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A study of logarithmic corrections and universal amplitude ratios in the two-dimensional 4-state Potts model

L.N. SHCHUR^{1,2}, B. BERCHE² and P. BUTERA³

¹ Landau Institute for Theoretical Physics, Russian Academy of Sciences, Chernogolovka 142432, Russia

² Laboratoire de Physique des Matériaux, UMR CNRS 7556, Université Henri Poincaré, Nancy 1,

B.P. 239, F-54506 Vandœuvre les Nancy Cedex, France

³ Istituto Nazionale di Fisica Nucleare, Sezione di Milano-Bicocca, Piazza delle Scienze 3, 20126, Milano, Italia

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Abstract. - Monte Carlo (MC) and series expansion (SE) data for the energy, specific heat, magnetization and susceptibility of the two-dimensional 4-state Potts model in the vicinity of the critical point are analysed. The role of logarithmic corrections is discussed and an approach is proposed in order to account numerically for these corrections in the determination of critical amplitudes. Accurate estimates of universal amplitude ratios A_+/A_- , Γ_+/Γ_- , Γ_T/Γ_- and R_C^\pm are given, which arouse new questions with respect to previous works.

Introduction. The concept of universality is of fundamental importance in the theory of phase transitions. Critical exponents and critical amplitudes describe the leading singularities of physical quantities in the vicinity of the critical point,

$$M_-(\tau) \approx B(-\tau)^\beta, \quad \chi_\pm(\tau) \approx \Gamma_\pm |\tau|^{-\gamma}, \quad C_\pm(\tau) \approx \frac{A_\pm}{\alpha} |\tau|^{-\alpha}, \quad (1)$$

($\tau = (T - T_c)/T$ is the reduced temperature and the labels \pm refer to the high-temperature (HT) and low-temperature (LT) sides of the critical temperature T_c) and universal combinations of critical amplitudes [1], as well as critical exponents characterize the universality class of the model. For the Potts models with $q > 2$, in addition to the above mentioned quantities, a transverse susceptibility is defined in the LT phase.

Analytical results for the critical amplitudes for the q -state Potts models with $q = 1, 2, 3$, and 4 were obtained by Delfino and Cardy [2], using the two-dimensional scattering field theory of Chim and Zamolodchikov [3]. In the case of the 4-state Potts model, the approach of Ref. [2] leads to the universal susceptibility amplitude ratios $\Gamma_+/\Gamma_- = 4.013$ and $\Gamma_T/\Gamma_- = 0.129$. Monte Carlo (MC) simulations also reported in [4] did not confirm conclusively these predictions. Another MC study due to Caselle et al. [5] leads to $\Gamma_+/\Gamma_- = 3.14(70)$, which is below the theoretical prediction of Delfino and Cardy. More recently Enting and Guttmann

analysed new (longer) series expansions for $q = 3$ and $q = 4$ obtained by the finite lattice method [6]. Their estimates $\Gamma_+/\Gamma_- = 3.5(4)$ and $\Gamma_T/\Gamma_- = 0.11(4)$ for $q = 4$ are in slightly better agreement with the results of [2] and [4]. An analysis by differential approximants however is successful only for $q = 3$ where the corrections to scaling are represented by pure powers, but meets with some difficulty in the $q = 4$ case, in which logarithmic corrections are expected. Therefore they had to resort to a slowly convergent direct analysis of the asymptotic behaviour of the expansion coefficients.

In this letter we present accurate Monte Carlo data supplemented by a re-analysis of the extended series derived in [6]. We are essentially concerned with the universal combinations

$$\frac{A_+}{A_-}, \frac{\Gamma_+}{\Gamma_-}, \frac{\Gamma_T}{\Gamma_-}. \quad (2)$$

To the various critical amplitudes of interest A_{\pm}, \dots we have associated appropriately defined "effective amplitudes", namely temperature dependent quantities $A_{\pm}(\tau), \dots$ which take as limiting values, when $|\tau| \rightarrow 0$ the critical amplitudes A_{\pm}, \dots . By analogy we have also considered "effective ratios" of critical amplitudes which tend to universal ratios as $\tau \rightarrow 0$ and exhibit smoother behaviours in the vicinity of the critical temperature than the quantities themselves. Considering effective ratios would even eliminate logarithmic corrections from the fit in the case of 4-state Potts model in absence of regular contributions, which unfortunately do exist! We also use the self-duality relation to check explicitly the cancellation of the dominant corrections to scaling in the case of the energy density evaluated at dual temperatures.

