptotic stability instead of a practical asymptotic stability we need the supplementary assumption:

(A5) The functions M, N and P are of class $\mathcal{C}^1$ and satisfy $\frac{\partial M}{\partial x}(\xi(E),E) < 0$, for all $0 < E < a$ and $\frac{\partial N}{\partial y}(\eta(E),E) < 0$, for all $0 < E < b$. We also have

$$\frac{\partial P}{\partial x}(x_\infty,y_\infty)\xi'(E_\infty) + \frac{\partial P}{\partial y}(x_\infty,y_\infty)\eta'(E_\infty) < 0.$$

Theorem 4.7. *Let assumptions* **(A1)** *to* **(A5)** *be satisfied. Then the equilibrium S is semi-globally asymptotically stable in the positive octant, as $\varepsilon \to 0$, that is, it is asymptotically stable and for any compact subset K of the positive octant there exists ε_0 such that, for all $\varepsilon < \varepsilon_0$, K is in the basin of attraction of S.*

Proof. From assumption **(A5)**, the equilibrium $(\xi(E),\eta(E))$ is uniformly exponentially stable for $a < E < b$. On the other hand, we have

$$\frac{\partial p}{\partial E}(E_\infty) = E_\infty\left(\frac{\partial P}{\partial x}(x_\infty,y_\infty)\xi'(E_\infty) + \frac{\partial P}{\partial y}(x_\infty,y_\infty)\eta'(E_\infty)\right) < 0,$$

where $p(E) = EP(\xi(E),\eta(E))$. Hence the equilibrium $E = E_\infty$ of the slow equation (2.4) is exponentially stable. Thus, Theorem A.8 applies and there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the equilibrium S is exponentially stable for system (1.4). Using Theorem 4.6, we see that the attractivity is semi global. Hence S is SGAS as $\varepsilon \to 0$. □

Instead of Assumption **(A4)**, we make now the following assumption:

(A4') We have $P(0,\eta(0)) < 0$ and for all $E > a_*$, $H(E) < G(E)$.

Under this assumption, we always have $H(E_0)) < G(E_0)$ and the asymptotic behavior of all solutions for which $E_0 > a_*$ is described by Proposition 4.3. We do not need the condition $P(\xi(0),0) < 0$, as it is the case in Fig. 5.1, right. So we have the following result:

Theorem 4.8. *The conclusions of Theorems 4.6 and 4.7 hold if assumption* **(A4)** *is replaced by assumption* **(A4')**.

4.3. Remarks. In this section, we give some comments on the assumptions **(A4)** and **(A4')**. In Section 2.3 we have seen that (1.4) has at least four equilibria (2.6) and (2.7). The equilibria (2.7) lie in the invariant xy-plane. From the description of the solutions in Section 4.1, we see that there are no solution of (1.4) with initial condition $x_0 > 0$, $y_0 > 0$ and $E_0 > 0$ that can approach the equilibria (2.7) as $\varepsilon \to 0$.

Let us examine what may happen if condition $P(0,\eta(0)) < 0$ of Assumption **(A4')** is relaxed. In the case where $P(0,\eta(0)) > 0$, (2.8) is an equilibrium of (1.4). Even if this equilibrium is unstable in the x-direction, there are solutions of (1.4) with initial condition $x_0 > 0$, $y_0 > 0$ and $E_0 > 0$ that can approach the equilibrium (2.8), as $\varepsilon \to 0$ (see remark following Theorem 3.6). Such solutions will stay for a very long time near the equilibrium $(0, \eta(c), c)$, before leaving its neighborhood, close to its unstable separatrix and jumping to the neighborhood of $(\xi(c), \eta(c), c)$ and then going towards the equilibrium S. Hence, it can stay for a long time near the equilibrium (2.8). In this case, the equilibrium S is not PSGAS because the limit given in Theorem 4.6 could be non uniform with respect to the initial conditions.

It is not very easy to verify that condition $H(E) < G(E)$ of assumption **(A4')** is satisfied, since functions H and G are not so easy to compute. This assumption was made to guarantee that a solution of (1.4) with initial condition $x_0 > 0$, $y_0 > 0$ and $E_0 > a_*$ will leave the neighborhood of point $(0, 0, E_1)$ and jumps to the neighborhood of point $(0, \eta(E_1), E_1)$ close to the orbit $x = 0$ of the unstable node $(0, 0)$.

