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q -Sturm-Liouville theory and the corresponding eigenfunction expansions

Lazhar Dhaouadi*

Abstract

The aim of this paper is to study the q -Schrödinger operator

$$L = q(x) - \Delta_q,$$

where $q(x)$ is a given function of x defined over $\mathbb{R}_q^+ = \{q^n, n \in \mathbb{Z}\}$ and Δ_q is the q -Laplace operator

$$\Delta_q f(x) = \frac{1}{x^2} \left[f(q^{-1}x) - \frac{1+q}{q} f(x) + \frac{1}{q} f(qx) \right].$$

1 Introduction

After the spectral analysis in [1] of the q -Laplace operator also called the second-order q -difference operator

$$\Delta_q f(x) = \frac{1}{x^2} \left[f(q^{-1}x) - \frac{1+q}{q} f(x) + \frac{1}{q} f(qx) \right],$$

it is natural to study the perturbed operator

$$L = q(x) - \Delta_q.$$

The eigenfunction expansion theory for the q -Sturm-Liouville equation (singular case) presented in this paper is based on the original works of Hermann Weyl in 1910 and of Edward Charles Titchmarsh in 1941, concerning Sturm-Liouville theory and the corresponding eigenfunction expansions. For this account the essential results of Weyl concern the regular, limit-circle and limit-point classifications of Sturm-Liouville differential equations (singular case); the eigenfunction expansion theory from Titchmarsh is based on classical function theory methods, in particular complex function theory. For more information on the classical theory, the reader can consult the references [3, 5, 7, 8].

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2 Basic definitions

Consider $0 < q < 1$. In what follows, the standard conventional notations from [2], will be used

$$\mathbb{R}_q^+ = \{q^n, \quad n \in \mathbb{Z}\}$$

$$(a, q)_0 = 1, \quad (a, q)_n = \prod_{i=0}^{n-1} (1 - aq^i).$$

The q -shift operator is

$$\Lambda_q f(x) = f(qx).$$

Next we introduce two concepts of q -analysis: the q -derivative and the q -integral. The q -derivative of a function f is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x},$$

and $D_q f(0) = f'(0)$, provided $f'(0)$ exists. The second order q -difference operator is

$$\Delta_q f(x) = \left[\frac{1 - q}{q} \right]^2 \Lambda_q^{-1} D_q^2 f(x) = \frac{1}{x^2} \left[f(q^{-1}x) - \frac{1 + q}{q} f(x) + \frac{1}{q} f(qx) \right].$$

The product rule for the q -derivative is

$$D_q(fg)(x) = D_q f(x)g(x) + \Lambda_q f(x)D_q g(x).$$

Jackson's q -integral (see [4]) in the interval $[a, b]$ and in the interval $[0, \infty[$ are defined by

$$\int_a^b f(x) d_q x = (1 - q) \sum_{n=0}^{\infty} q^n [bf(bq^n) - af(aq^n)]$$

$$\int_0^{\infty} f(x) d_q x = (1 - q) \sum_{n=-\infty}^{\infty} q^n f(q^n).$$

Also the rule of q -integration by parts is given by

$$\int_a^b D_q f(x)g(x) d_q x = [f(b)g(b) - f(a)g(a)] - \int_a^b \Lambda_q f(x)D_q g(x) d_q x.$$

In the end we denote by $L^2(\mathbb{R}_q^+)$ the Hilbert space of functions f defined on \mathbb{R}_q^+ and satisfy

$$\int_0^{\infty} |f(x)|^2 d_q x < \infty.$$

3 q -Sturm-Liouville difference equation

If F satisfies the q -difference equation

$$\Delta_q f(x) + [\lambda - q(x)]f(x) = 0, \quad (1)$$

and G the same equation with λ' instead of λ , then

$$\begin{aligned} & (\lambda' - \lambda) \int_0^b F(x)G(x)d_q x \\ &= \int_0^b [F(x) \{q(x)G(x) - \Delta_q G(x)\} - G(x) \{q(x)F(x) - \Delta_q F(x)\}]d_q x \\ &= - \int_0^b \{F(x)\Delta_q G(x) - G(x)\Delta_q F(x)\} d_q x = W_0(F, G) - W_b(F, G), \end{aligned}$$

where W_x is the q -Wronskian defined by

$$W_x(F, G) = \frac{(1-q)^2}{q} [F(x)\Lambda_q^{-1}D_q G(x) - G(x)\Lambda_q^{-1}D_q F(x)].$$

Note that the q -Wronskian defined here is slightly different from the q -Wronskian introduced in [6] and in [1]. In some cases we write $W[F(x), G(x)]$ instead of $W_x(F, G)$.

