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RING STRUCTURES FOR HOLOMORPHIC DISCRETE SERIES AND RANKIN-COHEN BRACKETS

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ABSTRACT. In the present note we discuss two different ring structures on the set of holomorphic discrete series of a causal symmetric space of Cayley type G/H and we suggest a new interpretation of Rankin-Cohen brackets in terms of intertwining operators arising in the decomposition of tensor products of holomorphic discrete series representations.

1. INTRODUCTION

When studying L-functions of quadratic characters H. Cohen [1] described in 1975 a particular family of bi-differential operators acting on smooth functions on the Poincaré upper half-plane Π. The initial interest in these operators, called henceforth the Rankin-Cohen brackets (RCB), is due to the fact that they give a powerful tool for producing new modular forms of higher weight.

More precisely, fix a positive integer k and define for every $f \in C^\infty(\Pi)$:

$$(f_{|k}\gamma)(z) := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right), \qquad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}).$$

One says that a function f holomorphic on Π is a modular form of weight k with respect to some arithmetic subgroup $\Gamma \subset G$ if it satisfies the identity $(f_{|k}\gamma) = f$ for all $\gamma \in \Gamma$.

Let k_1, k_2, j be three positive integers and $f, g \in C^\infty(\Pi)$. One sets

$$F_j(f,g) = \sum_{\ell=0}^{j} (-1)^\ell C^\ell_{k_1+j-1} C^{j-\ell}_{k_2+j-1} f^{(j-\ell)} g^{(\ell)}, \text{ where } f^{(\ell)} = \left(\frac{\partial}{\partial z}\right)^\ell f, \tag{1.1}$$

and C^ℓ_k denote the binomial coefficient $\frac{k!}{(k-\ell)!\ell!}$.

H. Cohen showed that the following identity holds:

$$F_j(f_{|k_1}\gamma, g_{|k_2}\gamma) = F_j(f,g)_{|k_1+k_2+2j}\gamma, \qquad \gamma \in SL(2,\mathbb{R}).$$

Therefore if f and g are modular of weight k_1 and k_2 respectively $F_j(f,g)$ is again a modular form of weight k_1+k_2+2j for every $j \in \mathbb{N}$. Notice that in case when $\Gamma = SL(2,\mathbb{Z})$ the only non trivial modular forms are of even weight.

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This construction was generalized in the setting of $Sp(n, \mathbb{Z})$-modular forms on the Siegel half plane, i.e. the symmetric space of positive definite symmetric matrices, by W.Eholzer and T.Ibukiyama [3]. An algebraic approach to RCB and their possible generalizations via the commutation relations that they should satisfy was developed by D. Zagier [26]. See also [27] for an overview of this subject from the number theoretic point of view.

From the other side, in 1996 A and J. Unterberger [25] showed that this family of bi-differential operators arises in an astonishing way in the context of the covariant quantization of one of the coadjoint orbits of the Lie group $G = SL(2, \mathbb{R})$. By developing a covariant symbolic calculus on the one-sheeted hyperboloid realized as the symmetric space $G/H = SL(2, \mathbb{R})/SO(1, 1)$ they proved that the composition $f \#_s g$ of two symbols f and g satisfying some regularity conditions (they are images by the inverse Laplace transform of holomorphic functions, square integrable with respect to some particular measure on the upper half plane) is again a symbol of the same kind and moreover it decomposes into a convergent sum $f \#_s g = \sum_j h_j$ where every summand h_j is related to the Rankin-Cohen bracket $F_j(f, g)$.

This result implies that the set of holomorphic discrete series representations with even parameter of the group $SL(2, \mathbb{R})$ is endowed with a graded non-commutative ring structure given by the so-called standard (or convolution-first) covariant symbolic calculus on $SL(2, \mathbb{R})/SO(1, 1)$.

The group of unimodular real matrices $G = SL(2, \mathbb{R})$ acts on the set of functions defined on Π and the modular forms are the invariants of this action restricted to $SL(2, \mathbb{Z}) \subset G$.

The fact that RCB's produce new modular forms from known ones fits with the standard techniques of transvectants developed in the classical invariant theory. This method allows us to construct new invariant analytic functions in two complex variables starting with a couple of known analytic functions invariant for the simultaneous linear action of $GL(2, \mathbb{C})$. This procedure involves some differential operators such that once restricted to homogeneous functions they coincide with the RCB's given by (1.1). P. Olver gives a very detailed overview of this construction in chapter 5 of his book [16] as well as in [17]. Notice that the basic lemma underlying the link between transvectants ant the Rankin-Cohen brackets was proved by S. Gundelfinger [8] already in 1886.

Inspired by this observation we shall gather in the present note these two different approaches to the RCB's using the representation theory of the group $SL(2, \mathbb{R})$. These techniques will make clear the way to generalize the notion of RCB's in the setting of para-Hermitian symmetric spaces of Hermitian type (some times called also causal symmetric spaces of Cayley type). The choice of this particular class of symplectic symmetric spaces is explained in the next section. We shall see that RCB's are related to the decomposition of tensor products of two holomorphic discrete series representations into irreducible components. Recent results by Peetre [18] and Peng and Zhang [19] give an explicit formula for the RCB in this case. The description of the Clebsh-Gordan coefficients of the group G is an important problem even from the physical point of view and we hope that this note will give a better understanding of what one calls now the Rankin-Cohen quantization [2].

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2. Geometric settings

Let G be a connected real semi-simple Lie group with finite center and K be a maximal compact subgroup. We assume that the Harish-Chandra condition $\operatorname{rank} G = \operatorname{rank} K$ holds what guarantees the existence of discrete series representations (i.e. unitary irreducible representations whose matrix coefficients are square integrable on G). Furthermore, we assume that G/K is a Hermitian symmetric space of tube type and thus G has holomorphic discrete series, i.e. discrete series realizable in holomorphic sections of holomorphic vector bundles over G/K. Equivalently the last condition means that the Harish-Chandra modules underlying these discrete series representations are highest weight modules.

Among such Lie groups we shall restrict our considerations to those which can be seen as automorphism groups of some semi-simple para-Hermitian symmetric space. More precisely, let σ be an involutive automorphism of G and H an open connected subgroup of the group of fixed points of σ. The coset space G/H (which is actually a coadjoint orbit of G and therefore is a symplectic manifold) is called para-Hermitian if its tangent bundle $T(G/H)$ splits into the sum of two G-invariant isomorphic sub-bundles (see [10] for a detailed study of such spaces).

This splitting induces a G-invariant polarization on $T(G/H)$ which is necessary in order to define a symbolic calculus. Indeed, this polarization will allow us to distinguish position and momenta variables on G/H which plays the role of the phase space, while the symmetric space G/K will be seen as the configuration space.

It turns out that the Lie groups G satisfying both conditions : G/K is Hermitian of tube type and G/H is para-Hermitian, have a nice description in terms of Euclidean Jordan algebras.

We shall briefly recall the link between Jordan algebras and the semi-simple Lie groups considered above.

An algebra V over $\mathbb{R}$ or $\mathbb{C}$ is said to be a Jordan algebra if for all elements x and y in V, one has $x{\cdot}y = y{\cdot}x$ and $x{\cdot}(x^2{\cdot}y) = x^2{\cdot}(x{\cdot}y)$. For an element $x \in V$ let $L(x)$ be the linear map of V defined by $L(x)y := x{\cdot}y$ and $P(x) = 2L(x)^2 - L(x^2)$ be the quadratic representation of V. For x and y in V one also defines an endomorphism $D(x, y)$ of V given by $D(x, y) = L(xy) - [L(x), L(y)]$.

We denote by $\beta(x, y)$ the symmetric bilinear form on V defined by $\beta(x, y) = \mathrm{Tr}L(x \cdot y)$.

Let r and n denote respectively the rank and the dimension of the Jordan algebra V. The integer d determined by $n = r + \frac{d}{2}r(r-1)$ is called *Peirce multiplicity*. For a regular element x, the minimal polynomial f_x is of degree r,

$$f_x(\lambda) = \lambda^r - a_1(x)\lambda^{r-1} + \cdots + (-1)^r a_r(x).$$

The coefficient a_j is a homogeneous polynomial of degree j, $\Delta(x) := a_r(x)$ is the *Jordan determinant*, and $\mathrm{tr}\,(x) := a_1(x)$ is the *Jordan trace* of x.

A Jordan algebra V is semi-simple if the form β is non-degenerate on V. A semi-simple Jordan algebra is unital, we denote by e its identity element.

A Jordan algebra V_o over $\mathbb{R}$ is said to be Euclidean if the bilinear form $\beta(x, y)$ is positive definite on V_o.

Let V_o be an Euclidean Jordan algebra (EJA) from now one. The set

$$\Omega := \{x^2 \mid x \text{ invertible in } V_o\}$$

is an open, convex, self-dual cone in V_o. Those properties of Ω actually characterize V_o as an EJA. The automorphism group of $G(\Omega)$ of the cone Ω is defined by

$$G(\Omega) = \{g \in GL(V_o) \mid g\Omega = \Omega\},$$

and it is a reductive Lie group.

Let V be the complexification of V_o. Consider the tube $T_\Omega = V_o + i\Omega \subset V$ and the Lie group $G = Aut(T_\Omega)$ of holomorphic automorphisms of T_Ω. According to general theory [5] Ch. X. §5, the group $G(\Omega)$ can be seen as a subgroup of G as well as the Jordan algebra V_o it-self. Indeed, for every $u \in V_o$, the translation $\tau_u : z \to z + u$ is a holomorphic automorphism of the tube T_Ω and the group of all real translations τ_u is an Abelian subgroup N of G isomorphic to the vector space V_o.

