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## A study of logarithmic corrections and universal amplitude ratios in the two-dimensional 4-state Potts model

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**Abstract.** – Monte Carlo (MC) and series expansion (SE) data for the energy, specific heat, magnetization and susceptibility of the two-dimensional 4-state Potts model in the vicinity of the critical point are analysed. The role of logarithmic corrections is discussed and an approach is proposed in order to account numerically for these corrections in the determination of critical amplitudes. Accurate estimates of universal amplitude ratios  $A_+/A_-$ ,  $\Gamma_+/\Gamma_-$ ,  $\Gamma_T/\Gamma_-$  and  $R_C^\pm$  are given, which arouse new questions with respect to previous works.

*Introduction.* The concept of universality is of fundamental importance in the theory of phase transitions. Critical exponents and critical amplitudes describe the leading singularities of physical quantities in the vicinity of the critical point,

$$M_-(\tau) \approx B(-\tau)^\beta, \quad \chi_\pm(\tau) \approx \Gamma_\pm |\tau|^{-\gamma}, \quad C_\pm(\tau) \approx \frac{A_\pm}{\alpha} |\tau|^{-\alpha}, \quad (1)$$

( $\tau$  is the reduced temperature  $\tau = (T - T_c)/T$  and the labels  $\pm$  refer to the high-temperature and low-temperature sides of the critical temperature  $T_c$ ) and universal combinations of critical amplitudes [1], as well as critical exponents characterize the universality class of the model. For the Potts models with  $q > 2$ , in addition to the above mentioned quantities, a transverse susceptibility can be defined in the low-temperature phase,  $\chi_T(\tau) \approx \Gamma_T |\tau|^{-\gamma}$  and  $\chi_-$  is also usually referred to as longitudinal susceptibility  $\chi_L$ .

Analytical results for the critical amplitudes for the  $q$ -state Potts models with  $q = 1, 2, 3$ , and 4 were obtained by Delfino and Cardy [2], using the two-dimensional scattering field theory of Chim and Zamolodchikov [3]. In the case of the 4-state Potts model, the approach of Ref. [2] leads for example to the universal susceptibility amplitude ratios  $\Gamma_+/\Gamma_- = 4.013$  and  $\Gamma_T/\Gamma_- = 0.129$ . Monte Carlo (MC) simulations were also reported in [4] but the authors found that the results were not fully conclusive. Another study due to Caselle et al. [5] leads to the estimate  $\Gamma_+/\Gamma_- = 3.14(70)$ , which is far below the theoretical prediction of Delfino

and Cardy. More recently Enting and Guttmann analysed new (longer) series expansions for  $q = 3$  and  $q = 4$  obtained by the finite lattice method [6]. Their estimates  $\Gamma_+/\Gamma_- = 3.5(4)$  and  $\Gamma_T/\Gamma_- = 0.11(4)$  for  $q = 4$  are in slightly better agreement with the results of [2] and [4]. An analysis of the series by differential approximants however is successful only in the  $q = 3$  case in which the corrections to scaling are represented by pure powers, but meets with some difficulty in the  $q = 4$  case, in which logarithmic corrections are expected. Therefore they had to resort to a slowly convergent direct analysis of the asymptotic behaviour of the expansion coefficients with respect to their order.

In this letter we present accurate Monte Carlo data supplemented by a re-analysis of the extended series derived by Enting and Guttmann [6]. We are essentially concerned with the following universal combinations

$$\frac{A_+}{A_-}, \frac{\Gamma_+}{\Gamma_-}, \frac{\Gamma_T}{\Gamma_-}, R_C^+ = \frac{A_+\Gamma_+}{B^2}, R_C^- = \frac{A_-\Gamma_-}{B^2}. \quad (2)$$

For all these quantities, effective ratios are defined which exhibit smoother behaviours in the vicinity of the critical temperature than the quantities themselves. This procedure *would even eliminate* logarithmic corrections from the fit in the case of 4-state Potts model in absence of regular contributions, which unfortunately do exist! We also use the self-duality relation to check explicitly the cancellation of the dominant corrections to scaling in the case of the energy density evaluated at dual temperatures.