If $\lambda = \mu + i\nu$, $\lambda' = \bar{\lambda}$ and $G = \bar{F}$, this gives

$$2\nu \int_0^b |F(x)|^2 d_q x = iW_0(F, \bar{F}) - iW_b(F, \bar{F}). \quad (2)$$

Now let $\phi(x) = \phi(x, \lambda)$ and $\theta(x) = \theta(x, \lambda)$ be two solutions of (1) such that

$$\begin{cases} \phi(0) = \frac{\sqrt{q}}{1-q} \sin \alpha, & \phi'(0) = -\frac{\sqrt{q}}{1-q} \cos \alpha \\ \theta(0) = \frac{\sqrt{q}}{1-q} \cos \alpha, & \theta'(0) = \frac{\sqrt{q}}{1-q} \sin \alpha \end{cases},$$

where α is real. Then it follows that

Theorem 1 *For every value of λ other than real values, there exist a constant $m(\lambda)$ such that (1) has a solution*

$$\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\phi(x, \lambda)$$

belonging to $L^2(\mathbb{R}_q^+)$.

Proof. In fact

$$W_x(\phi, \theta) = W_0(\phi, \theta) = \sin^2 \alpha + \cos^2 \alpha = 1.$$

and

$$W_0(\phi, \bar{\phi}) = W_0(\theta, \bar{\theta}) = 0.$$

Therefore

$$W_0(\theta + l\phi, \bar{\theta} + \bar{l}\bar{\phi}) = l - \bar{l} = 2i \operatorname{Im} l.$$

The general solution of (1) is of the form $\theta(x) + l\phi(x)$. Consider those solutions which satisfy a real boundary condition at $x = b$, say

$$\{\theta(b) + l\phi(b)\} \cos \beta + \{\Lambda_q^{-1} D_q \theta(b) + l\Lambda_q^{-1} D_q \phi(b)\} \sin \beta = 0,$$

where β is real. This gives

$$l = l(\lambda) = \frac{\theta(b) \cot \beta + \Lambda_q^{-1} D_q \theta(b)}{\phi(b) \cot \beta + \Lambda_q^{-1} D_q \phi(b)}.$$

For each b , as $\cot \beta$ varies, l describes a circle in the complex plane, say C_b . Replacing $\cot \beta$ by a complex variable z , we obtain

$$l = l(\lambda) = -\frac{\theta(b)z + \Lambda_q^{-1} D_q \theta(b)}{\phi(b)z + \Lambda_q^{-1} D_q \phi(b)}.$$

The centre of C_b correspond to

$$z = -\frac{\Lambda_q^{-1} D_q \bar{\phi}(b)}{\bar{\phi}(b)} = -\frac{W_b(\theta, \bar{\phi})}{W_b(\phi, \bar{\phi})}.$$

Since $-\Lambda_q^{-1} D_q \theta(b)/\Lambda_q^{-1} D_q \phi(b)$ is on C_b (for $z = 0$) the radius r_b of C_b is

$$r_b = \left| \frac{\Lambda_q^{-1} D_q \theta(b)}{\Lambda_q^{-1} D_q \phi(b)} - \frac{W_b(\theta, \bar{\phi})}{W_b(\phi, \bar{\phi})} \right| = \left| \frac{W_b(\theta, \phi)}{W_b(\phi, \bar{\phi})} \right| = \frac{1}{2\nu \int_0^b |\phi(x)|^2 d_q x}. \quad (3)$$

Now l is inside C_b if $\operatorname{Im} z < 0$, i.e. if

$$i \left\{ -\frac{l\Lambda_q^{-1} D_q \phi(b) + \Lambda_q^{-1} D_q \theta(b)}{l\phi(b) + \theta(b)} + \frac{\bar{l}\Lambda_q^{-1} D_q \bar{\phi}(b) + \Lambda_q^{-1} D_q \bar{\theta}(b)}{\bar{l}\bar{\phi}(b) + \bar{\theta}(b)} \right\} > 0,$$

i.e. if

$$i \left\{ |l|^2 W_b(\phi, \bar{\phi}) + lW_b(\phi, \bar{\theta}) + \bar{l}W_b(\theta, \bar{\phi}) + W_b(\theta, \bar{\theta}) \right\} > 0,$$

i.e. if

$$iW_b(\theta + l\phi, \bar{\theta} + \bar{l}\bar{\phi}) > 0,$$

i.e. if

$$2\nu \int_0^b |\theta + l\phi|^2 d_q x < iW_0(\theta + l\phi, \bar{\theta} + \bar{l}\bar{\phi}).$$