The subgroup of all affine linear transformations of the tube $P = G(\Omega) \ltimes N$ is a maximal parabolic subgroup of G.

The subgroups $G(\Omega)$ and N together with the inversion map $j : x \longrightarrow -x^{-1}$, generate the group G.

Let σ be the involution of G given by $\sigma(g) = j \circ g \circ j$, $g \in G$. In the case when V_o is a Euclidean Jordan algebra this is a Cartan involution. Let K be a maximal compact subgroup of G. Then the symmetric space $G/K \simeq T_\Omega$ is an Hermitian symmetric space of tube type.

For $w \in V$ the endomorphism $D(w,\bar{w})$ is Hermitian and one defines an invariant spectral norm $|w| = \|D(w,\bar{w})\|^{1/2}$. Let

$$\mathcal{D} = \{w \in V \,:\, |w| < 1\},$$

be the open unit ball for the spectral norm. Then the Cayley transform $p : z \mapsto (z - ie)(z + ie)^{-1}$ is a holomorphic isomorphism from the tube T_Ω onto the domain $\mathcal{D}$. Thus the group of holomorphic automorphisms of $\mathcal{D}$ that one denotes $G(\mathcal{D}) = Aut(\mathcal{D})$ is conjugate to $G : G(\mathcal{D}) = pGp^{-1}$. We shall refer to the domain $\mathcal{D}$ as to the Harish-Chandra bounded realization of the symmetric space G/K.

We denote $\overline{N} = \sigma(N)$ and $\overline{P} := G(\Omega) \ltimes \overline{N}$.

From the geometric point of view the subgroup $\overline{P}$ can be characterized in the following way:

$$\overline{P} = \{g \in G' \mid g(0) = 0\},$$

where G' is the subset of G of all transformations well defined at $0 \in V_o$. It is open and dense in G. Moreover $G' = NG(\Omega)\overline{N}$. The map $N \times G(\Omega) \times \overline{N} \to G'$ is a diffeomorphism. We shall refer to this decomposition as to the *Gelfand-Naimark decomposition* of the group G. Furthermore, for every transformation $g \in G$ which is well defined at $x \in V_o$, the transformation gn_x belongs to G' and its Gelfand-Naimark decomposition is given by :

$$gn_x = n_{g.x}(Dg)_x\bar{n}', \tag{2.1}$$

where $(Dg)_x \in G(\Omega)$ is the differential of the conformal map $x \to g.x$ at x and $\bar{n}' \in \overline{N}$ (see [20] Prop. 1.4).

The flag variety $\mathcal{M} = G/\overline{P}$, which is compact, is the *conformal compactification* of V_o. In fact the map $x \longrightarrow (n_x \circ j)P$ gives rise to an embedding of V_o into $\mathcal{M}$ as an open dense subset, and every transformation in G extends to $\mathcal{M}$.

Let $\mathfrak{g}$ be the Lie algebra of the automorphism group G. Euclidean Jordan algebras, corresponding Lie algebras of infinitesimal automorphisms of tube domains, and their maximal compact subalgebras are given by the first table.

$\mathfrak{g}$	$\mathfrak{k}$	V	V_o
$\mathfrak{su}(n,n)$	$\mathfrak{su}(n) \oplus \mathfrak{su}(n) \oplus \mathbb{R}$	$M(n,\mathbb{C})$	$Herm(n,\mathbb{C})$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{su}(n) \oplus \mathbb{R}$	$Sym(n,\mathbb{C})$	$Sym(n,\mathbb{R})$
$\mathfrak{so}^*(4n)$	$\mathfrak{su}(2n) \oplus \mathbb{R}$	$Skew(2n,\mathbb{C})$	$Herm(n,\mathbb{H})$
$\mathfrak{o}(n,2)$	$\mathfrak{o}(n) \oplus \mathbb{R}$	$\mathbb{C}^{n-1} \times \mathbb{C}$	$\mathbb{R}^{n-1} \times \mathbb{R}$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$Herm(3,\mathbb{O}) \otimes \mathbb{C}$	$Herm(3,\mathbb{O})$

Let us consider the involution η of the complex Jordan algebra V given by $\eta(x+iy)=-x+iy$ $(x,y\in V)$ and define the corresponding fix point sub-group in G by $H=\{g\in G\,|\,\eta g\eta=g\}$. Clearly $H=G(\Omega)$. The involutions η and σ commute.

Notice that the involution we introduced is a particular case of a conjugation of V satisfying the following properties.

- $\eta\,(V_0)=V_0$,
- $\eta\,(ie)=ie$,
- $-\eta$ is a real Jordan algebra automorphism of V.

The factor space G/H is a para-Hermitian symmetric space. It means that its tangent bundle splits into two G-invariant sub-bundles both isomorphic to the underlying Jordan algebra V_o.

We restricted all considerations to Euclidean Jordan algebras therefore the para-Hermitian spaces G/H that we get are of a particular type, one calls them causal symmetric spaces of Cayley type [6]. Their infinitesimal classification is given in the second table.

$\mathfrak{g}$	$\mathfrak{h}$	V	V_0
$\mathfrak{su}(n,n)$	$\mathfrak{sl}(n,\mathbb{C})\oplus\mathbb{R}$	$\mathrm{M}(n,\mathbb{C})$	$\mathrm{Herm}(n,\mathbb{C})$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sl}(n,\mathbb{R})\oplus\mathbb{R}$	$\mathrm{Sym}(n,\mathbb{C})$	$\mathrm{Sym}(n,\mathbb{R})$
$\mathfrak{so}^*(4n)$	$\mathfrak{su}^*(2n)\oplus\mathbb{R}$	$\mathrm{Skew}(2n,\mathbb{C})$	$\mathrm{Herm}(n,\mathbb{H})$
$\mathfrak{so}(n,2)$	$\mathfrak{so}(n-1,1)\times\mathbb{R}$	$\mathbb{C}^{n-1}\times\mathbb{C}$	$\mathbb{R}^{n-1}\times\mathbb{R}$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)}\oplus\mathbb{R}$	$\mathrm{Herm}(3,\mathbb{O})\otimes\mathbb{C}$	$\mathrm{Herm}(3,\mathbb{O})$

3. Two series of representations of G

3.1. **Holomorphic discrete series.** Holomorphic induction from a maximal compact subgroup leads to a series of unitary representations of G, called holomorphic discrete series representations, that one usually realizes on holomorphic sections of holomorphic vector bundles over G/K.

According to our convenience and easiness of presentation we shall use both bounded and unbounded realizations of the symmetric space G/K. We start with the simplest case of scalar holomorphic discrete series.

For a real parameter ν consider the weighted Bergman spaces $H^2_\nu(T_\Omega)$ of complex valued holomorphic functions $f\in\mathcal{O}(T_\Omega)$ such that

$$\|f\|^2_\nu=\int_{T_\Omega}|f(z)|^2\Delta^{\nu-2\frac{n}{r}}(y)dxdy<\infty,$$

where $z=x+iy\in T_\Omega$. Note that the measure $\Delta^{-2\frac{n}{r}}(y)dxdy$ on T_Ω is invariant under the action of the group G. For $\nu>1+d(r-1)$ these spaces are non empty Hilbert spaces with reproducing kernels. More precisely, the space $H^2_\nu(T_\Omega)$ has a reproducing kernel K_ν which is given by

$$K_\nu(z,w) = c_\nu \Delta\left(\frac{z-\bar{w}}{2i}\right)^{-\nu}, \tag{3.1}$$

where c_ν is some expression involving Gindikin's conical Γ-functions (see [5] p.261).

The action of G on $H^2_\nu(T_\Omega)$ given for every integer $\nu > 1 + d(r-1)$ by

$$\pi_\nu(g)f(z) = \mathrm{Det}^\nu(D_{g^{-1}}(z))f(g^{-1}.z) \tag{3.2}$$

is called a *scalar holomorphic discrete series representation.*[1]

In the above formula $D_g(z)$ denotes the differential of the conformal transformation $z \to g.z$ of the tube.

On the other hand side the corresponding action of the group $G(\mathcal{D})$ can be realized as follows. Let

$$B(z,w) = 1 - D(z,w) + P(z)P(w),$$

be the Bergman operator on V. Its determinant $\mathrm{det}B(z,w)$ is of the form $h(z,w)^{2n/r}$ where $h(z,w)$ is the so-called *canonical polynomial* (see [5] p.262). Notice that it is the pull back of $K_1(z,w)$ by the Cayley transform.

Then the group $G(\mathcal{D})$ acts on the space $H^2_\nu(\mathcal{D})$ of holomorphic functions f on $\mathcal{D}$ such that

$$\|f\|^2_{\nu,\mathcal{D}} = c'_\nu \int_{\mathcal{D}} |f(z)|^2 h(z,z)^{\nu - 2\frac{n}{r}} dxdy < \infty$$

by the similar formula $\pi_\nu(g)f(z) = \mathrm{Det}^\nu(D_{g^{-1}}(z))f(g^{-1}.z)$.