*Model and observables.* The Hamiltonian of the Potts model reads as  $H = -\sum_{\langle ij \rangle} \delta_{s_i s_j}$ , where  $s_i$  is a variable taking integer values between 0 and  $q - 1$ , and the sum is restricted to the nearest-neighbor sites  $\langle ij \rangle$  on the lattice. The partition function  $Z$  is defined by  $Z = \sum_{conf} e^{-\beta H}$  with  $\beta = 1/k_B T$ , and  $k_B$  the Boltzmann constant (fixed to unity). On the square lattice, in zero field, the model is self-dual. The duality relation  $(e^\beta - 1)(e^{\beta^*} - 1) = q$  determines the critical value of the inverse temperature  $\beta_c = \ln(1 + \sqrt{q})$ . Dual reduced temperatures  $\tau$  and  $\tau^*$  can be defined by  $\beta = \beta_c(1 - \tau)$  and  $\beta^* = \beta_c(1 + \tau^*)$ .

We use the Wolff algorithm [7] and work with square lattices of linear size  $L$  with periodic boundary conditions. Starting from an ordered state, we let the system equilibrate in  $10^5$  steps measured by the number of flipped Wolff clusters. The averages are computed over  $10^6$ – $10^7$  steps. We have simulated the model on square lattices with linear sizes  $L = 20, 40, 60, 80, 100$  and  $200$ . The data are measured in a range of reduced temperatures called the “critical window” and defined as follows: the lower limit is reached when  $|\tau|^{-\nu}$  reaches the size  $L$  of the system, and the upper limit of the critical window is fixed for convenience when the corrections to scaling in the Wegner asymptotic expansion [8] do not exceed a few percent, say 2–3%, of the leading critical behaviour Eq. (1) (forgetting about the logs).

The order parameter of a microstate  $\mathbf{M}(\mathbf{t})$  is evaluated at the time  $\mathbf{t}$  of the simulation as  $\mathbf{M} = \frac{qN_m/N-1}{q-1}$ , where  $N_m$  is the number of sites  $i$  with  $s_i = m$  and  $m \in [0, \dots, q - 1]$  is the spin value of the majority state.  $N = L^2$  is the total number of spins. The thermal average is denoted  $M = \langle \mathbf{M} \rangle$ . Thus, the longitudinal susceptibility in the low-temperature phase is measured by the fluctuation of the majority spin orientation  $k_B T \chi_- = \langle N_m^2 \rangle - \langle N_m \rangle^2$  and the transverse susceptibility is defined in the low-temperature phase as the fluctuations of the minority of the spins  $k_B T \chi_T = \frac{1}{(q-1)} \sum_{\mu \neq m} (\langle N_\mu^2 \rangle - \langle N_\mu \rangle^2)$ , while in the high-temperature phase  $\chi_+$  is given by the fluctuations in all  $q$  states,  $k_B T \chi_+ = \frac{1}{q} \sum_{\mu=0}^{q-1} (\langle N_\mu^2 \rangle - \langle N_\mu \rangle^2)$ , where  $N_\mu$  is the number of sites with the spin in the state  $\mu$ . Properly allowing for the finite-size effects, this definition of the susceptibility is, in both phases, completely consistent with the available series expansion data [9]. The internal energy density of a microstate is calculated as  $\mathbf{E} = -\frac{1}{N} \sum_{\langle ij \rangle} \delta_{s_i s_j}$ , and its ensemble average is denoted as  $E = \langle \mathbf{E} \rangle$ . The specific heat

measures the energy fluctuations,  $\beta^{-2}C = -\frac{\partial E}{\partial \beta} = (\langle \mathbf{E}^2 \rangle - \langle \mathbf{E} \rangle^2)$ .