Hence l is interior to C_b if $\nu > 0$, and

$$\int_0^b |\theta + l\phi|^2 d_q x < -\frac{\text{Im } l}{\nu}.$$

The same result is obtained if $\nu < 0$. It follows that, if l is interior to C_b , and $0 < b' < b$, then

$$\int_0^{b'} |\theta + l\phi|^2 d_q x < \int_0^b |\theta + l\phi|^2 d_q x < -\frac{\text{Im } l}{\nu}.$$

Hence l is also inside $C_{b'}$. Hence $C_{b'}$ includes C_b if $b' < b$. It follows that, as $b \rightarrow \infty$, the circle C_b converge either to a limit-circle or to a limit-point.

If $m = m(\lambda)$ is the limit-point, or any point on the limit-circle,

$$\int_0^b |\theta + m\phi|^2 d_q x < -\frac{\text{Im } m}{\nu},$$

for all values of b . Hence

$$\int_0^\infty |\theta + m\phi|^2 d_q x < -\frac{\text{Im } m}{\nu}.$$

This finish the proof. ■

Remark 1 *In the limit-circle case, r_b tends to a positive limit as $b \rightarrow \infty$. Hence, by (3) the function ϕ is $L^2(\mathbb{R}_q^+)$. So in fact, in this case every solution of (1) belongs to $L^2(\mathbb{R}_q^+)$.*

4 The eigenfunctions

On the circle C_b (if $\nu > 0$)

$$\frac{1}{2} |l|^2 \int_0^b |\phi|^2 d_q x - \int_0^b |\theta|^2 d_q x \leq \int_0^b |\theta + l\phi|^2 d_q x < -\frac{\text{Im } l}{\nu} \leq \frac{|l|}{\nu}.$$

Solving for $|l|$ we obtain

$$|l| \leq \frac{1}{\nu \int_0^b |\phi|^2 d_q x} + \left\{ \frac{2 \int_0^b |\theta|^2 d_q x}{\int_0^b |\phi|^2 d_q x} + \frac{1}{\left(\nu \int_0^b |\phi|^2 d_q x \right)^2} \right\}^{\frac{1}{2}}.$$

Since the above right-hand side is $O(\frac{1}{\nu})$. Hence as $\nu \rightarrow 0$, for any fixed b , it also follows that $m(\lambda) = O(\frac{1}{\nu})$. Hence, if $m(\lambda)$ has poles on the real axis, they are all simple. In this paper we assume that $m(\lambda)$ form a single analytic function, whose only singularities are poles on the real axis. Let them be $\lambda_0, \lambda_1, \dots$, and let the residues be r_0, r_1, \dots .

Lemma 1 *For any fixed complex λ and λ'*

$$\lim_{x \rightarrow \infty} W_x \{ \psi(\cdot, \lambda), \psi(\cdot, \lambda') \} = 0.$$

Proof. Since

$$W_b \{ \theta(x, \lambda) + l(\lambda)\phi(x, \lambda), \theta(x, \lambda') + l(\lambda')\phi(x, \lambda') \} = 0,$$

i.e.

$$W_b [\psi(x, \lambda) + \{l(\lambda) - m(\lambda)\}\phi(x, \lambda), \psi(x, \lambda') + \{l(\lambda') - m(\lambda')\}\phi(x, \lambda')] = 0,$$

i.e.

$$\begin{aligned} & W_b \{ \psi(x, \lambda), \psi(x, \lambda') \} + \{l(\lambda) - m(\lambda)\} W_b \{ \phi(x, \lambda), \psi(x, \lambda') \} \\ & + \{l(\lambda') - m(\lambda')\} W_b \{ \psi(x, \lambda), \phi(x, \lambda') \} \\ & + \{l(\lambda) - m(\lambda)\} \{l(\lambda') - m(\lambda')\} W_b \{ \phi(x, \lambda), \phi(x, \lambda') \} \\ & = 0. \end{aligned}$$

Now

$$\begin{aligned} W_b \{ \phi(x, \lambda), \psi(x, \lambda') \} &= (\lambda - \lambda') \int_0^b \phi(x, \lambda) \psi(x, \lambda') d_q x + W_0 \{ \phi(x, \lambda), \psi(x, \lambda') \} \\ &= O \left\{ \int_0^b |\phi(x, \lambda)|^2 d_q x \right\}^{\frac{1}{2}} + O(1), \end{aligned}$$

as $b \rightarrow \infty$, λ and λ' being fixed. In the limit-point case

$$|l(\lambda) - m(\lambda)| \leq 2r_b = \left\{ \nu \int_0^b |\phi(x, \lambda)|^2 d_q x \right\}^{-1},$$

so that

$$\lim_{b \rightarrow \infty} \{l(\lambda) - m(\lambda)\} W_b \{\phi(x, \lambda), \psi(x, \lambda')\} = 0.$$