Now, this is what may happen if condition $P(\xi(0), 0) < 0$ of Assumption **(A4)** is relaxed. If $P(\xi(0), 0) > 0$, then (2.9) is an equilibrium point of (1.4). Even if this equilibrium is unstable in the y-direction, there are solutions of (1.4) with initial condition $x_0 > 0$, $y_0 > 0$ and $E_0 > 0$ that can approach the equilibrium (2.9), as $\varepsilon \to 0$. Such solutions will stay for a very long time near the equilibrium $(\xi(d), 0, d)$, before leaving its neighborhood, close to its unstable separatrix and junping to the neighborhood of $(\xi(d), \eta(d), d)$ and then going towards the equilibrium S. Then the solution can stay for a long time near the equilibrium (2.9). In this case again, the equilibrium S is not PSGAS because the limit given in Theorem 4.6 could be non uniform with respect to the initial conditions.

5. Applications to Clarks's model. Assume that all parameters of (1.2) are positive. We have

$$\xi(E) = K\left(1 - \frac{q_1}{r}E\right), \quad \eta(E) = L\left(1 - \frac{q_2}{s}E\right), \quad a = \frac{r}{q_1}, \quad b = \frac{s}{q_2}.$$

The slow manifold is represented in FIG. 5.1.

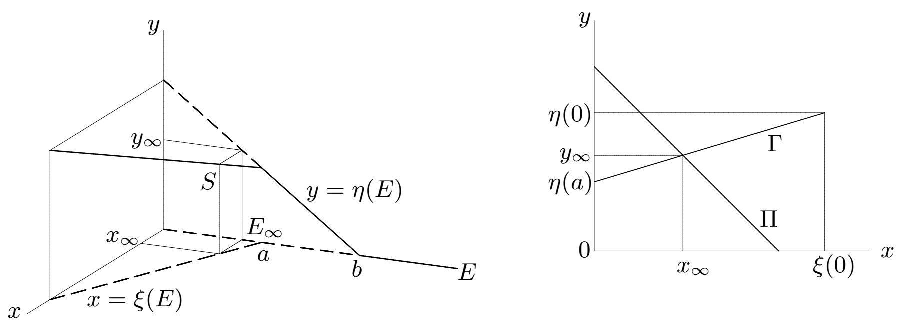

FIG. 5.1. *On the left: the slow manifold of system (1.2). Attracting parts of the slow manifold are indicated by a bold line, non attracting parts of the slow curve are indicated by a dashed line. On the right: the relative positions of* Γ *and* Π. *For the set of parameters (5.3) we have* $P(\xi(0), 0) > 0$.

5.1. Stability results. Assumptions **(A1)** and **(A2)** hold if and only if

$$sq_1 > rq_2. \tag{5.1}$$

On the other hand, we have $P(x, y) = p_1q_1x + p_2q_2y - c$, so assumption **(A3)** holds if and only if (see FIG. 5.1, right)

$$\frac{sq_1 - rq_2}{sq_1} p_2q_2L < c < p_1q_1K + p_2q_2L.$$

The equilibrium $S=(x_\infty,y_\infty,E_\infty)$ is given by $E_\infty=(p_1q_1K+p_2q_2L-c)\frac{rs}{\Delta}$,

$$x_\infty=(sq_1c-(sq_1-rq_2)p_2q_2L)\frac{K}{\Delta}\quad\text{and}\quad y_\infty=(rq_2c+(sq_1-rq_2)p_1q_1K)\frac{L}{\Delta},$$

where $\Delta=sp_1q_1^2K+rp_2q_2^2L$. Also, Assumption **(A4)** holds if and only if

$$p_2q_2L<c\ \text{ and }\ p_1q_1K<c. \tag{5.2}$$

By straightforward calculations, we can show that assumption **(A5)** also holds. As a corollary of Theorem 4.7, we deduce the following result:

THEOREM 5.1. *Assume that* $\max(p_1q_1K,p_2q_2L)<c<p_1q_1K+p_2q_2L$ *and* $sq_1>rq_2$. *Then the equilibrium* S *is semi-globally asymptotically stable in the positive octant, as* $\varepsilon\to0$.