Our MC study of the critical amplitudes will be supplemented by a reanalysis of the high-temperature (HT) and low-temperature (LT) expansions recently calculated through remarkably high orders by Enting, Guttmann and coworkers [10, 6]. In terms of these series, we can compute the effective critical amplitudes for the susceptibilities, the specific heat and the magnetization and extrapolate them by the current resummation techniques, namely simple Padé approximants (PA) and differential approximants (DA) properly biased with the exactly known critical temperatures and critical exponents. The LT expansion, expressed in terms of the variable  $z = \exp(-\beta)$ , extends through  $z^{43}$  in the case of the energy. The expansion of the longitudinal susceptibility extends through  $z^{59}$  ( $z^{47}$  for the transverse susceptibility) and the magnetization expansion extends through  $z^{43}$ . The HT expansions, computed in terms of the variable  $v = (1-z)/(1+(q-1)z)$ , extend to  $v^{43}$  in the case of the energy, and  $v^{24}$  for the susceptibility. As a general remark on our series analysis, we may point out that the accuracy of the amplitude estimates is questionable, since the mentioned resummation methods cannot reproduce the expected logarithmic corrections to scaling and therefore the extrapolations to the critical point are uncertain. In this case we have also tested a somewhat unconventional use of DA's: in computing the effective amplitudes, we only retain DA estimates outside some small vicinity of the critical point, where they appear to be stable and reliable. Finally we perform the extrapolations by fitting these data to an asymptotic form which includes logarithmic corrections.

*Logarithmic corrections.* In the usual parametrization  $\cos(\pi y/2) = \frac{1}{2}\sqrt{q}$  in terms of which the scaling dimensions are known, we have  $y = 0$  at  $q = 4$  and the second thermal exponent [11, 12]  $y_{\phi_2} = -4y/3(1-y)$  vanishes. Accordingly, the leading power-behaviour of the magnetization (and of other physical quantities) is modified [13] by a logarithmic factor

$$M_-(-|\tau|) = B|\tau|^{1/12}(-\ln|\tau|)^{-1/8}\mathcal{F}_{corr}(-\ln|\tau|), \quad (3)$$

and a correction function  $\mathcal{F}_{corr}(-\ln|\tau|)$  contains terms with integer powers of  $(-\ln|\tau|)$ , and  $(-\ln|\tau|)^{-1}\ln(-\ln|\tau|), \dots$ . Non-integer power corrections may also occur due to the higher (irrelevant) thermal exponents [11, 12, 14, 15]  $y_{\phi_n}$  or to other irrelevant fields, but let us first discuss the form of the logarithmic terms. Extending the pioneering works of Cardy, Nauenberg and Scalapino (CNS) [13, 16], Salas and Sokal (SS) [17] obtained a slowly convergent expansion of  $\mathcal{F}_{corr}(-\ln|\tau|)$  in logs, e.g. for the magnetization:

$$M_-(-|\tau|) = B|\tau|^{1/12}(-\ln|\tau|)^{-1/8} \left[ 1 - \frac{3}{16} \frac{\ln(-\ln|\tau|)}{-\ln|\tau|} + O\left(\frac{1}{\ln|\tau|}\right) \right]. \quad (4)$$

We provide below a re-examination of this and similar quantities. The non-linear RG equation for the relevant thermal and magnetic fields  $\phi$  and  $h$ , with corresponding RG eigenvalues  $y_\phi$  and  $y_h$ , and the marginal dilution field  $\psi$ , are given by

$$\frac{d\phi}{d\ln b} = (y_\phi + y_{\phi\psi}\psi)\phi, \quad (5)$$

$$\frac{dh}{d\ln b} = (y_h + y_{h\psi}\psi)h, \quad (6)$$

$$\frac{d\psi}{d\ln b} = g(\psi) \quad (7)$$

where  $b$  is the length rescaling factor and  $l = \ln b$ . The function  $g(\psi)$  may be Taylor expanded,  $g(\psi) = y_{\psi^2}\psi^2(1 + \frac{y_{\psi^3}}{y_{\psi^2}}\psi + \dots)$ . Accounting for marginality of the dilution field, there is no linear term at  $q = 4$ . The first term has been considered by Nauenberg and Scalapino [16],