Model and observables. The Hamiltonian of the Potts model reads as $H = -\sum_{\langle ij \rangle} \delta_{s_i s_j}$, where s_i takes integer values between 0 and $q - 1$, and the sum is restricted to the nearest-neighbor sites $\langle ij \rangle$ on the lattice. The partition function is defined by $Z = \sum_{conf} e^{-\beta H}$. On the square lattice, in zero field, the model is self-dual. The duality relation $(e^{\beta} - 1)(e^{\beta^*} - 1) = q$ determines the critical value of the inverse temperature $\beta_c = \ln(1 + \sqrt{q})$. Dual reduced temperatures τ and τ^* can be defined by $\beta = \beta_c(1 - \tau)$ and $\beta^* = \beta_c(1 + \tau^*)$.

In our simulations we use the Wolff algorithm [7] for square lattices of linear sizes $L = 20, 40, 60, 80, 100$ and 200 with periodic boundary conditions. Starting from an ordered state, we let the system equilibrate in 10^5 steps measured by the number of flipped clusters. The averages are computed over 10^6 – 10^7 steps. The data are measured in a range of reduced temperatures called the "critical window" and defined as follows: the lower limit is reached when $|\tau|^{-\nu}$ reaches the size L of the system, and the upper limit of the critical window is fixed for convenience when the corrections to scaling in the Wegner asymptotic expansion [8] do not exceed a few percent, say 2–3%, of the leading critical behaviour Eq. (1) (forgetting about the logs). This definition avoids finite size effects which would otherwise make our analysis more complex.

The order parameter of a microstate $\mathbf{M}(\mathbf{t})$ is evaluated at the time \mathbf{t} of the simulation as $\mathbf{M} = \frac{qN_m/N-1}{q-1}$, where N_m is the number of sites i with $s_i = m$ and $m \in [0, \dots, q - 1]$ is the spin value of the majority state. $N = L^2$ is the total number of spins. The thermal average is denoted $M = \langle \mathbf{M} \rangle$. Thus, the longitudinal susceptibility in the LT phase is measured by the fluctuation of the majority spin orientation $k_B T \chi_- = \langle N_m^2 \rangle - \langle N_m \rangle^2$ and the transverse susceptibility is defined in the LT phase in terms of the fluctuations of the minority of the spins $k_B T \chi_T = \frac{1}{(q-1)} \sum_{\mu \neq m} (\langle N_{\mu}^2 \rangle - \langle N_{\mu} \rangle^2)$, while in the HT phase χ_+ is given by the fluctuations in all q states, $k_B T \chi_+ = \frac{1}{q} \sum_{\mu=0}^{q-1} (\langle N_{\mu}^2 \rangle - \langle N_{\mu} \rangle^2)$, where N_{μ} is the number of sites with the spin in the state μ . This definition of the susceptibility is, in both phases, completely consistent with the available series expansion data [9]. The internal energy density of a microstate is calculated as $\mathbf{E} = -\frac{1}{N} \sum_{\langle ij \rangle} \delta_{s_i s_j}$, and its ensemble average is denoted as $E = \langle \mathbf{E} \rangle$. The specific heat measures the energy fluctuations, $(k_B T)^2 C = -\frac{\partial E}{\partial \beta} = (\langle \mathbf{E}^2 \rangle - \langle \mathbf{E} \rangle^2)$.

