



**HAL**  
open science

# **q-Sturm-Liouville theory and the corresponding eigenfunction expansions**

Lazhar Dhaouadi

► **To cite this version:**

Lazhar Dhaouadi. q-Sturm-Liouville theory and the corresponding eigenfunction expansions. 2007.  
hal-00163435v1

**HAL Id: hal-00163435**

**<https://hal.science/hal-00163435v1>**

Preprint submitted on 17 Jul 2007 (v1), last revised 16 Jul 2008 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# q-Sturm-Liouville theory and the corresponding eigenfunction expansions

Lazhar Dhaouadi\*

July 18, 2007

## Abstract

The eigenfunction expansion theory for the q-Sturm-Liouville equation (singular case) presented in this paper is based on the original works of Hermann Weyl in 1910 and of Edward Charles Titchmarsh in 1941, concerning Sturm-Liouville theory and the corresponding eigenfunction expansions. For this account the essential results of Weyl concern the regular, limit-circle and limit-point classifications of Sturm-Liouville differential equations (singular case); the eigenfunction expansion theory from Titchmarsh is based on classical function theory methods, in particular complex function theory.

## 1 Basic definitions

Consider  $0 < q < 1$ . In what follows, the standard conventional notations from [1], will be used

$$\mathbb{R}_q^+ = \{q^n, \quad n \in \mathbb{Z}\} \cup \{0\}$$

$$(a, q)_0 = 1, \quad (a, q)_n = \prod_{i=0}^{n-1} (1 - aq^i),$$

The  $q$ -shift operator is

$$\Lambda_q f(x) = f(qx).$$

---

\*Ecole Préparatoire d'Ingénieur, Bizerte, Tunisia. E-mail lazhardhaouadi@yahoo.fr

Next we introduce two concepts of  $q$ -analysis: the  $q$ -derivative and the  $q$ -integral. The  $q$ -derivative of a function  $f$  is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x},$$

and  $D_q f(0) = f'(0)$ , provided  $f'(0)$  exists. The second order  $q$ -difference operator is

$$\Delta_q f(x) = \left[ \frac{1-q}{q} \right]^2 \Lambda_q^{-1} D_q^2 f(x) = \frac{1}{x^2} \left[ f(q^{-1}x) - \frac{1+q}{q} f(x) + \frac{1}{q} f(qx) \right].$$

The product rule for the  $q$ -derivative is

$$D_q(fg)(x) = D_q f(x)g(x) + \Lambda_q f(x)D_q g(x).$$

Jackson's  $q$ -integral (see [3]) in the interval  $[a, b]$  and in the interval  $[0, \infty[$  are defined by

$$\int_a^b f(x) d_q x = (1-q) \sum_{n=0}^{\infty} q^n [bf(bq^n) - af(aq^n)].$$

$$\int_0^{\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} q^n f(q^n).$$

Also the rule of  $q$ -integration by parts is given by

$$\int_a^b D_q f(x)g(x) d_q x = [f(b)g(b) - f(a)g(a)] - \int_a^b \Lambda_q f(x)D_q g(x) d_q x.$$

## 2 $q$ -Sturm-Liouville difference equations

The object of this paper is to study the  $q$ -Schrödinger operator

$$L \equiv q(x) - \Delta_q,$$

where  $q(x)$  is a given function of  $x$  defined over  $\mathbb{R}_q^+$ . For more information on the classical theory, the reader can consult the references [2, 4, 5, 6].

If  $F(x)$  and  $G(x)$  satisfy the  $q$ -difference equation

$$\Delta_q f(x) + [\lambda - q(x)]f(x) = 0, \quad (1)$$

and the corresponding equation with  $\lambda'$  instead of  $\lambda$ , respectively, then

$$\begin{aligned} & (\lambda' - \lambda) \int_0^b F(x)G(x)d_q x \\ &= \int_0^b [F(x) \{q(x)G(x) - \Delta_q G(x)\} - G(x) \{q(x)F(x) - \Delta_q F(x)\}]d_q x \\ &= - \int_0^b \{F(x)\Delta_q G(x) - G(x)\Delta_q F(x)\} d_q x = W_0(F, G) - W_b(F, G). \end{aligned}$$

where  $W_x$  is the  $q$ -Wronskien defined by

$$W_x(F, G) = \frac{(1-q)^2}{q} [F(x)\Lambda_q^{-1}D_q G(x) - G(x)\Lambda_q^{-1}D_q F(x)].$$

If  $\lambda = \mu + i\nu$ ,  $\lambda' = \bar{\lambda}$  and  $G = \bar{F}$ , this gives

$$2\nu \int_0^b |F(x)|^2 d_q x = iW_0(F, \bar{F}) - iW_b(F, \bar{F}). \quad (2)$$