and later by Cardy, Nauenberg and Scalapino [13]. The second term was introduced by Salas and Sokal [17]. For convenience, we slightly change the notations of Salas and Sokal, denoting by  $y_{ij}$  the coupling coefficients between the scaling fields  $i$  and  $j$ . These parameters take the values  $y_{\phi\psi} = 3/(4\pi)$ ,  $y_{h\psi} = 1/(16\pi)$ ,  $y_{\psi^2} = 1/\pi$  and  $y_{\psi^3} = -1/(2\pi^2)$  [17], while the relevant scaling dimensions are  $y_\phi = 3/2$  and  $y_h = 15/8$ .

The fixed point is at  $\phi = h = 0$ . Starting from initial conditions  $\phi_0, h_0$ , the relevant fields grow exponentially with  $l$ . The field  $\phi$  is analytically related to the temperature, so the temperature behaviour follows from the renormalization flow from  $\phi_0 \sim |\tau|$  up to some  $\phi = O(1)$  outside the critical region. Notice also that the marginal field  $\psi$  remains of order  $O(\psi_0)$  and  $\psi_0$  is negative,  $|\psi_0| = O(1)$ . In zero magnetic field, under a change of length scale, the singular part of the free energy density transforms according to

$$f(\psi_0, \phi_0) = e^{-Dl} f(\psi, \phi), \quad (8)$$

where  $D = 2$  is the space dimension. Solving Eq. (5) leads to  $\ln(\phi/\phi_0) = y_\phi l + y_{\phi\psi} \int \psi dl$  where the last integral is obtained from Eq. (7) rewritten as  $\int_0^l \psi dl = \frac{1}{y_{\psi^2}} \ln(\psi/\psi_0) + \frac{1}{y_{\psi^2}} \ln G(\psi_0, \psi)$ . Note that  $G(\psi_0, \psi)$  takes the value 1 at the level of the approximation of Ref. [13] and the value  $\frac{y_{\psi^2} + y_{\psi^3} \psi_0}{y_{\psi^2} + y_{\psi^3} \psi}$  in Ref. [17]. Since this term appears always in the same combination, we write  $z = \frac{\psi_0}{\psi} \frac{1}{G(\psi_0, \psi)}$  and in the same way we set  $x = \phi_0/\phi$ . We thus obtain

$$l = -\frac{1}{y_\phi} \ln x + \frac{y_{\phi\psi}}{y_\phi y_{\psi^2}} \ln z, \quad (9)$$

(for brevity we will denote  $\nu = 1/y_\phi = \frac{2}{3}$ ,  $\mu = \frac{y_{\phi\psi}}{y_\phi y_{\psi^2}} = \frac{1}{2}$ ) and we deduce the following behaviour for the free energy density in zero magnetic field in terms of the thermal and dilution fields,

$$f(\phi_0, \psi_0) = x^{D\nu} z^{-D\mu} f(\phi, \psi). \quad (10)$$

The other thermodynamic properties follow from derivatives with respect to the scaling fields, e.g.  $E(\phi_0, \psi_0) = \frac{\partial}{\partial \phi_0} f(\psi_0, \phi_0) = x^{D\nu-1} z^{-D\mu} E(\phi, \psi)$ . What appears extremely useful is that the dependence on the quantity  $z$  cancels (due to the scaling relations among the critical exponents) in appropriate effective ratios<sup>(1)</sup>. This quantity  $z$  is precisely the only one where the log terms are hidden in the 4-state Potts model, and thus we may infer that *not only the leading log terms*, but *all the log terms hidden in the dependence on the marginal dilution field* disappear in the conveniently defined effective ratios. Now we proceed by iterations of Eq. (9). The asymptotic solution of Eq. (7) is