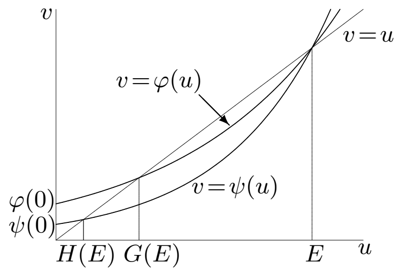

FIG. 5.2. *The functions H and G corresponding to (1.2) satisfy $H(E)<G(E)$ for all $E>b$.*

The result of Theorem 5.1 occurs for a larger range of parameters since $H(E)<G(E)$ is always true for (1.2) as shown in the following lemma:

LEMMA 5.2. *Under condition (5.1) we have $H(E)<G(E)$, for all $E>b$.*

Proof. We have

$$h(E)=\int_a^E\frac{r-q_1u}{-cu}du=\frac{q_1E-r\ln E-q_1a-r\ln a}{c}.$$

Let $E>b$, $u=H(E)$ is defined by

$$q_1u-r\ln u=q_1E-r\ln E\Leftrightarrow u=\varphi(u),\ \text{where}\ \varphi(u):=Ee^{\frac{u-E}{a}}.$$

Hence $H(E)$ is a fixed point of the function $v=\varphi(u)$. We have

$$\varphi(0)>0,\quad \varphi(E)=E,\quad \varphi'(E)=\frac{E}{a}>1\ \text{ and }\ \varphi''(u)>0.$$

Hence the function $v=\varphi(u)$ has a unique fixed point $u=H(E)$ in $]0,E[$. Similarly we have

$$g(E)=\int_b^E\frac{s-q_2u}{-cu}du=\frac{q_2E-s\ln E-q_2b-s\ln b}{c}.$$

Thus $u=G(E)$ is the unique fixed point, in $]0,E[$, of the function $v=\psi(u):=Ee^{\frac{u-E}{b}}$. We have $\psi(0)<\varphi(0)$, $\psi(E)=\varphi(E)=E$ and $\frac{1}{a}>\frac{1}{b}$. Hence the functions φ and ψ satisfy (see FIG. 5.2) $\psi(u)<\varphi(u)$ for all $u\in]0,E[$, so that $H(E)<G(E)$. ◻

From Lemma 5.2 we deduce that, if condition (5.1) holds, then assumption **(A4')** holds if and only if $p_2q_2L < c$ and this condition is less restrictive than (5.2). As a consequence of Theorem 4.8, we have:

THEOREM 5.3. *Assume that* $p_2q_2L < c < p_1q_1K + p_2q_2L$ *and* $sq_1 > rq_2$. *Then the equilibrium* S *is semi-globally asymptotically stable in the positive octant, as* $\varepsilon \to 0$.

The system (1.2) is dissipative, for any $\varepsilon > 0$. Thus any solution enters in a compact subset of the positive octant. Hence for any fixed $\varepsilon > 0$, the equilibrium S is GAS in the positive octant. Actually, we proved that, as $\varepsilon \to 0$, the attractivity is uniform with respect to the initial condition in any compact subset of the positive octant.

5.2. Numerical simulations. To illustrate our results, we carried some numerical experiments with $\varepsilon = 0.01$ and the following set of parameters:

$$K = 3, \quad L = 3, \quad r = 1, \quad s = 2, \quad p_1 = p_2 = q_1 = q_2 = 1, \quad c = 4. \tag{5.3}$$

For clarity, we draw the projections of the trajectories, related to two sets of initial values (see Fig. 5.3 and Fig. 5.4), on the planes (E, y), on the left, (E, x), in the center and (x, y), on the right. The behavior of the first trajectory with $x_0 = 4$, $y_0 = 4$ and $E_0 = 3 < a_*$ is in accordance with the results of Theorem 3.5 and Proposition 4.2. The behavior of the trajectory with $x_0 = 0$, $y_0 = 0$ and $E_0 = 6 > a_*$ supports the results of Theorem 3.5 and Proposition 4.3.

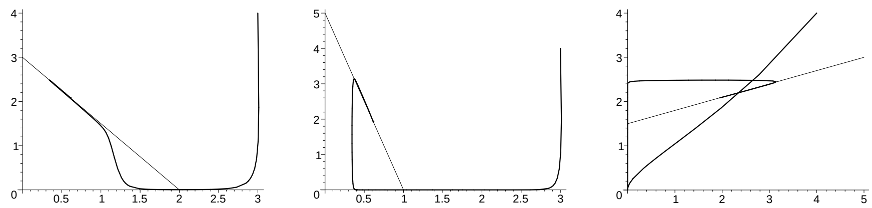

FIG. 5.3. *Numerical solutions of (1.2) with* $x_0 = 4$, $y_0 = 4$ *and* $E_0 = 3 < a_*$.