Now let  $\phi(x) = \phi(x, \lambda)$  and  $\theta(x) = \theta(x, \lambda)$  be two solutions of (1) such that

$$\begin{cases} \phi(0) = \sin \alpha, & \phi'(0) = -\cos \alpha \\ \theta(0) = \cos \alpha, & \theta'(0) = \sin \alpha \end{cases},$$

where  $\alpha$  is real. Then

$$W_x(\phi, \theta) = W_0(\phi, \theta) = \sin^2 \alpha + \cos^2 \alpha = 1.$$

and

$$W_0(\phi, \bar{\phi}) = W_0(\theta, \bar{\theta}) = 0.$$

Therefore

$$W_0(\theta + l\phi, \bar{\theta} + \bar{l}\bar{\phi}) = l - \bar{l} = 2i \operatorname{Im} l.$$

The general solution of (1) is of the form  $\theta(x) + l\phi(x)$ . Consider those solutions which satisfy a real boundary condition at  $x = b$ , say

$$\{\theta(b) + l\phi(b)\} \cos \beta + \{\Lambda_q^{-1}D_q \theta(b) + l\Lambda_q^{-1}D_q \phi(b)\} \sin \beta = 0,$$

where  $\beta$  is real. This gives

$$l = l(\lambda) = \frac{\theta(b) \cot \beta + \Lambda_q^{-1} D_q \theta(b)}{\phi(b) \cot \beta + \Lambda_q^{-1} D_q \phi(b)}.$$

For each  $b$ , as  $\cot \beta$  varies,  $l$  describes a circle in the complex plane, say  $C_b$ . Replacing  $\cot \beta$  by a complex variable  $z$ , we obtain

$$l = l(\lambda) = -\frac{\theta(b)z + \Lambda_q^{-1} D_q \theta(b)}{\phi(b)z + \Lambda_q^{-1} D_q \phi(b)}.$$

The centre of  $C_b$  correspond to

$$z = -\frac{\Lambda_q^{-1} D_q \bar{\phi}(b)}{\bar{\phi}(b)} = -\frac{W_b(\theta, \bar{\phi})}{W_b(\phi, \bar{\phi})}.$$

Since  $-\Lambda_q^{-1} D_q \theta(b) / \Lambda_q^{-1} D_q \phi(b)$  is on  $C_b$  (for  $z = 0$ ) the radius  $r_b$  of  $C_b$  is

$$r_b = \left| \frac{\Lambda_q^{-1} D_q \theta(b)}{\Lambda_q^{-1} D_q \phi(b)} - \frac{W_b(\theta, \bar{\phi})}{W_b(\phi, \bar{\phi})} \right| = \left| \frac{W_b(\theta, \phi)}{W_b(\phi, \bar{\phi})} \right| = \frac{1}{2\nu \int_0^b |\phi(x)|^2 d_q x}. \quad (3)$$

Now  $l$  is inside  $C_b$  if  $\text{Im } z < 0$ , i.e. if

$$i \left\{ -\frac{l \Lambda_q^{-1} D_q \phi(b) + \Lambda_q^{-1} D_q \theta(b)}{l \phi(b) + \theta(b)} + \frac{\bar{l} \Lambda_q^{-1} D_q \bar{\phi}(b) + \Lambda_q^{-1} D_q \bar{\theta}(b)}{\bar{l} \bar{\phi}(b) + \bar{\theta}(b)} \right\} > 0,$$

i.e. if

$$i \{ |l|^2 W_b(\phi, \bar{\phi}) + l W_b(\phi, \bar{\theta}) + \bar{l} W_b(\theta, \bar{\phi}) + W_b(\theta, \bar{\theta}) \} > 0,$$

i.e. if

$$i W_b(\theta + l\phi, \bar{\theta} + \bar{l}\bar{\phi}) > 0,$$

i.e. if

$$2\nu \int_0^b |\theta + l\phi|^2 d_q x < i W_b(\theta + l\phi, \bar{\theta} + \bar{l}\bar{\phi}).$$

Hence  $l$  is interior to  $C_b$  if  $\nu > 0$ , and

$$\int_0^b |\theta + l\phi|^2 d_q x < -\frac{\text{Im } l}{\nu}.$$

The same result is obtained if  $\nu < 0$ . It follows that, if  $l$  is interior to  $C_b$ , and  $0 < b' < b$ , then

$$\int_0^{b'} |\theta + l\phi|^2 d_q x < \int_0^b |\theta + l\phi|^2 d_q x < -\frac{\operatorname{Im} l}{\nu}.$$

Hence  $l$  is also inside  $C_{b'}$ . Hence  $C_{b'}$  includes  $C_b$  if  $b' < b$ . It follows that, as  $b \rightarrow \infty$ , the circle  $C_b$  converge either to a limit-circle or to a limit-point.