$$\frac{\psi}{\psi_0} = \frac{1}{1 - \psi_0 y_{\psi^2} l} \left( 1 + \frac{y_{\psi^3}}{(y_{\psi^2})^2} \frac{\ln l}{l} + O(1/l) \right), \quad (11)$$

and eventually one gets for the full correction to scaling variable the *heavy* expression  $z = \text{const} \times (-\ln |\tau|) \mathcal{E}(-\ln |\tau|) \mathcal{F}(-\ln |\tau|)$ , where  $\mathcal{E}(-\ln |\tau|)$  is a universal function

$$\mathcal{E}(-\ln |\tau|) = \left( 1 + \frac{3 \ln(-\ln |\tau|)}{4 - \ln |\tau|} \right) \left( 1 - \frac{3 \ln(-\ln |\tau|)}{4 - \ln |\tau|} \right)^{-1} \left( 1 + \frac{3}{4} \frac{1}{(-\ln |\tau|)} \right) \quad (12)$$

while  $\mathcal{F}(-\ln |\tau|)$  is a function of the variable  $(-\ln |\tau|)$  only, *where non universality enters through the constant  $\psi_0$* . Remember here that  $x \simeq |\tau|$ .

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<sup>(1)</sup> i.e. effective ratios which eventually tend towards universal limits when  $\tau \rightarrow 0$

In a given range of values of the reduced temperature, the function  $\mathcal{F}(-\ln|\tau|)$  should be fixed *and the only freedom that we have is to include background terms and possibly additive corrections to scaling* coming from irrelevant scaling fields. Among the additive correction terms, we may have those coming from the thermal sector  $\Delta_{\phi_n} = -\nu y_{\phi_n}$ , where the RG eigenvalues are  $y_{\phi_n} = D - \frac{1}{2}n^2$ ,  $n = 1, 2, 3, \dots$  [12]. The first dimension  $y_{\phi_1} = y_\phi = 3/2$  is the temperature RG eigenvalue. The next one is  $y_{\phi_2} = 0$  and this leads to the appearance of the logarithmic corrections, such that the first Wegner irrelevant correction to scaling in the thermal sector is  $\Delta_{\phi_3} = -\nu y_{\phi_3} = 5/3$ . One can also imagine a coupling to the scaling fields of the magnetic sector. The magnetic RG eigenvalues are given by  $y_{h_n} = D - \frac{1}{8}(2n - 1)^2$ . The first dimension  $y_{h_1} = y_h = 15/8$  is the magnetic field RG eigenvalue. The second one is still relevant,  $y_{h_2} = 7/8$ , and it could lead, if admissible by symmetry, to corrections generically governed by the *difference* of relevant eigenvalues  $(y_{h_1} - y_{h_2})/y_\phi = 2/3$ . The next contribution comes from  $y_{h_3} = -9/8$  and leads to a Wegner correction-to-scaling exponent  $\Delta_{h_3} = -\nu y_{h_3} = 3/4$ . Eventually, corrections coming from spatial inhomogeneities of primary fields (higher order derivatives) bring the extra possibility of integer correction exponents  $y_n = -n$  in the conformal tower of the identity. The first one of these irrelevant terms corresponds to a Wegner exponent  $\Delta_1 = -\nu(-1) = 2/3$  and it is always present. We may thus possibly include the following corrections:  $|\tau|^{2/3}$ ,  $|\tau|^{3/4}$ ,  $|\tau|^{4/3}$ ,  $|\tau|^{5/3}$ ,  $\dots$ , the first and third ones being always present, while the other corrections depend on the symmetry properties of the observables. Linear terms in  $|\tau|$  due to non linear relevant fields [1] could also be necessary.