Our MC study of the critical amplitudes will be supplemented by a reanalysis of the HT and LT expansions recently calculated through remarkably high orders by Enting, Guttmann and coworkers [6, 10]. In terms of these series, we can compute the effective critical amplitudes for the susceptibilities, the specific heat and the magnetization and extrapolate them by the current resummation techniques, namely simple Padé approximants (PA) and differential approximants (DA) properly biased with the exactly known critical temperatures and critical exponents. The LT expansion, expressed in terms of the variable $z = \exp(-\beta)$, extends through z^{43} in the case of E , z^{59} in the case of χ_- (z^{47} for χ_T) and z^{43} for M . The HT expansions, computed in terms of the variable $v = (1 - z)/(1 + (q - 1)z)$, extend to v^{43} in the case of E , and v^{24} for the χ_+ . As a general remark on our series analysis, we may point out that the accuracy of the amplitude estimates is questionable, since the mentioned resummation methods cannot reproduce the expected logarithmic corrections to scaling and therefore the extrapolations to the critical point are uncertain. In this case we have also tested a somewhat unconventional use of DA's: in computing the effective amplitudes, we only retain DA estimates outside some small vicinity of the critical point, where they appear to be stable and reliable. Finally we perform the extrapolations by fitting these data to an asymptotic form which includes logarithmic corrections.

Logarithmic corrections. In the usual parametrization $\cos(\pi y/2) = \frac{1}{2}\sqrt{q}$ in terms of which the scaling dimensions are known, we have $y = 0$ at $q = 4$ and the second thermal exponent [11, 12] $y_{\phi_2} = -4y/3(1 - y)$ vanishes. Accordingly, the leading power-behaviour of the magnetization (and of other physical quantities) is modified [13] by a logarithmic factor

$$M_-(-|\tau|) = B|\tau|^{1/12}(-\ln|\tau|)^{-1/8}\mathcal{F}_{corr}(-\ln|\tau|), \quad (3)$$

and a correction function $\mathcal{F}_{corr}(-\ln|\tau|)$ contains terms with integer powers of $(-\ln|\tau|)$, $\ln(-\ln|\tau|)/(-\ln|\tau|), \dots$. Non-integer power corrections may also occur due to the higher (irrelevant) thermal exponents [11, 12, 14, 15] y_{ϕ_n} or to other irrelevant fields, but let us first discuss the form of the logarithmic terms. Extending the pioneering works of Cardy, Nauenberg and Scalapino (CNS) [13, 16], Salas and Sokal (SS) [17] obtained a slowly convergent expansion of $\mathcal{F}_{corr}(-\ln|\tau|)$ in logs, e.g. for the magnetization:

$$M_-(-|\tau|) = B|\tau|^{1/12}(-\ln|\tau|)^{-1/8} \left[1 - \frac{3}{16} \frac{\ln(-\ln|\tau|)}{-\ln|\tau|} + O\left(\frac{1}{\ln|\tau|}\right) \right]. \quad (4)$$

We provide below a re-examination of this and similar quantities. The non-linear RG equation for the relevant thermal and magnetic fields ϕ and h , with corresponding RG eigenvalues y_ϕ and y_h , and the marginal dilution field ψ , are given by

$$\frac{d\phi}{d\ln b} = (y_\phi + y_{\phi\psi}\psi)\phi, \quad (5)$$

$$\frac{dh}{d\ln b} = (y_h + y_{h\psi}\psi)h, \quad (6)$$

$$\frac{d\psi}{d\ln b} = g(\psi) \quad (7)$$

where b is the length rescaling factor and $l = \ln b$. The function $g(\psi)$ may be Taylor expanded, $g(\psi) = y_{\psi^2}\psi^2(1 + \frac{y_{\psi^3}}{y_{\psi^2}}\psi + \dots)$. Accounting for marginality of the dilution field, there is no linear term at $q = 4$. The first term has been considered by Nauenberg and Scalapino [16], and later by Cardy, Nauenberg and Scalapino [13]. The second term was introduced by Salas and Sokal [17]. For convenience, we slightly change the notations of SS, denoting by y_{ij} the coupling coefficients between the scaling fields i and j . These parameters take the values $y_{\phi\psi} = 3/(4\pi)$, $y_{h\psi} = 1/(16\pi)$, $y_{\psi^2} = 1/\pi$ and $y_{\psi^3} = -1/(2\pi^2)$ [17], while the relevant scaling dimensions are $y_\phi = 3/2$ and $y_h = 15/8$.