If  $m = m(\lambda)$  is the limit-point, or any point on the limit-circle,

$$\int_0^b |\theta + m\phi|^2 d_q x < -\frac{\operatorname{Im} m}{\nu}$$

for all values of  $b$ . Hence

$$\int_0^\infty |\theta + m\phi|^2 d_q x < -\frac{\operatorname{Im} m}{\nu}.$$

It follows that

**Theorem 1** *For every value of  $\lambda$  other than real values, (1) has a solution*

$$\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\phi(x, \lambda),$$

*belonging to  $L^2(\mathbb{R}_q^+)$ .*

In the limit-circle case,  $r_b$  tends to a positive limit as  $b \rightarrow \infty$ ; hence, by (3), the function  $\phi(x)$  is  $L^2(\mathbb{R}_q^+)$ ; so in fact, in this case, every solution of (1) belongs to  $L^2(\mathbb{R}_q^+)$ .

### 3 The eigenfunctions

On the circle  $C_b$  (if  $\nu > 0$ )

$$\frac{1}{2} |l|^2 \int_0^b |\phi|^2 d_q x - \int_0^b |\theta|^2 d_q x \leq \int_0^b |\theta + l\phi|^2 d_q x < -\frac{\operatorname{Im} l}{\nu} \leq \frac{|l|}{\nu}.$$

Solving for  $|l|$  we obtain

$$|l| \leq \frac{1}{\nu \int_0^b |\phi|^2 d_q x} + \left\{ \frac{2 \int_0^b |\theta|^2 d_q x}{\int_0^b |\phi|^2 d_q x} + \frac{1}{\left( \nu \int_0^b |\phi|^2 d_q x \right)^2} \right\}^{\frac{1}{2}}.$$

Since the above right-hand side is  $O(\frac{1}{\nu})$ . Hence as  $\nu \rightarrow 0$ , for any fixed  $b$ , it also follows that  $m(\lambda) = O(\frac{1}{\nu})$ . Hence, if  $m(\lambda)$  has poles on the real axis, they are all simple. In this paper we assume that  $m(\lambda)$  form a single analytic function, whose only singularities are poles on the real axis. Let them be  $\lambda_0, \lambda_1, \dots$ , and let the residues be  $r_0, r_1, \dots$

**Lemma 1** *For any fixed complex  $\lambda$  and  $\lambda'$*

$$\lim_{x \rightarrow \infty} W_x \{ \psi(\cdot, \lambda), \psi(\cdot, \lambda') \} = 0.$$

Since

$$W_b \{ \theta(x, \lambda) + l(\lambda) \phi(x, \lambda), \theta(x, \lambda') + l(\lambda') \phi(x, \lambda') \} = 0,$$

i.e.

$$W_b [ \psi(x, \lambda) + \{ l(\lambda) - m(\lambda) \} \phi(x, \lambda), \psi(x, \lambda') + \{ l(\lambda') - m(\lambda') \} \phi(x, \lambda') ] = 0,$$

i.e.

$$\begin{aligned} & W_b \{ \psi(x, \lambda), \psi(x, \lambda') \} + \{ l(\lambda) - m(\lambda) \} W_b \{ \phi(x, \lambda), \psi(x, \lambda') \} \\ & + \{ l(\lambda') - m(\lambda') \} W_b \{ \psi(x, \lambda), \phi(x, \lambda') \} \\ & + \{ l(\lambda) - m(\lambda) \} \{ l(\lambda') - m(\lambda') \} W_b \{ \phi(x, \lambda), \phi(x, \lambda') \} \\ & = 0. \end{aligned}$$

Now

$$\begin{aligned} W_b \{ \phi(x, \lambda), \psi(x, \lambda') \} &= (\lambda - \lambda') \int_0^b \phi(x, \lambda) \psi(x, \lambda') d_q x + W_0 \{ \phi(x, \lambda), \psi(x, \lambda') \} \\ &= O \left\{ \int_0^b |\phi(x, \lambda)|^2 d_q x \right\}^{\frac{1}{2}} + O(1), \end{aligned}$$

as  $b \rightarrow \infty$ ,  $\lambda$  and  $\lambda'$  being fixed. In the limit-point case

$$|l(\lambda) - m(\lambda)| \leq 2r_b = \left\{ \nu \int_0^b |\phi(x, \lambda)|^2 d_q x \right\}^{-1},$$

so that

$$\lim_{b \rightarrow \infty} \{l(\lambda) - m(\lambda)\} W_b \{\phi(x, \lambda), \psi(x, \lambda')\} = 0.$$

This also holds in the limit-circle case, if  $l(\lambda) \rightarrow m(\lambda)$ , since then  $\int_0^b |\phi(x, \lambda)|^2 d_q x$  is bounded. Similar arguments apply to the other terms.