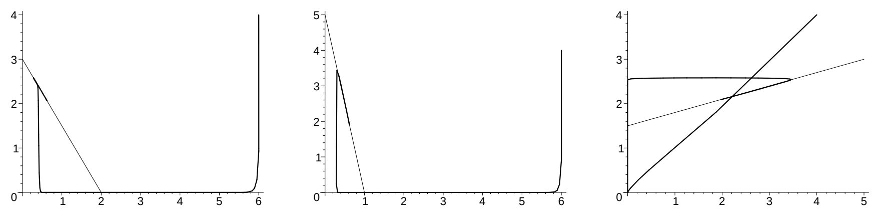

FIG. 5.4. *Numerical solutions of (1.2) with* $x_0 = 0$, $y_0 = 0$ *and* $E_0 = 6 > a_*$.

Appendix A. Slow and fast vector fields. In this appendix, we give the main definitions and tools of singular perturbation theory which are used in this paper. For more details and information the reader may consult [13, 17, 18, 24, 26]. We follow here the presentation given in [20, 21].

A.1. Stability in systems with parameters. The notion of uniform stability in systems with parameters is crucial in the hypothesis of Tikhonov's theorem. Let

$$z' = Z(z,a), \qquad z \in Z \subset \mathbb{R}^d$$

be a system depending on the parameter $a \in A \subset \mathbb{R}^m$. We assume that there exists a mapping $a \in A \mapsto h(a) \in Z$ such that for all $a \in A$ we have $Z(h(a),a) = 0$, so that $z = h(a)$ is an equilibrium point.

DEFINITION A.1. *The equilibrium $z = h(a)$ of equation $z' = Z(z,a)$ is said to be uniformly asymptotically stable for $a \in A$ if for all $\mu > 0$, there exists $\delta > 0$, such that for all $a \in A$ and for all solution $z(\tau,a)$*

$$\|z(0,a) - h(a)\| < \delta \;\Rightarrow\; \forall\tau > 0 \;\|z(\tau,a) - h(a)\| < \mu \text{ and } \lim_{\tau\to+\infty} z(\tau,a) = h(a)$$

DEFINITION A.2. *The equilibrium $z = h(a)$ of equation $z' = Z(z,a)$ is said to be uniformly exponentially stable in $a \in A$ if there exist $k > 0$, $\gamma > 0$ and $r > 0$ such that for all $a \in A$ any solution $z(\tau,a)$ with $\|z(0,a) - h(a)\| \le r$ satisfies*

$$\|z(\tau,a)\| \le k\|z(0,a)\|\mathrm{e}^{-\gamma\tau} \text{ for all } \tau \ge 0.$$

Note that for $\mathcal{C}^1$ systems, the uniform exponential stability holds if all eigenvalues of the linear part $\frac{\partial Z}{\partial z}(h(a),a)$ have negative real parts.

A.2. Practical asymptotic stability. To introduce the notion of practical asymptotic stability, consider the system depending on a parameter

$$\dot{x} = F(x,\varepsilon), \tag{A.1}$$

where $x \in X \subset \mathbb{R}^d$ and $\varepsilon > 0$ is a real parameter.

DEFINITION A.3. *The point $x = x_*$ is practically asymptotically stable (PAS) when $\varepsilon \to 0$ if there exists $A > 0$, such that for all $r > 0$ there exist $\varepsilon_0 > 0$ and $T > 0$ such that for all ε, for all solution $x(t,\varepsilon)$ of equation (A.1) and for all time t if $\varepsilon < \varepsilon_0$ $\|x(0,\varepsilon) - x_*\| < A$ and $t > T$ then, $\|x(t,\varepsilon) - x_*\| < r$.*

In the case where the solution $x(t;x_0,\varepsilon)$ of (A.1), with initial condition $x(0;x_0,\varepsilon) = x_0$, is unique the point $x = x_*$ is PAS if and only if

$$\lim_{t\to+\infty,\varepsilon\to 0} x(t,x_0,\varepsilon) = x_*,$$

the limit being uniform for x_0 in some ball of radius A independent on ε.

A.3. Semi global asymptotic stability. Here, we define the notion of semi global asymptotic stability.