More generally let $\mathfrak{g}$ be the Lie algebra of the automorphisms group $G(\mathcal{D})$ with complexification $\mathfrak{g}_c$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Let $\mathfrak{z}$ be the center of $\mathfrak{k}$. In our case the centralizer of $\mathfrak{z}$ in $\mathfrak{g}$ is equal to $\mathfrak{k}$ and the center of $\mathfrak{k}$ is one-dimensional. There is an element $Z_0 \in \mathfrak{z}$ such that $(\mathrm{ad}Z_0)^2 = -1$ on $\mathfrak{p}$. Fixing i a square root of -1, one has $\mathfrak{p}_c = \mathfrak{p} + i\mathfrak{p} = \mathfrak{p}_+ + \mathfrak{p}_-$ where $\mathrm{ad}Z_0|_{\mathfrak{p}_+} = i$, $\mathrm{ad}Z_0|_{\mathfrak{p}_-} = -i$. Then

$$\mathfrak{g}_c = \mathfrak{p}_+ \oplus \mathfrak{k}_c \oplus \mathfrak{p}_-. \tag{3.3}$$

and $[\mathfrak{p}_\pm, \mathfrak{p}_\pm] = 0, [\mathfrak{p}_+, \mathfrak{p}_-] = \mathfrak{k}_c$ and $[\mathfrak{k}_c, \mathfrak{p}_\pm] = \mathfrak{p}_\pm$. The vector space $\mathfrak{p}_+$ is isomorphic to V and furthermore it inherits its Jordan algebra structure. Let G_c be a connected, simply connected Lie group with Lie algebra $\mathfrak{g}_c$ and K_c, P_+,

[1]Notice that in general one shows, by use of analytic continuation, that the reproducing kernel (3.1) is positive-definite for a larger set of spectral parameters, namely for every ν in the so-called Wallach set $W(T_\Omega) = \left\{0, \frac{d}{2}, \ldots, (r-1)\frac{d}{2}\right\} \cup](r-1)\frac{d}{2}, \infty[$. However we restrict our considerations only to the subset of $W(T_\Omega)$ consisting of integer $\nu > 1 + d(r-1)$ in order to deal with spaces of holomorphic functions.

P_-, G, K, Z the analytic subgroups corresponding to $\mathfrak{k}_c, \mathfrak{p}_+, \mathfrak{p}_-, \mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{z}$ respectively. Then K_cP_- (and K_cP_+) is a maximal parabolic subgroup of G_c with split component $A = \exp i\mathbb{R}Z_0$. So the group $G = G(\mathcal{D})_o$ is closed in G_c.

Moreover, the exponential mapping is a diffeomorphism of $\mathfrak{p}_-$ onto P_- and of $\mathfrak{p}_+$ onto P_+ ([9] Ch.VIII, Lemma 7.8). Furthermore:

Lemma 3.1. *a.The mapping $(q, k, p) \mapsto qkp$ is a diffeomorphism of $P_+ \times K_c \times P_-$ onto an open dense submanifold of G_c containing G.*
b. The set GK_cP_- is open in $P_+K_cP_-$ and $G \cap K_cP_- = K$.

(see [9], Ch VIII, Lemmæ 7.9 and 7.10).

Thus G/K is mapped on an open, bounded domain $\mathcal{D}$ in $\mathfrak{p}_+$ This is an alternative description of the Harish-Chandra bounded realization of G/K. The group G acts on $\mathcal{D}$ via holomorphic transformations.

Everywhere in this section we shall denote $\bar{g}$ the complex conjugate of $g \in G_c$ with respect to G (do not confuse with the involution σ). Notice that P_+ is conjugate to P_-.

For $g \in P_+K_cP_-$ we shall write $g = (g)_+ (g)_0 (g)_-$, where $(g)_\pm \in P_\pm$, $(g)_0 \in K_c$. For $g \in G_c$, $z \in \mathfrak{p}_+$ such that $g.\exp z \in P_+K_cP_-$ we define

$$\exp g(z) = (g.\exp z)_+ \tag{3.4}$$

$$J(g, z) = (g.\exp z)_0. \tag{3.5}$$

$J(g, z) \in K_c$ is called the *canonical automorphic factor* of G_c (terminology of Satake).

Lemma 3.2. [23] *Ch.II, Lemma 5.1. The map J satisfies*
(i) $J(g, o) = (g)_0$, for $g \in P_+K_cP_-$,
(ii) $J(k, z) = k$ for $k \in K_c, z \in \mathfrak{p}_+$.
If for $g_1, g_2 \in G_c$ and $z \in \mathfrak{p}_+$, $g_1(g_2(z))$ and $g_2(z)$ are defined, then $(g_1g_2)(z)$ is also defined and
(iii) $J(g_1g_2, z) = J(g_1, g_2(z))\, J(g_2, z)$.

For $z, w \in \mathfrak{p}_+$ satisfying $(\exp \bar{w})^{-1}.\exp z \in P_+K_cP_-$ we define

$$K(z, w) = J((\exp \bar{w})^{-1}, z)^{-1} \tag{3.6}$$

$$= ((\exp \bar{w})^{-1}.\exp z)_0^{-1}. \tag{3.7}$$

This expression is always defined for $z, w \in \mathcal{D}$, for then

$$(\exp \bar{w})^{-1}.\exp z \in \overline{(GK_cP_-)}^{-1} GK_cP_- = P_+K_cGK_cP_- = P_+K_cP_-.$$

$K(z, w)$, defined on $\mathcal{D} \times \mathcal{D}$, is called the *canonical kernel* on $\mathcal{D}$ (by Satake). $K(z, w)$ is holomorphic in z, anti-holomorphic in w, with values in K_c. Here are a few properties:

Lemma 3.3. [23], *Ch.II, Lemma 5.2. The map K satisfies*
(i) $K(z,w) = \overline{K(w,z)}^{-1}$ *if* $K(z,w)$ *is defined,*
(ii)$K(o,w) = K(z,o) = 1$ *for* $z, w \in \mathfrak{p}_+$.
If $g(z), \bar{g}(w)$ *and* $K(z,w)$ *are defined, then* $K(g(z), \bar{g}(w))$ *is also defined and one has:*
(iii) $K(g(z), \bar{g}(w)) = J(g,z)\, K(z,w)\, \overline{J(\bar{g},w)}^{-1}$,

Lemma 3.4. [23], *Ch.II, Lemma 5.3. For $g \in G_c$ the Jacobian of the holomorphic mapping*
$z \mapsto g(z)$*, when it is defined, is given by*

$$\mathrm{Jac}\,(z \mapsto g(z)) = Ad_{\mathfrak{p}_+}(J(g,z)).$$

For any holomorphic character $\chi : K_c \mapsto \mathbb{C}$ we define:

$$j_\chi(g,z) = \chi(J(g,z)), \tag{3.8}$$

$$k_\chi(z,w) = \chi(K(z,w)). \tag{3.9}$$

Since $\chi(\bar{k}) = \overline{\chi(k)}^{-1}$ we have :

$$k_\chi(z,w) = \overline{k_\chi(w,z)}, \tag{3.10}$$

$$k_\chi(g(z),\bar{g}(w)) = j_\chi(g,z) k_\chi(z,w) \overline{j_\chi(\bar{g},w)} \tag{3.11}$$

in place of Lemma (3.3) (i) and (iii).

The character $\chi_1(k) = \det \mathrm{Ad}_{\mathfrak{p}_+}(k)$, $(k \in K_c)$ is of particular importance. We call the corresponding j_{χ_1}, k_{χ_1}: j_1 and k_1. Notice that

$$j_1(g,z) = \det(\mathrm{Jac}\,(z \mapsto g(z))). \tag{3.12}$$

Because of (3.12), $|k_1(z,z)|^{-1} d\mu(z)$, where $d\mu(z)$ is the Euclidean measure on $\mathfrak{p}_+$, is a $G-$invariant measure on $\mathcal{D}$. Indeed:

$$\begin{aligned} d\mu(g(z)) &= |j_1(g,z)|^2 d\mu(z), \\ k_1(g(z),g(z)) &= j_1(g,z)\, k_1(z,z)\, \overline{j_1(g,z)}, \qquad \text{for } g \in G. \end{aligned}$$

One can actually show that $k_1(z,z) > 0$ on $\mathcal{D}$. ([23], Ch.II, Lemma 5.8).

Let τ be an irreducible holomorphic representation of K_c on a finite dimensional complex vector space W with scalar product $\langle\,|\,\rangle$, such that $\tau_{|K}$ is unitary.

Lemma 3.5. *For every $k \in K_c$ one has the identity $\tau^*(k) = \tau(\bar{k})^{-1}$.*

This follows easily by writing $k = k_o \cdot \exp iX$ with $k_o \in K$, $X \in \mathfrak{k}$ and using that $\tau_{|K}$ is unitary.

Call $\pi_\tau = \mathrm{Ind}_K^G \tau$ and set W_τ for the representation space of π_τ. Then W_τ consists of maps $f : G \mapsto W$ satisfying

(i) f measurable,

(ii) $f(gk) = \tau^{-1}(k)f(g)$ for $g \in G,\ k \in K$,

(iii) $\int_{G/K} \|f(g)\|^2 d\dot{g} < \infty$, where $\|f(g)\|^2 = \langle f(g)|f(g)\rangle$ and $d\dot{g}$ is an invariant measure on G/K. Let us identify G/K with $\mathcal{D}$ and $d\dot{g}$ with $d_*z = k_1(z,z)^{-1}d\mu(z)$. Then W_τ can be identified with a space of maps on $\mathcal{D}$, setting

$$\varphi(z) = \tau(J(g,o))f(g), \tag{3.13}$$

if $z = g(o), f \in W_\tau$. Indeed, the right-hand side of (3.13) is clearly right $K-$invariant. The inner product becomes

$$(\varphi|\psi) = \int_{\mathcal{D}} \langle \tau^{-1}(J(g,o))\varphi(z)|\tau^{-1}(J(g,o))\psi(z)\rangle d_*z.$$

Since $\tau^{-1}(J(g,o))^*\tau^{-1}(J(g,o)) = \tau^{-1}(J(g,o)\overline{J(g,o)}^{-1}) = \tau^{-1}(K(z,z))$ by Lemma (3.3), we may also write

$$(\varphi|\psi) = \int_{\mathcal{D}} \langle \tau^{-1}(K(z,z))\varphi(z)|\psi(z)\rangle d_*z. \tag{3.14}$$

The G-action on the new space is given by

$$\pi_\tau(g)\varphi(z) = \tau^{-1}(J(g^{-1},z))\varphi(g^{-1}(z)), \qquad (g \in G,\ z \in \mathcal{D}). \tag{3.15}$$

Now we restrict to the closed sub-space of holomorphic maps and call the resulting Hilbert space $\mathcal{H}_\tau$. The space $\mathcal{H}_\tau$ is $\pi_\tau(G)$-invariant. We assume that $\mathcal{H}_\tau \neq \{0\}$.