**Lemma 2** *Let  $f_n(x)$  be a sequence of functions which converges in mean square to  $f(x)$  over any finite interval, while*

$$\int_0^\infty |f_n(x)|^2 d_q x \leq K$$

for all  $n$ . Then  $f(x)$  is  $L^2(\mathbb{R}_q^+)$ , and if  $g(x)$  belongs to  $L^2(\mathbb{R}_q^+)$ ,

$$\lim_{n \rightarrow \infty} \int_0^\infty f_n(x) g(x) d_q x = \int_0^\infty f(x) g(x) d_q x.$$

We have

$$\int_0^X |f(x)|^2 d_q x = \lim_{n \rightarrow \infty} \int_0^X |f_n(x)|^2 d_q x \leq K$$

for every  $X$ , so that  $f(x)$  is  $L^2(\mathbb{R}_q^+)$ . Now

$$\begin{aligned} \left| \int_0^\infty (f - f_n) g d_q x \right| &\leq \left| \int_0^X \right| + \left| \int_X^\infty \right| \\ &\leq \left\{ \int_0^X |f - f_n|^2 d_q x \int_0^\infty |g|^2 d_q x \right\}^{\frac{1}{2}} + \left\{ \int_0^\infty |f - f_n|^2 d_q x \int_X^\infty |g|^2 d_q x \right\}^{\frac{1}{2}}. \end{aligned}$$

This finish the proof.

Now by (2) if  $\lambda$  and  $\lambda'$  are not real, we have

$$\begin{aligned} (\lambda' - \lambda) \int_0^b \psi(x, \lambda) \psi(x, \lambda') d_q x &= W_0 \{\psi(x, \lambda), \psi(x, \lambda')\} - W_b \{\psi(x, \lambda), \psi(x, \lambda')\} \\ &= m(\lambda) - m(\lambda') - W_b \{\psi(x, \lambda), \psi(x, \lambda')\}, \end{aligned}$$



the second term in the right tends to zero as  $b \rightarrow \infty$ , by Lemma 2. Hence

$$\int_0^\infty \psi(x, \lambda)\psi(x, \lambda')d_qx = \frac{m(\lambda) - m(\lambda')}{\lambda' - \lambda}. \quad (4)$$

In particular, taking  $\lambda' = \bar{\lambda}$ , we obtain

$$\int_0^\infty |\psi(x, \lambda)|^2 d_qx = -\frac{\text{Im} \{m(\lambda)\}}{\nu}.$$

Now let  $\lambda_n$  be an eigenvalue, and let  $\lambda' = \lambda_n + i\nu$ ,  $\nu \rightarrow 0$ . Then for any fixed  $X$ ,

$$\begin{aligned} & \int_0^X |\nu\psi(x, \lambda') + ir_n\phi(x, \lambda_n)|^2 d_qx \\ &= \int_0^X |\nu\theta(x, \lambda') + \{\nu m(\lambda') + ir_n\}\phi(x, \lambda') - ir_n\{\phi(x, \lambda') - \phi(x, \lambda_n)\}|^2 d_qx \rightarrow 0. \end{aligned}$$

Also, by

$$\int_0^\infty |\nu\psi(x, \lambda')|^2 d_qx \leq |\nu m(\lambda')| = O(1)$$

as  $\nu \rightarrow 0$ , since the pole of  $m(\lambda')$  at  $\lambda_n$  is simple. On multiplying (4) by  $i\nu/r_m$ , making  $\nu \rightarrow 0$ , and using Lemma 2 we see that  $\phi(x, \lambda_n)$  is  $L^2(\mathbb{R}_q^+)$ , and

$$\int_0^\infty \psi(x, \lambda)\phi(x, \lambda_n)d_qx = \frac{1}{\lambda - \lambda_n}. \quad (5)$$

If  $\lambda$  tends to a different eigenvalue  $\lambda_m$ , on multiplying (5) by  $i\nu/r_m$  and making  $\nu \rightarrow 0$ , we obtain

$$\int_0^\infty \phi(x, \lambda_m)\phi(x, \lambda_n)d_qx = 0.$$

If  $\lambda$  tends to the same eigenvalue  $\lambda_n$ , it follows similarly that

$$\int_0^\infty \{\phi(x, \lambda_n)\}^2 d_qx = \frac{1}{r_n}.$$

Hence the functions

$$\psi_n(x) = \sqrt{r_n}\phi(x, \lambda_n)$$

form a normal orthogonal set of  $L^2(\mathbb{R}_q^+)$ .