The fixed point is at $\phi = h = 0$. Starting from initial conditions ϕ_0, h_0 , the relevant fields grow exponentially with l . The field ϕ is analytically related to the temperature, so

the temperature behaviour follows from the renormalization flow from $\phi_0 \sim |\tau|$ up to some $\phi = O(1)$ outside the critical region. Notice also that the marginal field ψ remains of order $O(\psi_0)$ and ψ_0 is negative, $|\psi_0| = O(1)$. In zero magnetic field, under a change of length scale, the singular part of the free energy density transforms according to

$$f(\psi_0, \phi_0) = e^{-Dl} f(\psi, \phi), \quad (8)$$

where $D = 2$ is the space dimension. Solving Eq. (5) and (7) leads to

$$l = -\frac{1}{y_\phi} \ln x + \frac{y_{\phi\psi}}{y_\phi y_{\psi^2}} \ln z, \quad (9)$$

where $z = \frac{\psi_0}{\psi} \frac{y_{\psi^2} + y_{\psi^3} \psi}{y_{\psi^2} + y_{\psi^3} \psi_0}$ and $x = \phi_0/\phi$ (for brevity we will denote $\nu = 1/y_\phi = \frac{2}{3}$, $\mu = \frac{y_{\phi\psi}}{y_\phi y_{\psi^2}} = \frac{1}{2}$) and we deduce the following behaviour for the free energy density in zero magnetic field in terms of the thermal and dilution fields,

$$f(\phi_0, \psi_0) = x^{D\nu} z^{-D\mu} f(\phi, \psi). \quad (10)$$

The other thermodynamic properties follow from derivatives with respect to the scaling fields, e.g. $E(\phi_0, \psi_0) = \frac{\partial}{\partial \phi_0} f(\psi_0, \phi_0) = x^{D\nu-1} z^{-D\mu} E(\phi, \psi)$. What appears extremely useful is that the dependence on the quantity z cancels (due to the scaling relations among the critical exponents) in appropriate effective ratios. This quantity z is precisely the only one which contains the log terms in the 4-state Potts model, and thus we may infer that *not only the leading log terms, but all the log terms hidden in the dependence on the marginal dilution field disappear* in the conveniently defined effective ratios. Now we proceed by iterations of Eq. (9) and eventually we get for the full correction to scaling variable the *heavy* expression $z = \text{const} \times (-\ln|\tau|) \mathcal{E}(-\ln|\tau|) \mathcal{F}(-\ln|\tau|)$, where $\mathcal{E}(-\ln|\tau|)$ is a universal function

$$\mathcal{E}(-\ln|\tau|) = \left(1 + \frac{3}{4} \frac{\ln(-\ln|\tau|)}{-\ln|\tau|}\right) \left(1 - \frac{3}{4} \frac{\ln(-\ln|\tau|)}{-\ln|\tau|}\right)^{-1} \left(1 + \frac{3}{4} \frac{1}{(-\ln|\tau|)}\right) \quad (11)$$

while $\mathcal{F}(-\ln|\tau|)$ is a function of the variable $(-\ln|\tau|)$ only, *where non universality enters through the constant ψ_0* . Remember here that $x \simeq |\tau|$.

In a given range of values of the τ , the function $\mathcal{F}(-\ln|\tau|)$ should be fixed *and the only freedom is to include background terms and possibly additive corrections to scaling* coming from irrelevant scaling fields. Among the additive correction terms, we may have those of the thermal sector $\Delta_{\phi_n} = -\nu y_{\phi_n}$, where the RG eigenvalues are $y_{\phi_n} = D - \frac{1}{2}n^2$, $n = 1, 2, 3, \dots$ [12]. The first dimension $y_{\phi_1} = y_\phi = 3/2$ is the temperature RG eigenvalue. The next one is $y_{\phi_2} = 0$ and this leads to the appearance of the logarithmic corrections, such that the first Wegner irrelevant correction to scaling in the thermal sector is $\Delta_{\phi_3} = -\nu y_{\phi_3} = 5/3$. In the magnetic sector, the RG eigenvalues are given by $y_{h_n} = D - \frac{1}{8}(2n-1)^2$. The first dimension $y_{h_1} = y_h = 15/8$ is the magnetic field RG eigenvalue. The second one is still relevant, $y_{h_2} = 7/8$, and it could lead, if admissible by symmetry, to corrections generically governed by the *difference* of relevant eigenvalues $(y_{h_1} - y_{h_2})/y_\phi = 2/3$. The next contribution comes from $y_{h_3} = -9/8$ and leads to a Wegner correction-to-scaling exponent $\Delta_{h_3} = -\nu y_{h_3} = 3/4$. Eventually, spatial inhomogeneities of primary fields (higher order derivatives) bring the extra possibility of integer correction exponents $y_n = -n$ in the conformal tower of the identity. The first one of these irrelevant terms corresponds to a Wegner exponent $\Delta_1 = -\nu(-1) = 2/3$ and it is always present. We may thus possibly include the following corrections: $|\tau|^{2/3}$, $|\tau|^{3/4}$, $|\tau|^{4/3}$, $|\tau|^{5/3}$, \dots , the first and third ones being always present, while the other corrections depend on the symmetry properties of the observables.