DEFINITION A.4. *The point $x = x_*$ is semi-globally asymptotically stable (SGAS) when $\varepsilon \to 0$ if it is asymptotically stable for all $\varepsilon \in]0,1]$ and for all compact $K \subset X$ there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, the basin of attraction of x_* contains K.*

DEFINITION A.5. *The point $x = x_*$ is practically semi-globally asymptotically stable (PSGAS) when $\varepsilon \to 0$ if for any compact $K \subset X$ and $r > 0$ there exist $\varepsilon_0 > 0$ and $T > 0$ such that for all ε, for all solution $x(t,\varepsilon)$ of equation (A.1) and for all time t $\varepsilon < \varepsilon_0$, $x(0,\varepsilon) \in K$ and $t > T$ then, $\|x(t,\varepsilon) - x_*\| < r$.*

In the case where the solution $x(t;x_0,\varepsilon)$ of (A.1) with initial condition $x(0,x_0,\varepsilon) = x_0$ is unique, the point $x = x_*$ is PSGAS if and only if

$$\lim_{t\to+\infty,\varepsilon\to 0} x(t,x_0,\varepsilon) = x_*,$$

the limit being uniform for x_0 in any prescribed compact subset of X.

A.4. Tikhonov's theorem. Let the initial value problem

$$\begin{array}{rclr} \dot{u} & = & U(u,v,\varepsilon), & u(0)=\alpha_\varepsilon, \\ \varepsilon\dot{v} & = & V(u,v,\varepsilon), & v(0)=\beta_\varepsilon. \end{array} \tag{A.2}$$

where $u \in \mathbb{R}^n$, $v \in \mathbb{R}^m$, $\dot{u} = \frac{du}{dt}$, $\dot{v} = \frac{dv}{dt}$ and ε is a real parameter. We assume that U and V are continuous functions and that problem (A.2) has a unique solution. We look at the solutions behavior when $\varepsilon \to 0$ and $t \in [0, T]$. The small parameter is multiplying the derivative and so the usual theory of continuous dependence of the solutions with respect to the parameters cannot be applied. If we write (A.2) at time scale $\tau = t/\varepsilon$ we obtain

$$\begin{array}{rclr} u' & = & \varepsilon U(u,v,\varepsilon), & u(0)=\alpha_\varepsilon, \\ v' & = & V(u,v,\varepsilon), & v(0)=\beta_\varepsilon. \end{array} \tag{A.3}$$

where $u' = \frac{du}{d\tau}$ and $v' = \frac{dv}{d\tau}$. The equation

$$v' = V(u,v,0). \tag{A.4}$$

where u a constant parameter, is called the *fast equation*. We assume that it has a unique solution with prescribed initial condition. The manifold $\mathcal{L}$ of equation

$$V(u,v,0) = 0, \tag{A.5}$$

is called the *slow manifold*: it is the set of equilibria of (A.4). We make the following assumptions which are listed by the letter (**H**) below.

(**H1**) For all u in some compact domain $K \subset \mathbb{R}^n$, all solutions of (A.4) tend towards an equilibrium $v = W(u)$, where $v = W(u)$ is an isolated root of equation (A.5). The equilibrium $v = W(u)$ is uniformly asymptotically stable for $u \in K$, (see definition in Appendix A.1).

(**H2**) We have $\alpha_0 \in K$ and β_0 belongs to the basin of attraction of $W(\alpha_0)$.

On the component $v = W(u)$ we define the *slow equation*

$$\dot{u} = U(u, W(u), 0), \qquad u \in K. \tag{A.6}$$

We assume that this equation has a unique solution with prescribed initial condition. Let $\overline{u}(t)$ be the solution of (A.6) with initial condition $u(0) = \alpha_0$.

(**H3**) We assume that $\overline{u}(t)$ is defined for $0 \leq t \leq T$.

THEOREM A.6. *Under hypothesis (**H1**) to (**H3**), the solution* $(u(t,\varepsilon), v(t,\varepsilon))$ *of (A.2) is defined at least on* $[0,T]$*. For all* $\delta > 0$*, it satisfies* $\lim_{\varepsilon\to 0} u(t,\varepsilon) = \overline{u}(t)$ *uniformly on* $[0,T]$ *and* $\lim_{\varepsilon\to 0} v(t,\varepsilon) = W(\overline{u}(t))$ *uniformly on* $[\delta, T]$.