## 4 Series expansions

Let  $f(y)$  be  $L^2(\mathbb{R}_q^+)$ , and let

$$\Phi(x, \lambda) = \psi(x, \lambda) \int_0^x \phi(y, \lambda) f(y) d_q y + \phi(x, \lambda) \int_x^\infty \psi(y, \lambda) f(y) d_q y,$$

where  $\phi$  and  $\psi$  are the functions defined above.

$$D_q \Phi(x, \lambda) = D_q \psi(x, \lambda) \int_0^{qx} \phi(y, \lambda) f(y) d_q y + D_q \phi(x, \lambda) \int_{qx}^\infty \psi(y, \lambda) f(y) d_q y$$

and

$$\begin{aligned} D_q^2 \Phi(x, \lambda) &= D_q^2 \psi(x, \lambda) \int_0^{q^2 x} \phi(y, \lambda) f(y) d_q y + D_q^2 \phi(x, \lambda) \int_{q^2 x}^\infty \psi(y, \lambda) f(y) d_q y \\ &\quad + q D_q \psi(x, \lambda) \phi(qx, \lambda) f(qx) - q D_q \phi(x, \lambda) \psi(qx, \lambda) f(qx). \end{aligned}$$

Therefore

$$\begin{aligned} \Delta_q \Phi(x, \lambda) &= \left[ \frac{1-q}{q} \right]^2 \Lambda_q^{-1} D_q^2 \Phi(x, \lambda) \\ &= \Delta_q \psi(x, \lambda) \int_0^{qx} \phi(y, \lambda) f(y) d_q y + \Delta_q \phi(x, \lambda) \int_{qx}^\infty \psi(y, \lambda) f(y) d_q y \\ &\quad + \frac{(1-q)^2}{q} [\Lambda_q^{-1} D_q \psi(x, \lambda) \phi(x, \lambda) - \Lambda_q^{-1} D_q \phi(x, \lambda) \psi(x, \lambda)] f(x) \\ &= [q(x) - \lambda] \Phi(x, \lambda) + W_x(\theta, \phi) f(x) = [q(x) - \lambda] \Phi(x, \lambda) + f(x). \end{aligned}$$

The function  $\Phi$  satisfies the boundary condition

$$\Phi(0, \lambda) \cos \alpha + \Phi'(0, \lambda) \sin \alpha = 0.$$

If  $\Phi_X(x, \lambda)$  is the corresponding function with  $f(y) = 0$  for  $y > X$ , then

$$\begin{aligned} \Phi_X(x, \lambda) &= \theta(x, \lambda) \int_0^x \phi(y, \lambda) f(y) d_q y + \phi(x, \lambda) \int_x^X \phi(y, \lambda) f(y) d_q y \\ &\quad + m(\lambda) \phi(x, \lambda) \int_0^X \phi(y, \lambda) f(y) d_q y. \end{aligned}$$

This is clearly regular everywhere except at  $\lambda = \lambda_0, \lambda_1, \dots$ , where it has simple poles with residues

$$r_n \phi(x, \lambda_n) \int_0^X \phi(y, \lambda_n) f(y) d_q y.$$

Hence making  $X \rightarrow \infty$  we find that  $\Phi(x, \lambda)$  has a simple pole at  $\lambda_n$ , its residue there being limit of the residue of  $\Phi_X(x, \lambda)$ , i.e.

$$r_n \phi(x, \lambda_n) \int_0^\infty \phi(y, \lambda_n) f(y) d_q y = \psi_n(x) \int_0^\infty \psi_n(y) f(y) d_q y = c_n \psi_n(x).$$

**Lemma 3** *If  $f(x)$  is any function of  $L^2(\mathbb{R}_q^+)$ ,*

$$\int_0^\infty |\Phi(x, \lambda)|^2 d_q x \leq \frac{1}{\nu^2} \int_0^\infty |f(x)|^2 d_q x.$$

Suppose first that  $f(x) = 0$  for  $x \geq X$ . Then the condition of self-adjointness

$$\int_0^\infty \Phi(x, \lambda) L \Phi(x, \lambda') d_q x = \int_0^\infty \Phi(x, \lambda') L \Phi(x, \lambda) d_q x,$$

is satisfied. For

$$\begin{aligned} & \int_0^\xi \{ \Phi(x, \lambda) L \Phi(x, \lambda') - \Phi(x, \lambda') L \Phi(x, \lambda) \} d_q x \\ &= - \int_0^\xi \{ \Phi(x, \lambda) \Delta_q \Phi(x, \lambda') - \Phi(x, \lambda') \Delta_q \Phi(x, \lambda) \} d_q x \\ &= -W[\Phi(x, \lambda), \Phi(x, \lambda')]_0^\xi. \end{aligned}$$