This also holds in the limit-circle case, if $l(\lambda) \rightarrow m(\lambda)$, since then $\int_0^b |\phi(x, \lambda)|^2 d_q x$ is bounded. Similar arguments apply to the other terms. ■

Lemma 2 *Let $\{f_n\}$ be a sequence of functions which converges in mean square to f over any finite interval, while*

$$\int_0^\infty |f_n(x)|^2 d_q x \leq K$$

for all n . Then f is $L^2(\mathbb{R}_q^+)$, and if g belongs to $L^2(\mathbb{R}_q^+)$,

$$\lim_{n \rightarrow \infty} \int_0^\infty f_n(x)g(x)d_q x = \int_0^\infty f(x)g(x)d_q x.$$

Proof. We have

$$\int_0^X |f(x)|^2 d_q x = \lim_{n \rightarrow \infty} \int_0^X |f_n(x)|^2 d_q x \leq K$$

for every X , so that f is $L^2(\mathbb{R}_q^+)$. Now

$$\begin{aligned} \left| \int_0^\infty (f - f_n)g d_q x \right| &\leq \left| \int_0^X (f - f_n)g d_q x \right| + \left| \int_X^\infty (f - f_n)g d_q x \right| \\ &\leq \left\{ \int_0^X |f - f_n|^2 d_q x \int_0^\infty |g|^2 d_q x \right\}^{\frac{1}{2}} + \left\{ \int_0^\infty |f - f_n|^2 d_q x \int_X^\infty |g|^2 d_q x \right\}^{\frac{1}{2}}. \end{aligned}$$

This finish the proof. ■

Proposition 1 *The functions*

$$\psi_n(x) = \sqrt{r_n} \phi(x, \lambda_n)$$

form a normal orthogonal set of $L^2(\mathbb{R}_q^+)$.

Proof. By (2) if λ and λ' are not real, we have

$$\begin{aligned} (\lambda' - \lambda) \int_0^b \psi(x, \lambda) \psi(x, \lambda') d_q x &= W_0 \{\psi(x, \lambda), \psi(x, \lambda')\} - W_b \{\psi(x, \lambda), \psi(x, \lambda')\} \\ &= m(\lambda) - m(\lambda') - W_b \{\psi(x, \lambda), \psi(x, \lambda')\}. \end{aligned}$$

By Lemma 1, the second term in the right tends to zero as $b \rightarrow \infty$. Hence

$$\int_0^\infty \psi(x, \lambda)\psi(x, \lambda')d_qx = \frac{m(\lambda) - m(\lambda')}{\lambda' - \lambda}. \quad (4)$$

In particular, taking $\lambda' = \bar{\lambda}$, we obtain

$$\int_0^\infty |\psi(x, \lambda)|^2 d_qx = -\frac{\text{Im}\{m(\lambda)\}}{\nu}.$$

Now let λ_n be an eigenvalue, and let $\lambda' = \lambda_n + i\nu$, $\nu \rightarrow 0$. Then for any fixed X ,

$$\begin{aligned} & \int_0^X |\nu\psi(x, \lambda') + ir_n\phi(x, \lambda_n)|^2 d_qx \\ &= \int_0^X |\nu\theta(x, \lambda') + \{\nu m(\lambda') + ir_n\}\phi(x, \lambda') - ir_n\{\phi(x, \lambda') - \phi(x, \lambda_n)\}|^2 d_qx \rightarrow 0. \end{aligned}$$

Also, by

$$\int_0^\infty |\nu\psi(x, \lambda')|^2 d_qx \leq |\nu m(\lambda')| = O(1)$$

as $\nu \rightarrow 0$, since the pole of $m(\lambda')$ at λ_n is simple. On multiplying (4) by $i\nu/r_m$, making $\nu \rightarrow 0$, and using Lemma 2 we see that $\phi(x, \lambda_n)$ is $L^2(\mathbb{R}_q^+)$, and

$$\int_0^\infty \psi(x, \lambda)\phi(x, \lambda_n)d_qx = \frac{1}{\lambda - \lambda_n}. \quad (5)$$