The pair $(\pi_\tau, \mathcal{H}_\tau)$ is called a *vector-valued holomorphic discrete series* of G.

In a similar way we can define the anti-holomorphic discrete series. We therefore start with $\bar{\tau}$ instead of τ and take anti-holomorphic maps. Then

$$\pi_{\bar{\tau}}(g)\psi(z) = \bar{\tau}^{-1}(J(g^{-1},z))\psi(g^{-1}(z)). \tag{3.16}$$

for $\psi \in \mathcal{H}_{\bar{\tau}}$. One easily sees that $\mathcal{H}_{\bar{\tau}} = \bar{\mathcal{H}}_\tau$ and $\pi_{\bar{\tau}} = \bar{\pi}_\tau$ in the usual sense. Notice that when the representation τ is one dimensional we recover scalar holomorphic discrete series representations introduced above.

The Hilbert space $\mathcal{H}_\tau$ is known to have a reproducing (or Bergman) kernel $\mathcal{K}_\tau(z,w)$. Its definition is as follows. Set

$$E_z : \varphi \mapsto \varphi(z) \qquad (\varphi \in \mathcal{H}_\tau)$$

for $z \in \mathcal{D}$. Then $E_z : \mathcal{H}_\tau \mapsto W$ is a continuous linear operator, and $\mathcal{K}_\tau(z,w) = E_zE_w^*$, being a End$(W)$-valued kernel, holomorphic in z, anti-holomorphic in w. In more detail :

$$\langle \varphi(w)|\,\xi\rangle = \int_{\mathcal{D}} \langle \tau^{-1}(K(z,z))\varphi(z)|\,\mathcal{K}_\tau(z,w)\xi\rangle d_*z \tag{3.17}$$

for any $\varphi \in \mathcal{H}_\tau,\ \xi \in W$ and $w \in \mathcal{D}$.

Since $\mathcal{H}_\tau$ is a $G-$module, one easily gets the following transformation property for $\mathcal{K}_\tau(z,w)$:
(3.18)
$$\mathcal{K}_\tau(g(z),g(w)) = \tau(J(g,z))\mathcal{K}_\tau(z,w)\tau(\overline{J(g,w)})^{-1} \qquad (g \in G, z, w \in \mathcal{D}).$$

Now consider $H(z,w) = \mathcal{K}_\tau(z,w)\cdot\tau^{-1}(K(z,w))$.

Clearly $H(g(z),g(w)) = \tau(J(g,z))H(z,w)\tau^{-1}(J(g,z))$ for all $z,w \in \mathcal{D}$. So, setting $z = w = o, g \in K$ we see that $H(o,o)$ is a scalar operator, and hence $H(z,z) = H(o,o)$ is so. But then $H(z,w) = H(o,o)$. So, we get

$$\mathcal{K}_\tau(z,w) = c\cdot\tau(K(z,w)), \tag{3.19}$$

where c is a scalar. The same reasoning yields that π_τ is *irreducible*. Indeed, if $\mathcal{H} \subset \mathcal{H}_\tau$ is a closed invariant subspace, then $\mathcal{H}$ has a reproducing kernel, say $K_{\mathcal{H}}$ and it follows that $K_{\mathcal{H}} = c\mathcal{K}_\tau$, so either $\mathcal{H} = \{0\}$ or $\mathcal{H} = \mathcal{H}_\tau$.

Let us briefly recall the analytic realization of (some of) vector-valued holomorphic discrete series representations of G. We start with the irreducible representations of the maximal compact subgroup K which can be realized on the space of polynomials $\mathcal{P}(V)$ and which are parameterized by the weights $\underline{\mathbf{m}} = (m_1,\ldots,m_r) \in \mathbb{Z}^r$ with $m_1 \geq \cdots \geq m_r \geq 0$ and $m_1+\cdots+m_r = m = |\underline{\mathbf{m}}|$. These representations do not exhaust all irreducible representations of K but they will produce all necessary components for our further discussion.

Let V' be the dual vector space of $V \simeq \mathfrak{p}_+$. Consider the $m-th$ symmetric tensor power of V'. It is naturally identified with the space $\mathcal{P}^m(V)$ of polynomials of degree m on V. It is well known (see for instance [4, 24]) that under the K-action this space decomposes multiplicity free into a direct sum of irreducible sub-representations :

$$\mathcal{P}^m(V) = \sum_{|\underline{\mathbf{m}}|=m}^{\oplus} \mathcal{P}^{\underline{\mathbf{m}}}(V),$$

where $\mathcal{P}^{\underline{\mathbf{m}}}(V)$ are irreducible representations of K of highest weight $\underline{\mathbf{m}}$. This decomposition is often called the Kostant-Hua-Schmid formula and we refer the reader to the paper [4] for a precise description of spaces $\mathcal{P}^{\underline{\mathbf{m}}}(V)$ and the corresponding highest weight vectors $\Delta_{\underline{\mathbf{m}}}$. We denote by $P_{\underline{\mathbf{m}}}$ the orthogonal projection of $\mathcal{P}^m(V)$ onto $\mathcal{P}^{\underline{\mathbf{m}}}(V)$.

Let $h(z,w)$ be as before the canonical polynomial on $V\times V$, then according to [4], for a real ν one has

$$h^{-\nu}(z,w) = \sum_{\underline{\mathbf{m}}}(\nu)_{\underline{\mathbf{m}}}K_{\underline{\mathbf{m}}}(z,w),$$

where $K_{\underline{\mathbf{m}}}(z,w)$ is the reproducing kernel of the space $\mathcal{P}^{\underline{\mathbf{m}}}(V)$, and $(\nu)_{\underline{\mathbf{m}}}$ stands for the generalized Pochhammer symbol:

$$(\nu)_{\underline{\mathbf{m}}} = \prod_{j=1}^{r}\left(\nu - \frac{d}{2}(j-1)\right)_{m_j} = \prod_{j=1}^{r}\prod_{k=1}^{m_j}\left(\nu - \frac{d}{2}(j-1)+k-1\right).$$

Denote $\mathcal{H}_\nu(\mathcal{P}^{\underline{\mathbf{m}}}(V))$ the Hilbert space of holomorphic functions on $\mathcal{D}$ with values in $\mathcal{P}^{\underline{\mathbf{m}}}(V)$ admitting the reproducing kernel

$$h^{-\nu}(z,w)\otimes^m K^t(z,w).$$

Then, for an integer $\nu > 1+d(r-1)$ and a given weight $\underline{\mathbf{m}}$ the group G acts on its unitarily and irreducibly by

$$\pi_{\nu,\underline{\mathbf{m}}}(g)f(z) = \mathrm{Det}(Dg^{-1}(z))^\nu \left(\otimes^m (dg^{-1})^t\right)\cdot f(g^{-1}.z), \tag{3.20}$$

where $\otimes^m(dg^{-1})^t$ on $\mathcal{P}^{\underline{\mathbf{m}}}(V)$ denotes the induced action of $(dg^{-1})^t$ on V.

3.2. **Maximal degenerate series.** Let $\mathrm{Det}(g)$ be the determinant of a linear transform $g \in G(\Omega) \subset GL(V_o)$. We denote by $\chi(g)$ a particular character of this reductive Lie group given by $\chi(g) := \mathrm{Det}(g)^{\frac{r}{n}}$.

This character can be trivially extended to the whole parabolic subgroup $\overline{P}$ by $\chi(h\bar{n}) := \chi(h)$ for every $h \in G(\Omega)$, $\bar{n} \in \overline{N}$.

For every $\mu \in \mathbb{C}$ we define a character χ_μ of $\overline{P}$ by $\chi_\mu(\bar{p}) := |\chi(\bar{p})|^\mu$.

The induced representation $\pi^-_\mu = \mathrm{Ind}_{\overline{P}}^G(\chi_\mu)$ of the group G acts on the space

$$\widetilde{I}_\mu := \{f \in C^\infty(G) \mid f(g\bar{p}) = \chi_\mu(\bar{p})f(g), \forall g \in G, \bar{p} \in \overline{P}\},$$

by left translations. A pre-Hilbert structure on $\widetilde{I}_\mu$ is given by $\|f\|^2 = \int_K |f(k)|^2\,dk$, where K is the maximal compact subgroup of G associated with the Cartan involution σ, and dk is the normalized Haar measure of K.