The integrated term vanishes at  $x = 0$ , and the integrated term at  $x = \xi$  tend to 0 as  $\xi \rightarrow \infty$ , since, if  $x > X$ ,

$$\Phi(x, \lambda) = \psi(x, \lambda) \int_0^X \phi(y, \lambda) f(y) d_q y,$$

$$\Lambda_q^{-1} D_q \Phi(x, \lambda) = \Lambda_q^{-1} D_q \psi(x, \lambda) \int_0^X \phi(y, \lambda) f(y) d_q y,$$

and the result follows from Lemma 1. Putting  $\lambda' = \bar{\lambda}$ , we obtain

$$\int_0^\infty \Phi(x, \lambda) \{\bar{\lambda} \Phi(x, \bar{\lambda}) - f(x)\} d_q x = \int_0^\infty \Phi(x, \bar{\lambda}) \{\lambda \Phi(x, \lambda) - f(x)\} d_q x,$$

i.e.

$$(\lambda - \bar{\lambda}) \int_0^\infty |\Phi(x, \lambda)|^2 d_q x = \int_0^\infty \{\Phi(x, \bar{\lambda}) - \Phi(x, \lambda)\} f(x) d_q x.$$

Hence, if  $\lambda = \mu + i\nu$ ,  $\nu > 0$ ,

$$2\nu \int_0^\infty |\Phi(x, \lambda)|^2 d_q x \leq 2 \int_0^\infty |\Phi(x, \lambda) f(x)| d_q x \leq 2 \left\{ \int_0^\infty |\Phi(x, \lambda)|^2 d_q x \int_0^\infty |f(x)|^2 d_q x \right\}.$$

This prove the result in the restricted case. If now  $f(x)$  is any function of  $L^2(\mathbb{R}_q^+)$ , for a fixed  $X'$  we have

$$\begin{aligned} \int_0^{X'} |\Phi_X(x, \lambda)|^2 d_q x &\leq \int_0^\infty |\Phi_X(x, \lambda)|^2 d_q x \\ &\leq \frac{1}{\nu^2} \int_0^\infty |f_X(x)|^2 d_q x = \frac{1}{\nu^2} \int_0^X |f(x)|^2 d_q x \leq \frac{1}{\nu^2} \int_0^\infty |f(x)|^2 d_q x. \end{aligned}$$

The result therefore follows on making first  $X \rightarrow \infty$ , then  $X' \rightarrow \infty$ .

We denot by  $\mathcal{L}_{q,2}$  the subspace of  $L^2(\mathbb{R}_q^+)$  of functions which satisfy

- The function  $Lf$  be  $L^2(\mathbb{R}_q^+)$ .
- $f'(0) \cos \alpha - f(0) \sin \alpha = 0$ .
- $\lim_{x \rightarrow \infty} W\{\psi(x, \lambda), f(x)\} = 0$ , for every non-real  $\lambda$ .

**Lemma 4** *If  $f(x)$  belongs to  $\mathcal{L}_{q,2}$  and  $\Phi(x, \lambda)$  defined above, is also denoted by  $\Phi(x, \lambda, f)$  then*

$$\Phi(x, \lambda, f) = \frac{1}{\lambda} \{f(x) + \Phi(x, \lambda, Lf)\}. \quad (6)$$

We have

$$\begin{aligned}\int_0^x \phi(y, \lambda) f(y) d_q y &= \frac{1}{\lambda} \int_0^x \{q(y)\phi(y) - \Delta_q \phi(y)\} f(y) d_q y \\ &= \frac{1}{\lambda} W[\phi(y), f(y)]_0^x + \frac{1}{\lambda} \int_0^x \{q(y)f(y) - \Delta_q f(y)\} \phi(y) d_q y.\end{aligned}$$

Similarly

$$\begin{aligned}\int_x^\infty \phi(y, \lambda) f(y) d_q y &= \frac{1}{\lambda} \int_x^\infty \{q(y)\phi(y) - \Delta_q \phi(y)\} f(y) d_q y \\ &= \frac{1}{\lambda} W[\psi(y), f(y)]_x^\infty + \frac{1}{\lambda} \int_x^\infty \{q(y)f(y) - \Delta_q f(y)\} \psi(y) d_q y,\end{aligned}$$

and the integrated term vanishes at the upper limit, which lead to the result.

The Green's function  $G(x, y, \lambda)$  is defined by

$$G(x, y, \lambda) = \begin{cases} -\psi(x, \lambda)\phi(y, \lambda), & \text{if } y \leq x \\ -\psi(y, \lambda)\phi(x, \lambda), & \text{if } y > x \end{cases},$$

then

$$\Phi(x, \lambda, f) = - \int_0^\infty G(x, y, \lambda) f(y) d_q y.$$

Therefore

$$f(x) = \int_0^\infty G(x, y, \lambda) \{Lf(y) - \lambda f(y)\} d_q y. \quad (7)$$