In fact, the inclusion of the most dominant correction in  $|\tau|^{2/3}$  appears to be sufficient. In the Baxter-Wu model, which belongs to the Potts  $q = 4$  model universality class <sup>(2)</sup>, the magnetization was shown to obey the asymptotic form [19, 20]  $M_-(-|\tau|) = B|\tau|^{1/12}(1 + \text{const} \times |\tau|^{2/3} + \text{const}' \times |\tau|^{4/3})$ , and Caselle et al. [5] also considered in the  $q = 4$  case a  $|\tau|^{2/3}$  term to fit the magnetization.

*Numerical results.* We eventually deduce the behaviour of the magnetization

$$M_-(-|\tau|) = B|\tau|^{1/12}(-\ln|\tau|)^{-1/8} \left[ \left( 1 + \frac{3}{4} \frac{\ln(-\ln|\tau|)}{-\ln|\tau|} \right) \left( 1 - \frac{3}{4} \frac{\ln(-\ln|\tau|)}{-\ln|\tau|} \right)^{-1} \left( 1 + \frac{3}{4} \frac{1}{-\ln|\tau|} \right) \mathcal{F}(-\ln|\tau|) \right]^{-1/8} (1 + a|\tau|^{2/3} + \dots). \quad (13)$$

Note that the whole bracket corresponds to the correction function of Eq. (3). Since all these log expressions are “lazy functions”, it is unsafe (for numerical purposes) to expand such terms, e.g.  $\left( 1 - \frac{3}{4} \frac{\ln(-\ln|\tau|)}{-\ln|\tau|} \right)^{-1} \simeq 1 + \frac{3}{4} \frac{\ln(-\ln|\tau|)}{-\ln|\tau|}$ , since the correction term is not small enough in the accessible temperature range  $|\tau| \simeq 0.05 - 0.25$ . We have thus to extract an effective function  $\mathcal{F}_{eff}(-\ln|\tau|)$  which mimics the real one  $\mathcal{F}(-\ln|\tau|)$  in the convenient temperature range. Defining various effective magnetization amplitudes at different levels of accuracy, namely  $B_{eff}^{(1)}(-|\tau|) = M_- \times |\tau|^{-1/12}(-\ln|\tau|)^{1/8}$  with the CNS leading log term,  $B_{eff}^{(2)}(-|\tau|) = M_- \times |\tau|^{-1/12}(-\ln|\tau|)^{1/8} \left( 1 - \frac{3}{16} \frac{\ln(-\ln|\tau|)}{-\ln|\tau|} \right)^{-1}$  with the SS correction or  $B_{eff}^{(3)}(-|\tau|) = M_- \times |\tau|^{-1/12}[-\ln|\tau|\mathcal{E}(-\ln|\tau|)]^{1/8}$  with our universal corrections, we are unable to recover a sensible  $B(1 + a|\tau|^{2/3} + b|\tau|^{4/3})$  behaviour. Of course, it is possible to fit the data to such an expression in a given range of temperatures, but the coefficients  $a$  and  $b$  thus obtained strongly depend on the temperature window and this is not acceptable. Improvement is only achieved through

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<sup>(2)</sup> It was proposed in Ref. [16] that  $\psi_0 = 0$  in the Baxter-Wu model and there are no log-corrections. Later Kinzel et al. [18] gave supporting considerations.

the following type of fit

$$B_{eff}^{(3)}(-|\tau|) = B \left( 1 + \frac{C_1}{-\ln|\tau|} + \frac{C_2 \ln(-\ln|\tau|)}{(-\ln|\tau|)^2} \right)^{1/8} (1 + a|\tau|^{2/3}). \quad (14)$$

The function  $\mathcal{F}(-\ln|\tau|)$  now takes the approximate expression

$$\mathcal{F}(-\ln|\tau|) \simeq \left( 1 + \frac{C_1}{-\ln|\tau|} + \frac{C_2 \ln(-\ln|\tau|)}{(-\ln|\tau|)^2} \right)^{-1}. \quad (15)$$