In the Baxter-Wu model, which belongs to the 4-state Potts model universality class ¹, the magnetization obeys the asymptotic form [19, 20] $M_-(-|\tau|) = B|\tau|^{1/12}(1 + \text{const} \times |\tau|^{2/3} + \text{const}' \times |\tau|^{4/3})$. Caselle et al. [5] also fit the magnetization with a $|\tau|^{2/3}$ term.

Numerical results. We eventually deduce the behaviour of the magnetization

$$M_-(-|\tau|) = B|\tau|^{1/12}(-\ln|\tau|)^{-1/8} \left[\left(1 + \frac{3 \ln(-\ln|\tau|)}{4 - \ln|\tau|} \right) \left(1 - \frac{3 \ln(-\ln|\tau|)}{4 - \ln|\tau|} \right)^{-1} \left(1 + \frac{3}{4} \frac{1}{-\ln|\tau|} \right) \mathcal{F}(-\ln|\tau|) \right]^{-1/8} (1 + a|\tau|^{2/3} + \dots). \quad (12)$$

Note that the whole bracket corresponds to the correction function of Eq. (3). It is unsafe (for numerical purposes) to expand it, since the correction term is not small enough in the accessible temperature range $|\tau| \simeq 0.05 - 0.25$. We have thus to extract an effective function $\mathcal{F}_{eff}(-\ln|\tau|)$ which mimics the real one $\mathcal{F}(-\ln|\tau|)$ in the convenient temperature range. Defining various effective magnetization amplitudes at different levels of accuracy, namely $B_{eff}^{(1)}(-|\tau|) = M_-|\tau|^{-1/12}(-\ln|\tau|)^{1/8}$ with the CNS leading log term, $B_{eff}^{(2)}(-|\tau|) = M_-|\tau|^{-1/12}(-\ln|\tau|)^{1/8} \left(1 - \frac{3 \ln(-\ln|\tau|)}{16 - \ln|\tau|} \right)^{-1}$ with the SS correction or $B_{eff}^{(3)}(-|\tau|) = M_-|\tau|^{-1/12}[-\ln|\tau| \mathcal{E}(-\ln|\tau|)]^{1/8}$ with our universal corrections, we are unable to recover a sensible

$$B(1 + a|\tau|^{2/3} + b|\tau|^{4/3}) \quad (13)$$

behaviour. Of course, it is possible to fit the data to any of these expressions in a given range of temperatures, but the coefficients a and b thus obtained strongly depend on the temperature window and this is not acceptable. Improvement is achieved through the following type of fit (instead of Eq. (13))

$$B_{eff}^{(3)}(-|\tau|) = B \left(1 + \frac{C_1}{-\ln|\tau|} + \frac{C_2 \ln(-\ln|\tau|)}{(-\ln|\tau|)^2} \right)^{1/8} (1 + a|\tau|^{2/3}). \quad (14)$$