**Lemma 5** *Let  $F(\lambda)$  be an analytic function of  $\lambda = \mu + i\nu$ , regular for  $-r \leq \mu \leq r, -r \leq \nu \leq r$ , and let*

$$|F(\lambda)| \leq \frac{M}{|\nu|},$$

*in this square. Then*

$$|F(\lambda)| \leq \frac{3M}{r}.$$

Let

$$G(\lambda) = (\lambda^2 - r^2)F(\lambda).$$

On the upper and lower sides of the square

$$|G(\lambda)| \leq (|\lambda|^2 + r^2) \frac{M}{r} \leq 3rM.$$

On the left-hand and right-hand sides

$$|G(\lambda)| \leq |\nu|(|\lambda| + r) \frac{M}{|\nu|} \leq 3rM.$$

Hence  $|G(\lambda)| \leq 3rM$  throughout the square. Hence on the imaginary axis

$$|F(\lambda)| \leq \frac{3rM}{|\lambda^2 - r^2|} = \frac{3rM}{\nu^2 + r^2} \leq \frac{3M}{r}.$$

**Theorem 2** *Let  $f(x)$  be a function belonging in  $\mathcal{L}_{q,2}$  then*

$$\int_0^\infty \{f(x)\}^2 d_q x = \sum_{n=0}^\infty c_n^2,$$

Suppose that  $f(x)$  satisfies the conditions of the above theorem, and also that  $f(x) = 0$  for sufficient large values of  $x$ . Let

$$\Psi(\lambda) = \int_0^\infty f(x) \Phi(x, \lambda) d_q x,$$

then  $\Psi(\lambda)$  is regular except for simple poles at the points  $\lambda_n$ , where it has residues

$$c_n \int_0^\infty \psi_n(x) f(x) d_q x = c_n^2.$$

By (6)

$$\Psi(\lambda) = \frac{1}{\lambda} \int_0^\infty \{f(x)\}^2 d_q x + \frac{1}{\lambda} \int_0^\infty \Phi(x, \lambda, Lf) f(x) d_q x, \quad (8)$$

and the last term is

$$O \left\{ \frac{1}{|\lambda|} \left[ \int_0^\infty |\Phi(x, \lambda, Lf)|^2 d_q x \int_0^\infty \{f(x)\}^2 d_q x \right]^{\frac{1}{2}} \right\} = O \left( \frac{1}{|\lambda\nu|} \right),$$

by Lemma 3, applied to  $Lf$ .

Let  $C(R)$  denote the contour formed by the segments of lines  $(R-i, R+i)$  and  $(-R-i, -R+i)$ , joined by semicircles of radius  $R$  and centres  $\pm i$ . Then

$$\int_{C(R)} \Psi(\lambda) d\lambda = \sum_{-R < \lambda_n < R} c_n^2.$$

On the part of the upper semicircle in the first quadrant, we have

$$\lambda = i + R e^{i\theta} \quad (0 \leq \theta \leq \frac{\pi}{2}).$$

Hence the last term in (8), integrated round this quadrant, gives

$$O \left\{ \int_0^{\frac{\pi}{2}} \frac{R}{R(1 + R \sin \theta)} d\theta \right\} = O \left\{ \int_0^{\frac{1}{R}} d\theta \right\} + O \left\{ \int_{\frac{1}{R}}^{\frac{\pi}{2}} \frac{d\theta}{R\theta} \right\} = O \left( \frac{1}{R} \right) + O \left( \frac{\log R}{R} \right) = o(1).$$

A similar argument applies to the other quadrants. Hence the integral of  $\Psi(\lambda)$  round each semicircle tends to

$$\pi i \int_0^\infty \{f(x)\}^2 d_q x$$

as  $R \rightarrow \infty$ . To prove the theorem for the class of functions considered, it is therefore sufficient to prove that

$$\lim_{R \rightarrow \infty} \int_{R-i}^{R+i} \Psi(\lambda) d\lambda = 0,$$

and a similar result with  $-R$  in place of  $R$ .

Let

$$\chi(\lambda) = \Psi(\lambda) - \sum_{R-1 \leq \lambda_n \leq R+1} \frac{c_n^2}{\lambda - \lambda_n}.$$

Then  $\chi(\lambda)$  is regular for  $R-1 \leq \lambda \leq R+1$ , and

$$|\chi(\lambda)| < \frac{K}{|\lambda\nu|} + \frac{1}{\nu} \sum_{R-1 \leq \lambda_n \leq R+1} c_n^2 \leq \frac{\epsilon(R)}{|\nu|},$$

where  $\epsilon(R) \rightarrow 0$  as  $R \rightarrow \infty$ . Hence, by Lemma 5

$$|\chi(\lambda)| \leq 3\epsilon(R)$$

on the segment  $(R - i, R + i)$ . Hence

$$\lim_{R \rightarrow \infty} \int_{R-i}^{R+i} \chi(\lambda) d\lambda = 0.$$

Also

$$\int_{R-i}^{R+i} \frac{d\lambda}{\lambda - \lambda_n} d\lambda = O(1).$$

Hence

$$\int_{R-i}^{R+i} \sum_{R-1 \leq \lambda_n \leq R+1} \frac{c_n^2}{\lambda - \lambda_n} d\lambda = O \left\{ \sum_{R-1 \leq \lambda_n \leq R+1} c_n^2 \right\} = o(1).$$