If λ tends to a different eigenvalue λ_m , on multiplying (5) by $i\nu/r_m$ and making $\nu \rightarrow 0$, we obtain

$$\int_0^\infty \phi(x, \lambda_m)\phi(x, \lambda_n)d_qx = 0.$$

If λ tends to the same eigenvalue λ_n , it follows similarly that

$$\int_0^\infty \{\phi(x, \lambda_n)\}^2 d_qx = \frac{1}{r_n},$$

which leads to the result. ■

5 Series expansions

Let f be $L^2(\mathbb{R}_q^+)$ and let

$$\Phi(x, \lambda) = \psi(x, \lambda) \int_0^x \phi(y, \lambda) f(y) d_q y + \phi(x, \lambda) \int_x^\infty \psi(y, \lambda) f(y) d_q y,$$

where ϕ and ψ are the functions defined above. In the following we denote by c_n the n th Fourier coefficient of the function f

$$c_n = \int_0^\infty \psi_n(y) f(y) d_q y.$$

Proposition 2 *The function*

$$\lambda \mapsto \Phi(x, \lambda)$$

has a simple pole at λ_n , its residue is $c_n \psi_n(x)$.

Proof. We have

$$D_q \Phi(x, \lambda) = D_q \psi(x, \lambda) \int_0^{qx} \phi(y, \lambda) f(y) d_q y + D_q \phi(x, \lambda) \int_{qx}^\infty \psi(y, \lambda) f(y) d_q y$$

and

$$\begin{aligned} D_q^2 \Phi(x, \lambda) &= D_q^2 \psi(x, \lambda) \int_0^{q^2 x} \phi(y, \lambda) f(y) d_q y + D_q^2 \phi(x, \lambda) \int_{q^2 x}^\infty \psi(y, \lambda) f(y) d_q y \\ &\quad + q D_q \psi(x, \lambda) \phi(qx, \lambda) f(qx) - q D_q \phi(x, \lambda) \psi(qx, \lambda) f(qx). \end{aligned}$$

Therefore

$$\begin{aligned} \Delta_q \Phi(x, \lambda) &= \left[\frac{1-q}{q} \right]^2 \Lambda_q^{-1} D_q^2 \Phi(x, \lambda) \\ &= \Delta_q \psi(x, \lambda) \int_0^{qx} \phi(y, \lambda) f(y) d_q y + \Delta_q \phi(x, \lambda) \int_{qx}^\infty \psi(y, \lambda) f(y) d_q y \\ &\quad + \frac{(1-q)^2}{q} [\Lambda_q^{-1} D_q \psi(x, \lambda) \phi(x, \lambda) - \Lambda_q^{-1} D_q \phi(x, \lambda) \psi(x, \lambda)] f(x) \\ &= [q(x) - \lambda] \Phi(x, \lambda) + W_x(\theta, \phi) f(x) = [q(x) - \lambda] \Phi(x, \lambda) + f(x). \end{aligned}$$

The function Φ satisfies the boundary condition

$$\Phi(0, \lambda) \cos \alpha + \Phi'(0, \lambda) \sin \alpha = 0. \quad (6)$$

If $\Phi_X(x, \lambda)$ is the corresponding function with $f(y) = 0$ for $y > X$, then

$$\begin{aligned}\Phi_X(x, \lambda) &= \theta(x, \lambda) \int_0^x \phi(y, \lambda) f(y) d_q y + \phi(x, \lambda) \int_x^X \phi(y, \lambda) f(y) d_q y \\ &\quad + m(\lambda) \phi(x, \lambda) \int_0^X \phi(y, \lambda) f(y) d_q y.\end{aligned}$$

This is clearly regular everywhere except at $\lambda = \lambda_0, \lambda_1, \dots$, where it has simple poles with residues

$$r_n \phi(x, \lambda_n) \int_0^X \phi(y, \lambda_n) f(y) d_q y.$$

Hence making $X \rightarrow \infty$ we find that $\Phi(x, \lambda)$ has a simple pole at λ_n , its residue there being limit of the residue of $\Phi_X(x, \lambda)$, i.e.

$$r_n \phi(x, \lambda_n) \int_0^\infty \phi(y, \lambda_n) f(y) d_q y = \psi_n(x) \int_0^\infty \psi_n(y) f(y) d_q y = c_n \psi_n(x).$$

This finish the proof. ■

Lemma 3 *If f is any function of $L^2(\mathbb{R}_q^+)$ then*

$$\int_0^\infty |\Phi(x, \lambda)|^2 d_q x \leq \frac{1}{\nu^2} \int_0^\infty |f(x)|^2 d_q x.$$