The function $\mathcal{F}(-\ln|\tau|)$ in Eq. (12) now takes the approximate expression $\mathcal{F}(-\ln|\tau|) \simeq \left(1 + \frac{C_1}{-\ln|\tau|} + \frac{C_2 \ln(-\ln|\tau|)}{(-\ln|\tau|)^2} \right)^{-1}$. What is remarkable is the stability of the fit to Eq. (14). Analysing MC data, we obtain (fit a) $C_1 = -0.757(1)$ and $C_2 = -0.522(11)$ which yields an amplitude $B = 1.1570(1)$. It is also possible to try a simpler choice in the narrow temperature window, fixing $C_2 = 0$ and approximating the whole series by the C_1 -term only (now $C_1 = -0.88(5)$, called fit b), which then leads to a very close magnetization amplitude $B = 1.1559(12)$. An analysis of SE data gives very similar results. By the way, in the case of the magnetization the coefficient b is found to be almost zero and we did not include it in Eq. (14) [21]. Note that these estimates follow from a coherent analysis of both MC data and SE extrapolations. The errors reported are the standard deviations resulting from the fits, since our definition of the temperature window is such that there is no finite-size-effect in the τ -range considered. For MC data we perform weighted fits (i.e. each point is weighted with the inverse statistical error of the point) while the fits of the SE data are unweighted. Our major improvement (compared to previous references using MC and/or SE data) is not in the quality of the data themselves, but in the functional form of fit employed which incorporates in an effective function (dependent on the temperature window) the effect of the *non-universal* part of the series of log terms, all universal terms being explicitly taken into account. Nevertheless, another source of error comes from the effective function itself. We can estimate this additional error in the following way: we compute the amplitudes when changing the coefficients C_1 and C_2 of an amount as large as

¹It was proposed in Ref. [16] that $\psi_0 = 0$ in the Baxter-Wu model and there are no log-corrections. Later Kinzel et al. [18] gave supporting considerations.

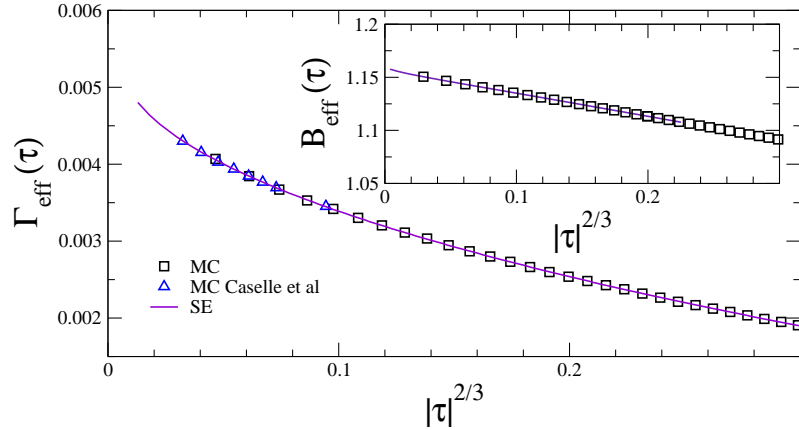


Fig. 1: Effective susceptibility amplitude (and magnetization amplitude in the inset) in the LT phase.

half to twice their optimal values reported above, leading to two estimates for the amplitudes. We arbitrarily define the difference of these estimates as the additional error. Γ_- is the only amplitude for which this error is significant.

We thus obtain a closed expression for the dominant logarithmic corrections which is more suitable than previously proposed forms to describe the temperature range accessible in a numerical study:

$$\text{Obs.}(\pm|\tau|) \simeq \text{Ampl.} \times |\tau|^\triangleleft \times [\mathcal{E}(-\ln|\tau|)\mathcal{F}(-\ln|\tau|)]^\# \times (1 + \text{Corr. terms}) + \text{Backgr. terms}, \quad (15)$$

$$\text{Corr. terms} = a|\tau|^{2/3} + b|\tau|^{4/3} + \dots, \quad (16)$$

$$\text{Backgr. terms} = D_0 + D_1|\tau| + \dots \quad (17)$$

where \triangleleft and $\#$ are exponents which depend on the observable considered, and take the values $1/12$ and $-1/8$, respectively, in the case of the magnetization. The dots represent higher order terms which theoretically do exist, but practically do not need to be included.