According to the Gelfand-Naimark decomposition a function $f \in \widetilde{I}_\mu$ is determined by its restriction $f_{V_o}(x) = f(n_x)$ on $N \simeq V_o$. Let I_μ be the subspace of $C^\infty(V_o)$ of functions f_{V_o} with $f \in \widetilde{I}_\mu$. The group G acts on I_μ by:

$$\pi^-_\mu(g)f(x) = |A(g,x)|^\mu f(g^{-1}.x),\ g \in G,\ x \in V_o, \tag{3.21}$$

where $A(g,x) := \chi_\mu\big((Dg^{-1})_x\big)$. These representations are usually called the *maximal degenerate series representations* of G.

One shows that the norm of a function $f(n_x) = f_{V_o}(x) \in I_\mu$ is given by:

$$\|f\|^2 = \int_{V_o} |f_{V_o}(x)|^2 h(x,-x)^{2\Re\mu+\frac{n}{r}}\,dx, \tag{3.22}$$

where $h(z,w)$ is the *canonical polynomial* introduced above. Formula (3.22) implies that for $\Re\mu = -\frac{n}{2r}$ the space I_μ is contained in $L^2(V_o)$ and the representation π^-_μ extends as a unitary representation on $L^2(V_o)$.

Analogously the character χ can be extended to the subgroup P and one defines in a similar way the representation $\pi_\mu^+ = \mathrm{Ind}_P^G(\chi_{-\mu})$.

Following the standard procedure we introduce an intertwiner between π_μ^- and $\pi^+_{\mu-\frac{n}{r}}$. Consider the map $\widetilde{A}_\mu$ defined on $\widetilde{I}_\mu$ by

$$f \longrightarrow (\widetilde{A}_\mu f)(g) := \int_N f(gn)dn, \quad \forall g \in G, \tag{3.23}$$

where dn is a left invariant Haar measure on N. One shows that this integral converges for $\Re\mu > \frac{n}{2r_0}$.

Proposition 3.6. *For every $f \in \widetilde{I}_\mu$ the function $\widetilde{A}_\mu f$ belongs to $\widetilde{I}_{-\mu}$ and the map $\widetilde{A}_\mu$ given by (3.23) intertwines the corresponding representations of G:*

$$\widetilde{\pi}^+_{\mu-\frac{n}{r}}(g)(\widetilde{A}_\mu f) = \widetilde{A}_\mu(\widetilde{\pi}^-_\mu(g)f),\ \forall f \in \widetilde{I}_\mu,\ g \in G. \tag{3.24}$$

4. Ring structures on the holomorphic discrete series

In this section we discuss two different ring structures that one can endow on the set of holomorphic discrete series.

4.1. **Laplace transform and the point-wise product.** We start with a generalization of a result on the usual point-wise product due to A. and J. Unterberger (cf. [25] Lemma 3.1) in the case when $G = SL(2,\mathbb{R})$.

Theorem 4.1. *Let V_0 be a Euclidean Jordan algebra and T_Ω be the corresponding tube domain $V_0 + i\Omega$. Consider two real numbers ν_1 and ν_2 such that $\nu_1, \nu_2 > 1 + d(r-1) = 2\frac{n}{r} - 1$ and two functions $F_1 \in H^2_{\nu_1}(T_\Omega)$ and $F_2 \in H^2_{\nu_2}(T_\Omega)$. Then their point-wise product $F_1 \cdot F_2$ belongs to $H^2_{\nu_1+\nu_2}(T_\Omega)$.*

In order to prove this statement recall the following result ([5], Theorem XIII 1.1). Let Γ_Ω denote the Gindikin conical Γ-function.

Lemma 4.2. *Let ν be a real number, $\nu > \frac{2n}{r} - 1$. Let $L^2_\nu(\Omega)$ be the space $L^2(\Omega, \Delta(2u)^{-\nu+\frac{n}{r}}du)$. For any $f \in L^2_\nu(\Omega)$, set*

$$F(z) = (2\pi)^{-n/2} \int_\Omega f(u) \mathrm{e}^{(z|u)} du. \tag{4.1}$$

Then $F \in H^2_\nu(T_\Omega)$ and $f \mapsto F$ is a linear isomorphism from $L^2_\nu(\Omega)$ onto $H^2_\nu(T_\Omega)$. Moreover

$$\|F\|^2_\nu = \Gamma_\Omega\left(\nu - \frac{n}{r}\right)\|f\|^2_\nu.$$

Let now $F_1 \in H^2_{\nu_1}(T_\Omega)$, $F_2 \in H^2_{\nu_2}(T_\Omega)$ and let u and v correspond to F_1 and F_2 respectively by the lemma, so $u \in L^2_{\nu_1}(\Omega)$, $v \in L^2_{\nu_2}(\Omega)$. Then we shall show:

$$\|u * v\|_{\nu_1+\nu_2} \le C(\nu_1,\nu_2)\,\|u\|_{\nu_1}\,\|v\|_{\nu_2},$$

where $C(\nu_1,\nu_2)$ is a constant. This is sufficient to prove the theorem since the map $f \mapsto F$ sends convolutions to point-wise products. Observe that f is extended to V by setting it zero outside Ω. More precisely we have

Lemma 4.3. *Let $\nu_1,\nu_2 > \frac{2n}{r}-1$, $\|u\|_{\nu_1}<\infty$, $\|v\|_{\nu_2}<\infty$ for the measurable functions u and v on Ω. Set*

$$(u*v)(\tau)=\int_{\Omega\cap(\tau-\Omega)} u(\tau-\eta)\,v(\eta)d\eta \quad (\tau\in\Omega).$$

Then $(u*v)(\tau)$ exists for almost all τ, is measurable and

$$\|u*v\|_{\nu_1+\nu_2}\le C(\nu_1,\nu_2)\,\|u\|_{\nu_1}\,\|v\|_{\nu_2}.$$

We only prove the estimate, since the rest of this lemma follows from the same proof, applying Fubini's theorem at each step.

We have to give an estimate for the integral

$$\begin{aligned} I &= \int_\Omega (u*v)(\tau)\,\Delta(2\tau)^{\frac{n}{r}-\nu_1-\nu_2}\overline{w}(\tau)d\tau \\ &= \int_\Omega\int_\Omega \Delta(2(\xi+\eta))^{\frac{n}{r}-\nu_1-\nu_2}v(\eta)u(\xi)\overline{w}(\xi+\eta)\,d\xi d\eta \end{aligned}$$

under the assumption that $\|w\|_{\nu_1+\nu_2}<\infty$ $(w\in L^2_{\nu_1+\nu_2}(\Omega))$.

For any $t>0$, using the inequality

$$2|u(\xi)v(\eta)|\le t\,\Delta(2(\xi+\eta))^{\frac12(\nu_2-\nu_1)}\,|u(\xi)|^2+$$
$$t^{-1}\Delta(2(\xi+\eta))^{\frac12(\nu_1-\nu_2)}\,|v(\eta)|^2$$

we get

$$\begin{aligned} 2|I| &\le t\int_\Omega |u(\xi)|^2\,d\xi\int_{\xi+\Omega}\Delta(2\tau)^{\frac{n}{r}-\nu_1-\nu_2+\frac12(\nu_2-\nu_1)}\,|w(\tau)|\,d\tau \\ &+ t^{-1}\int_\Omega |v(\eta)|^2\,d\eta\int_{\eta+\Omega}\Delta(2\tau)^{\frac{n}{r}-\nu_1-\nu_2+\frac12(\nu_1-\nu_2)}\,|w(\tau)|\,d\tau \\ &\le \|w\|_{\nu_1+\nu_2}\Big[t\int_\Omega |u(\xi)|^2\,d\xi\,(\int_{\xi+\Omega}\Delta(2\tau)^{\frac{n}{r}-2\nu_1}\ d\tau)^{1/2} \\ &+ t^{-1}\int_\Omega |v(\eta)|^2\,d\eta\,(\int_{\eta+\Omega}\Delta(2\tau)^{\frac{n}{r}-2\nu_2}\,d\tau)^{1/2}\Big]. \end{aligned}$$

Let us compute the expression $\int_{\xi+\Omega}\Delta(2\tau)^{\frac{n}{r}-2\nu_2}d\tau$ for $\xi\in\Omega$. Set $\xi=g\cdot e$ for $g\in G(\Omega)$. The $G(\Omega)$-invariant measure on Ω is equal to $\Delta(\tau)^{-\frac{n}{r}}d\tau$, so that we get

$$\int_{\xi+\Omega}\Delta(2\tau)^{\frac{n}{r}-2\nu_1}d\tau = 2^{n-2\nu_1 r}\int_{\xi+\Omega}\Delta(\tau)^{\frac{2n}{r}-2\nu_1}\Delta(\tau)^{-\frac{n}{r}}d\tau$$

$$= 2^{n-2\nu_1 r}\int_{e+\Omega}\Delta(g\cdot\tau)^{\frac{2n}{r}-2\nu_1}\Delta(\tau)^{-\frac{n}{r}}d\tau \tag{4.2}$$

Now $\Delta(g\cdot\tau) = (\mathrm{Det}\, g)^{r/n}\,\Delta(\tau) = \Delta(g\cdot\tau)\,\Delta(\tau)$, so we obtain for the latter expression

$$= \int_{e+\Omega}\Delta(2\tau)^{\frac{n}{r}-2\nu_1}\,d\tau\;\Delta(\xi)^{\frac{2n}{r}-2\nu_1}.$$

The term $\int_{e+\Omega}\Delta(2\tau)^{\frac{n}{r}-2\nu_1}d\tau$ has finally to be computed.