Proof. Suppose first that $f(x) = 0$ for $x \geq X$. Then the condition of self-adjointness

$$\int_0^\infty \Phi(x, \lambda) L\Phi(x, \lambda') d_q x = \int_0^\infty \Phi(x, \lambda') L\Phi(x, \lambda) d_q x,$$

is satisfied. Indeed

$$\begin{aligned}& \int_0^\xi \{ \Phi(x, \lambda) L\Phi(x, \lambda') - \Phi(x, \lambda') L\Phi(x, \lambda) \} d_q x \\ &= - \int_0^\xi \{ \Phi(x, \lambda) \Delta_q \Phi(x, \lambda') - \Phi(x, \lambda') \Delta_q \Phi(x, \lambda) \} d_q x \\ &= -W[\Phi(x, \lambda), \Phi(x, \lambda')]_0^\xi.\end{aligned}$$

By (6) the integrated term vanishes at $x = 0$. The integrated term at $x = \xi$ tend to 0 as $\xi \rightarrow \infty$. Since, if $x > X$ we have

$$\Phi(x, \lambda) = \psi(x, \lambda) \int_0^X \phi(y, \lambda) f(y) d_q y,$$

then the result follows from Lemma 1.

Putting $\lambda' = \bar{\lambda}$ we obtain

$$\int_0^\infty \Phi(x, \lambda) \{\bar{\lambda} \Phi(x, \bar{\lambda}) - f(x)\} d_q x = \int_0^\infty \Phi(x, \bar{\lambda}) \{\lambda \Phi(x, \lambda) - f(x)\} d_q x,$$

i.e.

$$(\lambda - \bar{\lambda}) \int_0^\infty |\Phi(x, \lambda)|^2 d_q x = \int_0^\infty \{\Phi(x, \bar{\lambda}) - \Phi(x, \lambda)\} f(x) d_q x.$$

Hence, if $\lambda = \mu + i\nu, \nu > 0$,

$$2\nu \int_0^\infty |\Phi(x, \lambda)|^2 d_q x \leq 2 \int_0^\infty |\Phi(x, \lambda) f(x)| d_q x \leq 2 \left\{ \int_0^\infty |\Phi(x, \lambda)|^2 d_q x \int_0^\infty |f(x)|^2 d_q x \right\}.$$

This prove the result in the restricted case. If now f is any function of $L^2(\mathbb{R}_q^+)$, for a fixed X' we have

$$\begin{aligned} \int_0^{X'} |\Phi_X(x, \lambda)|^2 d_q x &\leq \int_0^\infty |\Phi_X(x, \lambda)|^2 d_q x \\ &\leq \frac{1}{\nu^2} \int_0^\infty |f_X(x)|^2 d_q x = \frac{1}{\nu^2} \int_0^X |f(x)|^2 d_q x \leq \frac{1}{\nu^2} \int_0^\infty |f(x)|^2 d_q x. \end{aligned}$$

The result therefore follows on making first $X \rightarrow \infty$, then $X' \rightarrow \infty$. ■

We denote by $\mathcal{L}_{q,2}$ the subspace of $L^2(\mathbb{R}_q^+)$ of functions which satisfy

- The function Lf be $L^2(\mathbb{R}_q^+)$.
- $f'(0) \cos \alpha - f(0) \sin \alpha = 0$.
- $\lim_{x \rightarrow \infty} W\{\psi(x, \lambda), f(x)\} = 0$, for every non-real λ .

Lemma 4 *If f belongs to $\mathcal{L}_{q,2}$ and $\Phi(x, \lambda)$ defined above is also denoted by $\Phi(x, \lambda, f)$ then*

$$\Phi(x, \lambda, f) = \frac{1}{\lambda} \{f(x) + \Phi(x, \lambda, Lf)\}. \quad (7)$$

Proof. We have

$$\begin{aligned}\int_0^x \phi(y, \lambda) f(y) d_q y &= \frac{1}{\lambda} \int_0^x \{q(y)\phi(y) - \Delta_q \phi(y)\} f(y) d_q y \\ &= \frac{1}{\lambda} W[\phi(y), f(y)]_0^x + \frac{1}{\lambda} \int_0^x \{q(y)f(y) - \Delta_q f(y)\} \phi(y) d_q y.\end{aligned}$$

Similarly

$$\begin{aligned}\int_x^\infty \phi(y, \lambda) f(y) d_q y &= \frac{1}{\lambda} \int_x^\infty \{q(y)\phi(y) - \Delta_q \phi(y)\} f(y) d_q y \\ &= \frac{1}{\lambda} W[\psi(y), f(y)]_x^\infty + \frac{1}{\lambda} \int_x^\infty \{q(y)f(y) - \Delta_q f(y)\} \psi(y) d_q y,\end{aligned}$$

and the integrated term vanishes at the upper limit. ■

The Green's function $G(x, y, \lambda)$ is defined by

$$G(x, y, \lambda) = \begin{cases} -\psi(x, \lambda)\phi(y, \lambda), & \text{if } y \leq x \\ -\psi(y, \lambda)\phi(x, \lambda), & \text{if } y > x \end{cases},$$

then

$$\Phi(x, \lambda, f) = - \int_0^\infty G(x, y, \lambda) f(y) d_q y.$$

Therefore

$$f(x) = \int_0^\infty G(x, y, \lambda) \{Lf(y) - \lambda f(y)\} d_q y. \quad (8)$$