This prove the theorem for the special class of functions. The theorem can be extended to all functions of integrable square. It also follows that, if  $g(x)$  is another function of  $L^2(\mathbb{R}_q^+)$ , with Fourier coefficients  $d_n$ , then

$$\int_0^\infty f(x)g(x)d_qx = \sum_{n=0}^\infty c_n d_n. \quad (9)$$

**Theorem 3** *Let  $f(x)$  be a function blogin in  $\mathcal{L}_{q,2}$  then*

$$f(x) = \sum_{n=0}^\infty c_n \psi_n(x), \quad \forall x \in \mathbb{R}_q^+.$$

From the formula

$$(\lambda_n - \lambda) \int_0^b \psi(x, \lambda) \phi(x, \lambda_n) d_qx = W_0\{\psi(x, \lambda), \phi(x, \lambda_n)\} - W_b\{\psi(x, \lambda), \phi(x, \lambda_n)\},$$

and (5), it follows that the second term on the right-hand side tends to 0 as  $b \rightarrow \infty$ . Hence we may take  $f(x) = \psi_n(x)$  in (7), which gives

$$\int_0^\infty G(x, y, \lambda) \psi_n(x) d_qy = \frac{\psi_n(x)}{\lambda_n - \lambda}.$$

Also

$$\begin{aligned} & \int_0^\infty |G(x, y, \lambda)|^2 d_qy \\ &= |\psi(x, \lambda)|^2 \int_0^x |\phi(x, \lambda)|^2 d_qy + |\phi(x, \lambda)|^2 \int_x^\infty |\psi(x, \lambda)|^2 d_qy \leq K \end{aligned}$$



say, for  $x$  in a finite interval, and  $\lambda$  not real. Hence by the Bessel inequality

$$\sum_{n=0}^{\infty} \left| \frac{\psi_n(x)}{\lambda_n - \lambda} \right|^2 \leq K. \quad (10)$$

Now let  $f(x)$  satisfy the conditions of Theorem 3, and let

$$g(x) = Lf(x) - \lambda f(x).$$

If  $\lambda$  is not real,

$$\int_0^{\infty} [\psi(y, \lambda)Lf(y) - f(y)L\{\psi(y, \lambda)\}]d_qy = W[f, \psi]_0^{\infty} = f'(0) \cos \alpha - f(0) \sin \alpha = 0,$$

and  $L\psi = \lambda\psi$ . Put  $\lambda = \lambda_n + i\nu$ , multiply by  $\nu$ , and make  $\nu \rightarrow 0$ . Using Lemma 3, we obtain

$$\int_0^{\infty} \psi_n(y)Lf(y)d_qy = \lambda_n \int_0^{\infty} \psi_n(y)f(y)d_qy = \lambda_n c_n.$$

Hence, if  $d_n$  is the Fourier coefficient of  $g(x)$ ,

$$d_n = (\lambda_n - \lambda)c_n,$$

and

$$\sum_{n=0}^{\infty} |\lambda_n - \lambda|^2 c_n^2, \quad (11)$$

is convergent. From (7) and (9) with  $f(y)$  replaced by  $G(x, y, \lambda)$ , it follows that

$$f(x) = \int_0^{\infty} G(x, y, \lambda)g(y)d_qy = \sum_{n=0}^{\infty} \frac{\psi_n(x)}{\lambda_n - \lambda} (\lambda_n - \lambda)c_n = \sum_{n=0}^{\infty} c_n \psi_n(x),$$

the required result. The absolute and uniform convergence of the series follows from (10) and the convergence of (11).

## References

- [1] G. Gasper and M. Rahman, Basic hypergeometric series, Encyclopedia of mathematics and its applications 35, Cambridge university press, 1990.
- [2] P. Hartman, On differential equations with non-oscillatory eigenfunctions. Duke math. J.15 697–709 (1948)
- [3] F. H. Jackson, On a q-Definite Integrals, Quarterly Journal of Pure and Application Mathematics 41, 1910, 193-203.
- [4] D.B. Sears, Note on the uniqueness of the Green's function associated with certain second order differential equations. Canadian J. Math.4, 314–325 (1950).
- [5] E.C. Titchmarsh, On the uniqueness of the Green's function associated with a second order differential equations. Canadian J. Math.1, 191–198 (1949).
- [6] H. Weyl, Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörige Entwicklung willkürlicher Functionen. Math. Ann.68, 220–269 (1910).