Proof. See [21, 25, 26]. □

The limit for the fast variable v does not hold at 0 since there is a boundary layer in z. More precisely, let $\tilde{u}(\tau)$ be the solution of the *boundary layer equation*

$$v' = V(\alpha_0, v, 0), \qquad v(0) = \beta_0.$$

This solution is defined for all $\tau \geq 0$ and tends to $W(\alpha_0)$. We have:

$$\lim_{\varepsilon\to 0} (v(t,\varepsilon) - \tilde{v}(t/\varepsilon)) = W(\overline{v}(t)) - W(\alpha_0) \text{ uniformly on } [0,T].$$

A.5. Approximation for all $t \geq 0$. To obtain approximations on the infinite interval, we need a supplementary assumption.

(**H4**) The slow equation (A.6) has an asymptotically stable equilibrium $u = u_\infty$ and the initial condition α_0 is in its basin of attraction.

THEOREM A.7. *Under hypothesis (**H1**) to (**H4**), the solution* $(u(t,\varepsilon), v(t,\varepsilon))$ *of (A.2) is defined for all* $t \geq 0$*. For all* $\delta > 0$*, it satisfies* $\lim_{\varepsilon\to 0} u(t,\varepsilon) = \overline{u}(t)$ *for all* $t \geq 0$ *and* $\lim_{\varepsilon\to 0} v(t,\varepsilon) = W(\overline{u}(t))$ *for all* $t \geq \delta$*.*

Proof. See [13, 17, 21]. □

From theorem A.7 we deduce that

$$\lim_{t\to+\infty,\varepsilon\to 0} u(t,\varepsilon) = x_\infty, \qquad \lim_{t\to+\infty,\varepsilon\to 0} v(t,\varepsilon) = v_\infty,$$

where $v_\infty = W(u_\infty)$. This result does not imply that (u_∞, v_∞) is an asymptotically stable equilibrium of (A.2) as it is illustrated by the following example:

EXAMPLE. Consider the planar slow-fast system

$$\dot{u} = u^2(\varepsilon - u), \qquad \varepsilon\dot{v} = -v. \tag{A.7}$$

The slow manifold is $v = 0$ and $v = 0$ is GAS for the fast equation $v' = -v$. The equilibrium $u = 0$ is GAS for the reduced equation $\dot{u} = -u^3$. However, the equilibrium $(0,0)$ of (A.7) is unstable.

A.6. Stability results. To obtain stability results of the equilibrium (u_∞, v_∞), we need supplementary conditions on the system. Assume that for all $\varepsilon > 0$, we have $U(u_\infty, v_\infty, \varepsilon) = 0$ and $V(u_\infty, v_\infty, \varepsilon) = 0$, so that, for all $\varepsilon > 0$, (u_∞, v_∞) is an equilibrium of system (A.2). We assume that asymptotic stability is replaced by exponential stability in assumptions (**H1**) and (**H4**). We have the following result:

THEOREM A.8. *Let hypothesis (**H1**) to (**H4**) be satisfied, where asymptotic stability is replaced by exponential stability, that is,* $v = W(u)$ *is an uniformly exponentially stable point for* $v' = V(u,v,0)$ *on* $u \in K$ *and* u_∞ *is an exponentially stable point for* $\dot{u} = U(u, W(u), 0)$*. Then, there exists* $\varepsilon^* > 0$ *such that for all* $0 < \varepsilon < \varepsilon^*$*,* $(u_\infty, W(u_\infty))$ *is exponentially stable for system (A.2).*

Proof. See [17], Section 9.4 or [18], Section 7.5. □

Exponential stability cannot be replaced by asymptotic stability as shown by (A.7). In this example $x = 0$ is GAS but not exponentially stable for the reduced equation $\dot{x} = -x^3$ and the origin is unstable for the complete system.

REMARK. Similar results was obtained using geometric singular perturbation theory (GSPT) (see [14], Section B.3). In GSPT [12, 15, 16], system (A.2) is called the *slow system* and system (A.3) is called the *fast system.* In this paper we adopted the terminology of the classical singular perturbation theory [21, 25, 26] and we refer to (A.4), which is the limit of (A.3) when $\varepsilon \to 0$, as the fast equation and to (A.6), which is the limit of (A.2) when $\varepsilon \to 0$, as the slow equation. Note that in GSPT, the slow manifold is not necessarily attracting, as in Tikhonov's theorem. In GSPT the results hold for the more general case of slow and fast systems for which the slow manifold is normally hyperbolic, i.e., the real part of the eigenvalues of $\frac{\partial V}{\partial v}(u, W(u), 0)$ are non 0, possibly positive or negative. However, in GSPT, the functions U and V in (A.2) must be smooth, not only continuous, as it is the case in Tikhonov's theory. GSPT shows that for small $\varepsilon > 0$, (A.2) has a locally invariant manifold which is $O(\varepsilon)$-close to the slow manifold.