Lemma 5 *Let $F(\lambda)$ be an analytic function of $\lambda = \mu + i\nu$, regular for $-r \leq \mu \leq r, -r \leq \nu \leq r$, and let*

$$|F(\lambda)| \leq \frac{M}{|\nu|},$$

in this square. Then

$$|F(\lambda)| \leq \frac{3M}{r}.$$

Proof. Let

$$G(\lambda) = (\lambda^2 - r^2)F(\lambda).$$

On the upper and lower sides of the square

$$|G(\lambda)| \leq (|\lambda|^2 + r^2) \frac{M}{r} \leq 3rM.$$

On the left-hand and right-hand sides

$$|G(\lambda)| \leq |\nu|(|\lambda| + r) \frac{M}{|\nu|} \leq 3rM.$$

Hence $|G(\lambda)| \leq 3rM$ throughout the square. Hence on the imaginary axis

$$|F(\lambda)| \leq \frac{3rM}{|\lambda^2 - r^2|} = \frac{3rM}{\nu^2 + r^2} \leq \frac{3M}{r}.$$

This finish the proof. ■

Theorem 2 *Let f be a function belonging in $\mathcal{L}_{q,2}$ then*

$$\int_0^\infty \{f(x)\}^2 d_q x = \sum_{n=0}^\infty c_n^2,$$

Proof. Suppose that f satisfies the conditions of the above theorem, and also that $f(x) = 0$ for sufficient large values of x . Let

$$\Psi(\lambda) = \int_0^\infty f(x)\Phi(x, \lambda) d_q x,$$

then $\Psi(\lambda)$ is regular except for simple poles at the points λ_n , where it has residues

$$c_n \int_0^\infty \psi_n(x) f(x) d_q x = c_n^2.$$

By (7)

$$\Psi(\lambda) = \frac{1}{\lambda} \int_0^\infty \{f(x)\}^2 d_q x + \frac{1}{\lambda} \int_0^\infty \Phi(x, \lambda, Lf) f(x) d_q x, \quad (9)$$

and the last term is

$$O \left\{ \frac{1}{|\lambda|} \left[\int_0^\infty |\Phi(x, \lambda, Lf)|^2 d_q x \int_0^\infty \{f(x)\}^2 d_q x \right]^{\frac{1}{2}} \right\} = O \left(\frac{1}{|\lambda\nu|} \right),$$

by Lemma 3, applied to Lf .

Let $C(R)$ denote the contour formed by the segments of lines $(R-i, R+i)$ and $(-R-i, -R+i)$, joined by semicircles of radius R and centres $\pm i$. Then

$$\int_{C(R)} \Psi(\lambda) d\lambda = \sum_{-R < \lambda_n < R} c_n^2.$$

On the part of the upper semicircle in the first quadrant, we have

$$\lambda = i + R e^{i\theta} \quad (0 \leq \theta \leq \frac{\pi}{2}).$$

Hence the last term in (9), integrated round this quadrant, gives

$$O \left\{ \int_0^{\frac{\pi}{2}} \frac{R}{R(1 + R \sin \theta)} d\theta \right\} = O \left\{ \int_0^{\frac{1}{R}} d\theta \right\} + O \left\{ \int_{\frac{1}{R}}^{\frac{\pi}{2}} \frac{d\theta}{R\theta} \right\} = O \left(\frac{1}{R} \right) + O \left(\frac{\log R}{R} \right) = o(1).$$

A similar argument applies to the other quadrants. Hence the integral of $\Psi(\lambda)$ round each semicircle tends to

$$\pi i \int_0^\infty \{f(x)\}^2 dx$$

as $R \rightarrow \infty$. To prove the theorem for the class of functions considered, it is therefore sufficient to prove that

$$\lim_{R \rightarrow \infty} \int_{R-i}^{R+i} \Psi(\lambda) d\lambda = 0,$$

and a similar result with $-R$ in place of R .