We make the change of variable $\tau\mapsto\tau^{-1}$. Observe that $(e+\Omega)^{-1} = (e-\Omega)\cap\Omega$. The differential of $\tau\mapsto\tau^{-1}$ is $-P(\tau)^{-1}$ and $|\mathrm{Det}\,(-P(\tau))^{-1}| = \Delta(\tau)^{\frac{2n}{r}}$, see ([5],Prop. II 3.3 and Prop. III 4.2). So

$$\int_{e+\Omega}\Delta(2\tau)^{\frac{n}{r}-2\nu_1}d\tau = 2^{n-2\nu_1 r}\int_{(e-\Omega)\cap\Omega}\Delta(\tau)^{-\frac{n}{r}+2\nu_1}\Delta(\tau)^{-\frac{2n}{r}}d\tau$$

$$= 2^{n-2\nu_1 r}\int_{(e-\Omega)\cap\Omega}\Delta(\tau)^{-\frac{2n}{r}+2\nu_1}d\tau = 2^{n-2\nu_1 r}B_\Omega(-\frac{2n}{r}+2\nu_1,\frac{n}{r})$$

$$= 2^{n-2\nu_1 r}\frac{\Gamma_\Omega(-\frac{2n}{r}+2\nu_1)\,\Gamma_\Omega(\frac{n}{r})}{\Gamma_\Omega(-\frac{n}{r}+2\nu_1)}.$$

So we obtain

$$|I| = \Big[t\|u\|^2_{\nu_1}\Big\{2^{n-2\nu_1 r}\frac{\Gamma_\Omega(-\frac{2n}{r}+2\nu_1)\,\Gamma_\Omega(\frac{n}{r})}{\Gamma_\Omega(-\frac{n}{r}+2\nu_1)}\Big\}^{1/2}+$$

$$t^{-1}\|v\|^2_{\nu_2}\Big\{2^{n-2\nu_2 r}\frac{\Gamma_\Omega(-\frac{2n}{r}+2\nu_2)\,\Gamma_\Omega(\frac{n}{r})}{\Gamma_\Omega(-\frac{n}{r}+2\nu_2)}\Big\}^{1/2}\Big]\,\|w\|_{\nu_1+\nu_2}.$$

Taking the minimum for $t>0$, we get

$$\|u*v\|_{\nu_1+\nu_2}\le 2^{\frac{n}{r}-(\nu_1+\nu_2)\frac{r}{2}}\Big\{\frac{\Gamma_\Omega(-\frac{2n}{r}+2\nu_1)\,\Gamma_\Omega(\frac{n}{r})}{\Gamma_\Omega(-\frac{n}{r}+2\nu_1)}\Big\}^{1/4}.$$

$$\Big\{\frac{\Gamma_\Omega(-\frac{2n}{r}+2\nu_2)\,\Gamma_\Omega(\frac{n}{r})}{\Gamma_\Omega(-\frac{n}{r}+2\nu_2)}\Big\}^{1/4}\|u\|_{\nu_1}\,\|v\|_{\nu_2}.$$

Remark 4.4. If $G = SL(2,\mathbb{R})$, then $T_\Omega = \Pi$ and for every $f\in H^2_\nu(\Pi)$ $\dfrac{df}{dz}\in H^2_{\nu+2}(\Pi)$.

4.2. **Product structure on $L^2(G/H)$.** There exists a G-equivariant embedding of square-integrable functions on the causal symmetric space G/H into the composition algebra of Hilbert-Schmidt operators by means of the following diagram:

$$L^2(G/H) \hookrightarrow \pi_\mu^+ \otimes \pi_\mu^- \hookrightarrow \pi_\mu^+ \otimes \overline{\pi_\mu^+} \simeq HS(L^2(V_o), dx),$$

where dx is the usual Lebesgue measure on V_o.

The first arrow is of geometric nature and it is given by the fact that the symmetric space G/H is an open dense subset of $G/P \cap G/\bar{P}$. The last isomorphism is given by

$$L^2(V_0, dx) \otimes \overline{L^2(V_0, dx)} \simeq HS(L^2(V_0, dx).$$

This embedding gives rise to a covariant symbolic calculus on G/H.

In order to introduce the covariant symbolic calculus on G/H we start with the case of $G = SL(2, \mathbb{R})$ and $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a \in \mathbb{R}^* \right\}$.

Let P^- be the parabolic subgroup of G consisting of the lower triangular matrices

$$P^- : \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix},$$

with $c \in \mathbb{R}, a \in \mathbb{R}^*$ and let P^+ be the group of upper triangular matrices

$$P^+ : \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix},$$

with $b \in \mathbb{R}, a \in \mathbb{R}^*$. The group G acts on the sphere $S = \left\{ s \in \mathbb{R}^2 : \|s\|^2 = 1 \right\}$ and acts transitively on $\widetilde{S} = S/\sim$, where $s \sim s'$ if and only if $s = \pm s'$, by

$$g.s = \frac{g(s)}{\|g(s)\|}.$$

Clearly $\mathrm{Stab}(\widetilde{0,1}) = P^-$. So $\widetilde{S} \simeq G/P^-$. Similarly $\widetilde{S} \simeq G/P^+$: $\widetilde{S} = G.(\widetilde{1,0})$. If ds is the usual normalized surface measure on S, then

$$d(g.s) = \|g(s)\|^{-2} ds.$$

For $\mu \in \mathbb{C}$, define the character ω_μ of $P^\pm$ by

$$\omega_\mu(p) = |a|^\mu.$$

Consider $\pi_\mu^\pm = \mathrm{Ind}_{P^\pm}^G \omega_{\mp\mu}$.

Both π_μ^+ and π_μ^- can be realized on $C^\infty(\widetilde{S})$, the space of smooth functions ϕ on S satisfying

$$\phi(-s) = \phi(s), \quad (s \in S).$$

The formula for π_μ^- is

$$\pi_\mu^-(g)\phi(s) = \phi(g^{-1}.s)\|g^{-1}(s)\|^\mu.$$

Let θ be the Cartan involution of G given by $\theta(g) = {}^t g^{-1}$. Then

$$\pi_\mu^+(g)\phi(s) = \phi(\theta(g^{-1}).s)\|\theta(g^{-1})(s)\|^\mu.$$

Since here

$$\theta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = w\begin{pmatrix} a & b \\ c & d \end{pmatrix} w^{-1}$$

with $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, one has that $\pi_\mu^- \sim \pi_\mu^+$.

Let (,) denote the standard inner product on $L^2(S)$:

$$(\phi,\psi) = \int_S \phi(s)\overline{\psi(s)}ds.$$

Then this form is invariant with respect to the pairs

$$(\pi_\mu^-, \pi_{-\bar\mu-2}^-), \quad \text{and} \quad (\pi_\mu^+, \pi_{-\bar\mu-2}^+).$$

Therefore if $\Re\mu = -1$, then the representations $\pi_\mu^\pm$ are unitary, the inner product being (,).

G acts also on $\widetilde{S} \times \widetilde{S}$ by

$$g.(u,v) = (g.u, \theta(g)v). \tag{4.3}$$

This action is <u>not</u> transitive: the orbit

$$(\widetilde{S} \times \widetilde{S})_s^\# = \{(u,v) : \langle u,v\rangle \neq 0\} = G.(\widetilde{(0,1)}, \widetilde{(0,1)})$$

is dense. Moreover $(\widetilde{S} \times \widetilde{S})_s^\# \simeq G/H$.

The map

$$f \to f(u,v)|\langle u,v\rangle|^{-1+is}, \quad (s \in \mathbb{R}),$$

is a unitary G-isomorphism between $L^2(G/H)$ and

$$\pi_{-1+is}^- \hat{\otimes}_2 \pi_{-1+is}^+$$

acting on $L^2(\widetilde{S}\times\widetilde{S})$. The latter space is provided with the usual inner product.

Define the operator A_μ on $C^\infty(\widetilde{S})$ by the formula

$$A_\mu\phi(s) = \int_S |\langle s,t\rangle|^{-\mu-2}\phi(t)dt.$$

This integral is absolutely convergent for $\Re\mu < -1$, and can be analytically extended to the whole complex plane as a meromorphic function. It is easily checked that A_μ is an intertwining operator

$$A_\mu \pi_\mu^\pm(g) = \pi_{-\mu-2}^\mp(g)A_\mu.$$

The operator $A_{-\mu-2} \circ A_\mu$ intertwines $\pi_\mu^\pm$ with itself, and is therefore a scalar $c(\mu)$. It can be computed using K-types:

$$c(\mu) = \pi \frac{\Gamma\left(\frac{\mu+1}{2}\right)\Gamma\left(-\frac{\mu+1}{2}\right)}{\Gamma\left(\frac{-\mu}{2}\right)\Gamma\left(1+\frac{\mu}{2}\right)}.$$

One also shows that : $A_\mu^* = A_{\bar{\mu}}$. So that for $\mu = -1 + is$ we get (by abuse of notation):

$$c(s) = \pi \frac{\Gamma\left(\frac{is}{2}\right)\Gamma\left(-\frac{is}{2}\right)}{\Gamma\left(\frac{1-is}{2}\right)\Gamma\left(\frac{1+is}{2}\right)},$$

and moreover

$$A_{(-1+is)} \circ A^*_{(-1+is)} = c(s)I,$$

so that $\pi^{-\frac{1}{2}} \frac{\Gamma\left(\frac{1+is}{2}\right)}{\Gamma\left(\frac{is}{2}\right)} A_{(-1+is)} = d(s)A_{(-1+is)}$ is a unitary intertwiner between π^+_{-1+is} and π^-_{-1-is}.