The susceptibility (see Fig. 1) and the energy density can also be fitted to the expression above. Our results are summarized in table 1. The efficiency of the fits relies on the asymptotic form Eq. (15) which in our opinion is based on sufficiently safe theoretical grounds. Its validity can furthermore easily be checked (indirectly) through the computation of effective amplitude ratios for which all logarithmic corrections have to cancel. A specific example is given by the leading behaviour of the energy density ratio. The values $E(\beta)$ and $E(\beta^*)$ of the internal energy at dual temperatures are related through $(1 - e^{-\beta})E(\beta) + (1 - e^{-\beta^*})E(\beta^*) = -2$. Defining the quantity $A_+/A_- = (E(\beta) - E_0)\tau^{\alpha-1}/(E_0 - E(\beta^*))(\tau^*)^{\alpha-1}$, the constant E_0 being the value of the energy at the transition temperature [22], $E_0 = E(\beta_c) = -1 - 1/\sqrt{q}$, we may expand close to the transition point $A_+(\tau)/A_-(\tau^*) = 1 + (3 - \alpha)\alpha_q\tau + O(\tau^{1+\alpha})$ with $\alpha_q = -E_0\beta_c e^{-\beta_c} = \frac{\ln(1+\sqrt{q})}{\sqrt{q}}$. This relation, checked numerically, shows that the leading corrections to scaling vanish.

The universal combinations of amplitudes follow from the results listed in table 1 and are summarized in table 2. Fits a and b in these tables refer to the two possible choices for the constants C_1 and C_2 in $\mathcal{F}(-\ln|\tau|)$ as explained above.

Conclusion. The main outcome of this work is the surprisingly high values of the ratios Γ_+/Γ_- , Γ_T/Γ_- and R_C^+ , clearly far above the predictions of Delfino and Cardy. Note that our results are also supported by a direct extrapolation of *effective amplitude ratios* for which most of the corrections to scaling disappear. In the case of the conflicting quantities, this

Table 1: Critical amplitudes in the 4-state Potts model. The amplitudes reported correspond to the estimates which follow from the analysis of MC data and of SE data with both types of fits. The second figures in parenthesis for Γ_- refer to the additional error discussed in the text.

F fit #	B	Γ_+	Γ_-	Γ_T
MC a	1.1570(1)	0.03144(15)	0.00454(2)(20)	0.00076(1)
MC b	1.1559(12)	0.03178(30)	0.00484(3)(5)	0.00073(1)
SE a	1.1575(1)	0.03041(1)	0.00483(1)(20)	0.00073(1)
SE b	1.1575(1)	0.03039(1)	0.00493(1)(5)	0.00073(1)

technique leads to $\Gamma_+/\Gamma_- = 6.6(3)$ and $6.5(1)$, and $\Gamma_T/\Gamma_- = 0.160(8)$ and $0.152(2)$, using respectively fits a and b to fit the MC data. The corresponding figures resulting from fits of SE data are $\Gamma_+/\Gamma_- = 6.30(1)$ and $6.16(1)$, and $\Gamma_T/\Gamma_- = 0.151(3)$ and $0.148(3)$. Note that the additional source of error is not taken into account in these estimates.

We believe that our fitting procedure is reliable, and since the disagreement with theoretical calculations can hardly be resolved, we suspect that the discrepancy might be attributed to the assumptions made in Ref. [2] in order to predict the susceptibility ratios. Even more puzzling is the fact that Delfino and Cardy argue in favour of a higher robustness of their results for Γ_T/Γ_- than for Γ_+/Γ_- , but the disagreement is indisputable in both cases.

Finally, in favour of our results, one may mention a work of W. Janke and one of us (LNS) on the amplitude ratios in the Baxter-Wu model (in the 4-state Potts model universality class), according to which $\Gamma_+/\Gamma_- \simeq 6.9$ [23]. These results, obtained from an analysis of MC data show a similar discrepancy with Delfino and Cardy's results and a further analysis still seems to be necessary.

* * *

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Table 2: Universal combinations of the critical amplitudes in the 4-state Potts model. In the last two rows, the figure on the left follow from MC results and that on the right from SE results.

A_+/A_-	Γ_+/Γ_-	Γ_T/Γ_-	source
1.	4.013	0.129	[2, 4]
–	3.14(70)	–	[5]
–	3.5(4)	0.11(4)	[6]
1.00(1)	6.93(6)(35) – 6.30(2)(27)	0.167(3)(9) – 0.151(2)(8)	fit a
1.00(1)	6.57(10)(13) – 6.16(1)(6)	0.151(3)(4) – 0.148(2)(4)	fit b

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