Let

$$\chi(\lambda) = \Psi(\lambda) - \sum_{R-1 \leq \lambda_n \leq R+1} \frac{c_n^2}{\lambda - \lambda_n}.$$

Then $\chi(\lambda)$ is regular for $R-1 \leq \lambda \leq R+1$, and

$$|\chi(\lambda)| < \frac{K}{|\lambda\nu|} + \frac{1}{\nu} \sum_{R-1 \leq \lambda_n \leq R+1} c_n^2 \leq \frac{\epsilon(R)}{|\nu|},$$

where $\epsilon(R) \rightarrow 0$ as $R \rightarrow \infty$. Hence, by Lemma 5

$$|\chi(\lambda)| \leq 3\epsilon(R)$$

on the segment $(R-i, R+i)$. Hence

$$\lim_{R \rightarrow \infty} \int_{R-i}^{R+i} \chi(\lambda) d\lambda = 0.$$

Also

$$\int_{R-i}^{R+i} \frac{d\lambda}{\lambda - \lambda_n} d\lambda = O(1).$$

Hence

$$\int_{R-i}^{R+i} \sum_{R-1 \leq \lambda_n \leq R+1} \frac{c_n^2}{\lambda - \lambda_n} d\lambda = O \left\{ \sum_{R-1 \leq \lambda_n \leq R+1} c_n^2 \right\} = o(1).$$

This prove the theorem for the special class of functions. The theorem can be extended to all functions of integrable square. ■

Remark 2 *It also follows that, if g is another function of $L^2(\mathbb{R}_q^+)$, with Fourier coefficients d_n , then*

$$\int_0^\infty f(x)g(x)d_q x = \sum_{n=0}^\infty c_n d_n. \quad (10)$$

Theorem 3 *Let f be a function belonging in $\mathcal{L}_{q,2}$ then*

$$f(x) = \sum_{n=0}^\infty c_n \psi_n(x), \quad \forall x \in \mathbb{R}_q^+.$$

Proof. From the formula

$$(\lambda_n - \lambda) \int_0^b \psi(x, \lambda) \phi(x, \lambda_n) d_q x = W_0\{\psi(x, \lambda), \phi(x, \lambda_n)\} - W_b\{\psi(x, \lambda), \phi(x, \lambda_n)\},$$

and (5), it follows that the second term on the right-hand side tends to 0 as $b \rightarrow \infty$. Hence we may take $f(x) = \psi_n(x)$ in (8), which gives

$$\int_0^\infty G(x, y, \lambda) \psi_n(x) d_q y = \frac{\psi_n(x)}{\lambda_n - \lambda}.$$

Also

$$\begin{aligned} & \int_0^\infty |G(x, y, \lambda)|^2 d_q y \\ &= |\psi(x, \lambda)|^2 \int_0^x |\phi(x, \lambda)|^2 d_q y + |\phi(x, \lambda)|^2 \int_x^\infty |\psi(x, \lambda)|^2 d_q y \leq K, \end{aligned}$$

say, for x in a finite interval, and λ not real. Hence by the Bessel inequality

$$\sum_{n=0}^\infty \left| \frac{\psi_n(x)}{\lambda_n - \lambda} \right|^2 \leq K. \quad (11)$$

Now let f satisfies the conditions of Theorem 3, and let

$$g(x) = Lf(x) - \lambda f(x).$$

If λ is not real,

$$\int_0^\infty [\psi(y, \lambda)Lf(y) - f(y)L\{\psi(y, \lambda)\}]d_q y = W[f, \psi]_0^\infty = f'(0) \cos \alpha - f(0) \sin \alpha = 0,$$

and $L\psi = \lambda\psi$. Put $\lambda = \lambda_n + i\nu$, multiply by ν , and make $\nu \rightarrow 0$. Using Lemma 3, we obtain

$$\int_0^\infty \psi_n(y)Lf(y)d_q y = \lambda_n \int_0^\infty \psi_n(y)f(y)d_q y = \lambda_n c_n.$$

Hence, if d_n is the Fourier coefficient of $g(x)$,

$$d_n = (\lambda_n - \lambda)c_n,$$

then

$$\sum_{n=0}^{\infty} |\lambda_n - \lambda|^2 c_n^2, \quad (12)$$

is convergent. From (8) and (10) with $f(y)$ replaced by $G(x, y, \lambda)$, it follows that

$$f(x) = \int_0^\infty G(x, y, \lambda)g(y)d_q y = \sum_{n=0}^{\infty} \frac{\psi_n(x)}{\lambda_n - \lambda} (\lambda_n - \lambda)c_n = \sum_{n=0}^{\infty} c_n \psi_n(x),$$

the required result. The absolute and uniform convergence of the series follows from (11) and the convergence of (12). ■

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