A.7. Practical semi global asymptotic stability. In general, the attractivity of the full system is not global as shown by the following example:

EXAMPLE. Consider the two dimensional slow fast system

$$\dot{u} = -u + u^2 v, \qquad \varepsilon \dot{v} = -v. \tag{A.8}$$

The fast equation $v' = -v$ has $v = 0$ as a globally exponentially stable equilibrium. The corresponding slow equation $\dot{u} = -u$ has $u = 0$ as a globally exponentially stable equilibrium. By theorem A.8, the origin of (A.8) is exponentially stable for small ε. Actually, the origin is exponentially stable for all $\varepsilon > 0$, but the attractivity is not global since

$$\frac{d}{dt}(uv) = uv(uv - 1 - 1/\varepsilon),$$

shows that the hyperbola $uv = 1 + 1/\varepsilon$ is invariant. The basin of attraction of the origin is the set $B := \{(u, v) \in \mathbb{R}^2 : uv < 1 + 1/\varepsilon\}$. Thus, the origin is not GAS for (A.8). However, the origin of (A.8) is semi-globally asymptotically stable when $\varepsilon \to 0$.

Let us give, now a result of practical semi global asymptotic stability in the slow-fast system (A.2). We do not assume that $U(u_\infty, v_\infty, \varepsilon) = 0$ and $V(u_\infty, v_\infty, \varepsilon) = 0$ as in the previous section. Hence the point (u_∞, v_∞) is not necessarily an equilibrium of (A.2).

THEOREM A.9. *Assume that $U(u_\infty, v_\infty, 0) = 0$ and $V(u_\infty, v_\infty, 0) = 0$, the equilibrium $u = u_\infty$ is GAS for $\dot{u} = U(u, W(u), 0)$ and the equilibrium $v = W(u)$ is GAS for $v' = V(u, v, 0)$. Then the point (u_∞, v_∞) is PSGAS for (A.2) as $\varepsilon \to 0$.*

Proof. See [20]. □

A.8. Canard Solutions and Delayed Loss of Stability. The results of the previous section do not apply easily, since in most examples met in applications, it is rare that the slow manifold has a unique component and that the equilibrium of the slow equation is GAS. In general, the slow manifold has several components and the solution jumps from one to the other. These jumps are often accompanied by the phenomenon of *canard solutions* or *delayed loss of stability phenomenon*. For details and complements see [1], p. 179–192 and [2]. Stability loss delay in dynamical bifurcations is an important and newly discovered phenomenon [23]. Consider again the slow-fast system,

$$x' = \varepsilon, \qquad z' = Z(x, z, \varepsilon) \tag{A.9}$$

where $x \in \mathbb{R}$ and where system $z' = Z(x, z, 0)$ has an equilibrium $z = h(x)$ for each value of the parameter x. Suppose that there exists a bifurcation value $x = x_*$ of the parameter, at which the equilibrium loses stability, that is, $z = h(x)$ is asymptotically stable for $x < x_*$ and unstable for $x > x_*$. Assume that $x(0, \varepsilon) = x_0 < x_*$. Then the solution of system (A.9) will go quickly near the equilibrium $z = h(x_0)$ and then remains close to the curve $z = h(x)$, until x reaches some value $x_1 > x_*$. This means that the loss of stability which must occur at $x = x_*$ is delayed until $x = x_1$. The general theory in [23] does not apply in our system. This theory requires that a pair of eigenvalues crosses the imaginary axis, when in our case a real eigenvalue crosses zero. The stability loss delay phenomenon is closely related to the phenomenon of canard solutions. Canard solutions are special trajectories of slow and fast systems that first move near the stable part of the slow manifold, then move near the unstable part of it. These solutions were first discovered in the framework of Nonstandard Analysis. See [3, 9, 29] for historical comments and references. The study of canard solutions has also been made in the framework of classical asymptotic analysis [11] and also in the framework of GSPT, using center manifolds and blow-up. See [10, 27] for details and references.

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