We thus get a $\pi^-_{-1+is}\hat{\otimes}_2\bar{\pi}^-_{-1+is}$ invariant map from $L^2(G/H)$ onto $L^2(\widetilde{S} \times \widetilde{S})$ given by

$$\begin{aligned} f &\to d(s) \int_S f(u,w)|\langle u,w\rangle|^{-1+is}|\langle v,w\rangle|^{-1-is}dw \\ &= (T_s f)(u,v), \qquad s \neq 0. \end{aligned}$$

This integral does not converge: it has to be considered as obtained by analytic continuation.

Define the product $f\#_s g$ for $f, g \in L^2(G/H)$ as follows: $(T_s f)(u,v)$ is the kernel of a Hilbert-Schmidt operator $Op(f)$ on $L^2(\widetilde{S})$. Then we set:

$$Op(f\#_s g) = Op(f) \circ Op(g).$$

We then have:
- $\|f\sharp_s g\|_2 \leq \|f\|_2 \cdot \|g\|_2$.
- $Op(L_x f) = \pi^-_{-1+is}(x)Op(f)\pi^-_{-1+is}(x^{-1})$, so

$$L_x(f\#_s g) = (L_x f)\#_s(L_x g),$$

for $x \in G$.

Let us write down a formula for $f\#_s g$; we have:

$$d^{-2}(s)(f\#_s g)(u,v) \tag{4.4}$$

$$= \int_S \int_S f(u,x)g(y,v)|[u,y,x,v]|^{-1+is}d\mu(x,y), \tag{4.5}$$

where $d\mu(x,y) = |\langle x,y\rangle|^{-2}dxdy$ is a G-invariant measure on $\widetilde{S} \times \widetilde{S}$ for the G-action (4.3). Here

$$[u,y,x,v] = \frac{\langle u,x\rangle\langle y,v\rangle}{\langle u,v\rangle\langle y,x\rangle}.$$

For a generic causal symmetric space of Cayley type G/H the composition formula of two symbols $f, g \in L^2(G/H)$ is defined in a similar way. In order to keep a reasonable size of this note we just indicate the flavor of the explicit formula. Recall that G/H is para-Hermitian, $T_{eH}(G/H) \simeq V_0 \oplus V_0$, and hence functions on it can be seen as functions on $V_0 \times V_0$. Therefore $f \sharp_s g$ is given by a double integral on V_0, of f and g against an appropriate power of the quotient of four functions that are integral kernels of the intertwining operator (3.23), exactly as in (4.4).

4.3. **Product structure on $L^2(G/H)_{\mathrm{hol}}$.** One says that a symmetric space G/H has discrete series representations if the set of representations of G on minimal closed invariant subspaces of $L^2(G/H)$ is nonempty. According to a fundamental result of Flensted-Jensen [7] the discrete series for G/H is nonempty and infinite if

$$rank(G/H) = rank(K/K \cap H).$$

For a causal symmetric space of Cayley type G/H this condition is fulfilled and on can realize part of its discrete series as holomorphic discrete series representations of the group G.

More precisely assume that π is a scalar holomorphic discrete series representation of G, i.e. it acts on $\mathcal{H}_\pi \subset \mathcal{O}(D) \cap L^2(D, dm_\pi)$ where D is some symmetric domain (the image of the tube T_Ω by the Cayley transform) and where $dm_\pi(w)$ is a measure on D associated to π. In such a case the Hilbert space $\mathcal{H}_\pi$ has a reproducing kernel $K_\pi(z, w)$.

Assume that the representation π occurs as a multiplicity free closed subspace in the Plancherel formula for $L^2(G/H)$ (actually this is the case in our setting, see [15] Theorem 5.9).

Consider $\xi_\pi \in \mathcal{H}_\pi^{-\infty}$ the unique up to scalars H-fixed distribution vector associated to π (see [15] p. 142 for the definition of $\xi_\pi = \phi_\lambda(z)$). It gives rise to a continuous embedding map

$$\mathcal{J}_\pi : \mathcal{H}_\pi \hookrightarrow L^2(G/H) \subset \mathcal{D}'(G/H)$$

given for any analytic vector $v \in \mathcal{H}_\pi^\infty$ by

$$(\mathcal{J}_\pi v)(x) = \langle v, \pi(x)\xi_\pi \rangle, \qquad x \in G/H, \tag{4.6}$$

where by abusing notations we write $\pi(x)$ instead of $\pi(g)$ with $x = g.H \in G/H$.

For any fixed $w \in D$ let us define the function $v_w := K_\pi(\cdot, w)$ which is actually a real analytic vector in $\mathcal{H}_\pi$.

Consider now the following function :

$$g_w(x) := (\mathcal{J} v_w)(x), \qquad x \in G/H,\ w \in D.$$

Because of the reproducing property of the Hilbert space $\mathcal{H}_\pi$ for every $f \in \mathcal{H}_\pi$ one can write

$$f(z) = \int_D K_\pi(z,w) f(w) dm_\pi(w).$$

Furthermore, if such a function f is an analytic vector for the representation $\pi : f \in \mathcal{H}_\pi^\infty$, then

$$(\mathcal{J}_\pi f)(x) = \int_D f(w) g_w(x) dm_\pi(w).$$

Choosing an appropriate normalization in (4.6) one can get the embedding $\mathcal{J}_\pi$ isometric. Therefore the subspace generated by $g_w(x)$, $w \in D$ is a closed subspace of $L^2(G/H)$ isometric to some holomorphic discrete series representation of G. (see [15] Theorem 5.4 for the precise statement).

The dual map $\mathcal{J}_\pi^* : \mathcal{D}(G/H) \mapsto \mathcal{H}_\pi$ is defined by

$$\begin{aligned} \langle \mathcal{J}_\pi^* \phi, f\rangle &= \langle \phi, \mathcal{J}_\pi f\rangle \\ &= \int_{G/H}\int_D \phi(x)\overline{f(w)g_w(x)}dm(w)d\nu(x),\ \forall \phi \in \mathcal{D}(G/H), \end{aligned}$$

where $d\nu(x)$ denotes the invariant measure on G/H. Therefore we have,

$$(\mathcal{J}_\pi^* \phi)(w) = \int_{G/H} \phi(x)\overline{g_w(x)}d\nu(x).$$

Similar observations are valid for vector-valued holomorphic discrete series representations as well.

Define the set

$$L^2(G/H)_{\rm hol} = \bigoplus_{\pi \in \hat{G}'_{\rm hol}} \mathcal{J}_\pi(\mathcal{H}_\pi)$$

where $\hat{G}'_{\rm hol}$ denotes the set of equivalence classes of unitary irreducible holomorphic discrete series representations of G with corresponding character τ trivial on $H \cap Z$. Notice that the space $L^2(G/H)_{hol}$ decomposes multiplicity free into irreducible subspaces ([14]).

According to [15] the H-fixed distribution vector $\xi_k = \xi_{\pi_\nu}$, associated with the scalar holomorphic discrete series representation π_ν (see (3.2)) is given up to a constant by

$$\xi_k(z) = \Delta\left(\frac{\eta(z) - \bar{z}}{2i}\right)^{-\frac{\nu}{2}}, \quad z \in V_0 + i\Omega. \tag{4.7}$$

Example. If $G = SL(2,\mathbb{R})$ the holomorphic discrete series of G are only scalar and according to (3.2) act on $H^2_k(\Pi)$ $(k \geq 2, k \in \mathbb{N})$ by

$$\pi_k(g)f(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right), \qquad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The involution η is given here by $\eta(z) = -\bar{z}$, the subgroup H is isomorphic to $SO(1,1)$ and according to (4.7) $\xi_k(z) = (-\bar{z})^{-\frac{k}{2}}$ what corresponds precisely to $\frac{k}{2}$-th power of the Unterbergers generating function ([25] Prop 3.3.)

$$g_z(s,t) = \frac{s-t}{(s-\bar{z})(t-\bar{z})},$$

evaluated at the base point $(0,\infty)$ of the orbit G/H.

In this case it is well known [15] that only π_k with k **even** can be uniquely realized on $L^2(G/H)$. Therefore we have

$$L^2(G/H)_{hol} = \bigoplus_{k\,\text{even}} \mathcal{J}_k(H^2_k(\Pi)),$$

In [25] Theorem 3.6 the authors showed that the set $L^2(G/H)_{hol}$ is closed under the non commutative product $\#_s$ (4.4). They give an explicit formula for the components of $f\#_s g$ for $f \in \mathcal{J}_k(H^2_k(\Pi))$ and $g \in \mathcal{J}_\ell(H^2_\ell(\Pi))$ in terms of Rankin-Cohen brackets (1.1). The method they developed is elegant but technical and from our point of view not well adapted for generalization.

To get more insight in the product structure of $L^2(G/H)_{\text{hol}}$, we rely on some recent results by T.Kobayashi ([11],Theorem 7.4),

We are going to show that $L^2(G/H)_{hol}$ is closed under the product $\#_s$. It is, because of the continuity of the product, sufficient to show the following theorem.

Theorem 4.5. *Let $\mathcal{H}_\pi$ and $\mathcal{H}_{\pi'}$ be two irreducible closed subspaces of $L^2(G/H)_{\text{hol}}$. Then*

$$\mathcal{J}_\pi(f)\#_s\mathcal{J}_{\pi'}(g) \in L^2(G/H)_{\text{hol}}.$$

for every $f \in \mathcal{H}_\pi$ and $g \in \mathcal{H}_{\pi'}$.

Even in the case of $G = SL(2,\mathbb{R})$ this result reduces the computations of [25] in an interesting way. The proof of this theorem follows from a fairly recent result by Kobayashi saying:

Theorem 4.6. *(Kobayashi). Let π and π' be holomorphic discrete series representations of G. Then the representation*

$$\pi\hat{\otimes}_2\pi'$$

decomposes discretely into holomorphic discrete series representations of G with finite multiplicities. Moreover, $\pi\hat{\otimes}_2\pi'$ is K-admissible, i.e. every irreducible representation of K occurs in it with finite multiplicity.

In general we do not have a multiplicity free decomposition.

Now let us show our theorem. The map $f\otimes g \to \mathcal{J}_\pi(f)\#_s\mathcal{J}_{\pi'}(g)$ clearly gives rise to a $K-$ and $\mathcal{U}(\mathfrak{g})-$equivariant linear map

$$(\mathcal{H}_\pi\hat{\otimes}_2\mathcal{H}_{\pi'})^K = \mathcal{H}_\pi^K \otimes \mathcal{H}_{\pi'}^K \to L^2(G/H),$$

and thus the result follows for f and g $K-$finite, and then, by continuity of the product, for all f and g.

Example The decomposition of the tensor product of two holomorphic discrete series for $SL(2,\mathbb{R})$ was obtained by J.Repka ([22]) in full generality using the Harish-Chandra-modules techniques, and it is given by

$$\pi_n\hat{\otimes}_2\pi_m = \bigoplus_{k=0}^{\infty}\pi_{m+n+2k},$$

so that this reduces the computations in [25] even more.

At the same time V.F.Molchanov obtained the same result (decomposition of all possible tensor products of unitary irreducible representations of $SO(2,1)$) in [13]. He realized such tensor products on functions defined on the one-sheeted hyperboloid and gave all the Fourier coefficients of the positive-definite kernel that defines the Hilbert structure of these unitary representations.

In the general situation we have to consider also vector-valued holomorphic discrete series representations. Indeed, according to the theorem (4.6) and particularly to the result stated in the theorem 3.3 in [19] the tensor product of two scalar holomorphic discrete series representations decomposes multiplicity free in the direct sum of unitary irreducible vector-valued holomorphic discrete series representations:

$$\mathcal{H}_{\nu_1}\otimes\mathcal{H}_{\nu_2} = \sum_{\underline{\mathbf{m}}\geq 0}\mathcal{H}_{\nu_1+\nu_2}(\mathcal{P}^{\underline{\mathbf{m}}}(V')),$$

in the case when $\nu_1 \geq \nu_2 > 1 + d(r-1)$.

In order to understand the previous decomposition we have to identify its different ingredients.

First, we see an element of the tensor product $\mathcal{H}_{\nu_1}\otimes\mathcal{H}_{\nu_2}$ as a holomorphic function $F(z,w)$ on $D\times D$. Therefore, according to [21] Coroll. 6.26, p. 269 for any positive integer m one can write a Taylor expansion formula:

$$F(z,w) = \sum_{j=0}^{m}(F^{(j)}(z), \otimes^j(w-z)) + (F^{(m+1)}(z,w), \otimes^{m+1}(z-w)),$$

where $F^{(j)}(z)$ are $\mathcal{P}^j(V')$-valued holomorphic functions on D, $F^{(m+1)}(z,w)$ is a $\mathcal{P}^{m+1}(V')$-valued holomorphic function on $D\times D$ uniquely determined by the data of $F(z,w)$, and $(\, , \,)$ denotes the standard pairing of corresponding vector spaces.

Second, according to Peetre [18] consider an $End(V)$-valued holomorphic differential form on D defined for every fixed $w_1, w_2 \in V$ and $z \in D$ by

$$\Omega(z; , w_1, w_2) = d_z B(z, w_1) B(z, w_1)^{-1} - d_z B(z, w_2) B(z, w_2)^{-1},$$

and denote by $\omega(z; w_1, w_2)$ it trace $-\frac{r}{2n} \mathrm{tr} \Omega(z; w_1, w_2)$. The former differential form plays a crucial role in the construction of intertwining operators for tensor products.

Namely, for fixed w_1 and w_2 the expression

$$h(z, w_1)^{-\nu_1} h(z, w_2)^{-\nu_2} P_{\underline{\mathbf{m}}} \otimes^{|\underline{\mathbf{m}}|} \omega(z; w_1, w_2)$$

can be seen as an element of the space $\overline{\mathcal{H}_{\nu_1}} \otimes \overline{\mathcal{H}_{\nu_2}}$ dual of $\mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2}$. Let $\langle \, , \, \rangle$ stand for the corresponding pairing. Then the operator $I_{\underline{\mathbf{m}}}$ given by

$$I_{\underline{\mathbf{m}}}(f \otimes g)(z) = \langle h(z, \cdot)^{-\nu_1} h(z, \cdot)^{-\nu_2} P_{\underline{\mathbf{m}}} \otimes^{|\underline{\mathbf{m}}|} \omega(z; \cdot, \cdot), f \otimes g \rangle, \tag{4.8}$$

is a G-equivariant map from $(\pi_{\nu_1} \otimes \pi_{\nu_2}, \mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2})$ to the space of $\mathcal{P}^{\underline{\mathbf{m}}}(V)$-valued holomorphic functions on D seen as the representation space of $\pi_{\nu_1+\nu_2, \underline{\mathbf{m}}}$ (see (3.20).

Theorem 4.4 in [19] which is an extended version of the main theorem in [18] gives a description of this map. Summarizing and using Theorem (4.5), we get

Proposition 4.7. *Let $\nu_1 \geq \nu_2 > 1 + d(r-1)$ and $f \in \mathcal{H}_{\nu_1}$, $g \in \mathcal{H}_{\nu_2}$. Assume that $\left(\mathcal{H}_{\nu_1}^{-\infty}\right)^H$ and $\left(\mathcal{H}_{\nu_2}^{-\infty}\right)^H$ are not reduced to $\{0\}$. Then*

$$\mathcal{J}_{\pi_{\nu_1}}(f) \#_s \mathcal{J}_{\pi_{\nu_2}}(g) = \sum_{\underline{\mathbf{m}} \geq 0} c_{\underline{\mathbf{m}}, s} \mathcal{J}_{\pi_{\nu_1+\nu_2}}(B_{\underline{\mathbf{m}}, \nu_1, \nu_2}(f, g)),$$

where $c_{\underline{\mathbf{m}},s}$ are fundamental constants given by the $\#_s$ product of the reproducing kernels of the corresponding Bergman spaces $\mathcal{H}_{\nu_1}$ and $\mathcal{H}_{\nu_2}$ and $B_{\underline{\mathbf{m}}, \nu_1, \nu_2}$ is such a bi-differential operator on $\mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2}$ that

$$I_{\underline{\mathbf{m}}}(B_{\underline{\mathbf{m}}, \nu_1, \nu_2}(f, g)) = \sum_{|\underline{\mathbf{n}}| + |\underline{\mathbf{n}}'| = m} C_{|\underline{\mathbf{m}}|}^{|\underline{\mathbf{n}}|} \cdot \frac{(-1)^{|\underline{\mathbf{n}}|}}{(\nu_1)_{\underline{\mathbf{n}}} (\nu_2)_{\underline{\mathbf{n}}'}} \cdot P_{\underline{\mathbf{m}}} \left(P_{\underline{\mathbf{n}}} \partial^{|\underline{\mathbf{n}}|} f \otimes P_{\underline{\mathbf{n}}'} \partial^{|\underline{\mathbf{n}}'|} g \right), \tag{4.9}$$

with $\underline{\mathbf{n}}$ and $\underline{\mathbf{n}}'$ being all possible weights such that $|\underline{\mathbf{n}}| + |\underline{\mathbf{n}}'| = |\underline{\mathbf{m}}|$.

In case when the group under consideration is $SU(1,1)$ formula (4.9) reduces to the Rankin-Cohen brackets initially introduced for the $SL(2,\mathbb{R})$-action. The fact that the expression remains the same in both compact and non-compact realizations of the Riemannian symmetric space G/K, is due to the fact that the groups $SU(1,1)$ and $SL(2,\mathbb{R})$ are real forms of the same complex Lie group $SL(2,\mathbb{C})$ and therefore covariant differential operators on

$SU(1,1)/S(U(1)\times U(1))$ and $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ are isomorphic via the analytic continuation in the complexification of these symmetric spaces. This phenomenon holds in general for covariant differential operators on D and on T_Ω and for transvectants in particular as was noticed by Peetre [18] p. 1076.

For this reason it would be natural to call the bi-differential operators occurring in (4.9) *generalized Rankin-Cohen brackets.*

Open questions

- The construction we described is valid for holomorphic discrete series representations with spectral parameter $\nu > 1 + d(r-1)$. However it would be interesting to understand whether this can be extended to the whole Wallach set.
- A possible relationship with Vertex algebras, already mentioned in [18] and [26], were pointed out to us by I.Cherednik. It is a challenge to investigate this link.
- We have seen that the ring structure on $L^2(G/H)_{\mathrm{hol}}$ is related to the tensor product of holomorphic discrete series representations. Does it reflect, via the Tannaka-Krein duality, the existence of a certain Hopf algebra that would govern the non-commutative product $\sharp_s$?
- According to the Beilinson-Bernstein classification of $(\mathfrak{g},K)$-modules one can study representations of semi-simple Lie groups in terms of D-modules on associated flag varieties. Can the ring structure on the set of holomorphic discrete series be interpreted as a cup-product on the sheaves associated with closed H-orbits on the flag variety?

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