What is remarkable is the stability of the fit to Eq. (14). Analysing MC data, we obtain (fit a)  $C_1 = -0.757(1)$  and  $C_2 = -0.522(11)$  which yields an amplitude  $B = 1.1570(1)$ . It is also possible to try a simpler choice in the narrow temperature window, fixing  $C_2 = 0$  and approximating the whole series by the  $C_1$ -term only (now  $C_1 = -0.88(5)$ , called fit b), which then leads to a very close magnetization amplitude  $B = 1.1559(12)$ . An analysis of SE data gives very similar results. By the way, the coefficient  $b$  is found to be almost zero and we did not include it in Eq. (14). Note that these estimates follow from a coherent analysis of both MC data and SE extrapolations [21].

We thus obtain a closed expression for the dominant logarithmic corrections which is more suitable than previously proposed forms to describe the temperature range accessible in a numerical study:

$$\text{Obs.}(\pm|\tau|) \simeq \text{Ampl.} \times |\tau|^\triangleleft \times [\mathcal{E}(-\ln|\tau|)\mathcal{F}(-\ln|\tau|)]^\square \times (1 + \text{Corr. terms}) + \text{Backgr. terms}, \quad (16)$$

$$\text{Corr. terms} = a|\tau|^{2/3} + \dots, \quad (17)$$

$$\text{Backgr. terms} = D_0 + D_1|\tau| + \dots \quad (18)$$

where  $\triangleleft$  and  $\square$  are exponents which depend on the observable considered, and take the values  $1/12$  and  $-1/8$ , respectively, in the case of the magnetization. The dots represent higher order terms which theoretically do exist, but practically do not need to be included in the fits.

The susceptibility and the energy density can also be fitted to the expression above. Our results are summarized in table I. The fits are generally very stable, with high confidence levels. This shows that the asymptotic form of Eq. (16) we have chosen, works well and moreover in our opinion is based on sufficiently safe theoretical grounds. The validity of Eq. (16) can furthermore easily be checked (indirectly) through the computation of convenient effective amplitude ratios (e.g.  $\chi_T/\chi_-$ ) for which all logarithmic corrections have to cancel. A direct test of the cancellation of these logarithmic terms is also demonstrated by the leading behaviour of the energy density ratio. The values  $E(\beta)$  and  $E(\beta^*)$  of the internal energy at dual temperatures are related through  $(1 - e^{-\beta})E(\beta) + (1 - e^{-\beta^*})E(\beta^*) = -2$ . Defining the quantity  $\frac{A_+(\tau)}{A_-(\tau^*)} = \frac{E(\beta) - E_0}{E_0 - E(\beta^*)}$ , the constant  $E_0$  being the value of the energy at the transition temperature [22],  $E_0 = E(\beta_c) = -1 - 1/\sqrt{q}$ , we may expand close to the transition point:  $\frac{A_+(\tau)}{A_-(\tau^*)} = 1 + 2\alpha_q\tau + O(\tau^{1+\alpha})$  with  $\alpha_q = -E_0\beta_c e^{-\beta_c} = \frac{\ln(1+\sqrt{q})}{\sqrt{q}}$ . This relation, checked numerically, shows that the leading corrections to scaling vanish.

The universal combinations of amplitudes follow from the results listed in table I and are summarized in table II. Fits a and b in these tables refer to the two possible choices for the constants  $C_1$  and  $C_2$  in Eq. (15) as explained above, and the figures quoted are obtained from the average of MC and SE estimates.

*Conclusion.* The main outcome of this work is the surprisingly high values of the ratios  $\Gamma_+/\Gamma_-$ ,  $\Gamma_T/\Gamma_-$  and  $R_C^+$ , clearly far above the predictions of Delfino and Cardy.

TABLE I. – *Critical amplitudes in the 4-state Potts model. The amplitudes reported correspond to an average between the estimates which follow from the analysis of MC data and of SE data.*

fit #	$B$	$A_+$	$A_-$	$\Gamma_+$	$\Gamma_-$	$\Gamma_T$
a	1.1580(1)	1.338(3)	1.338(3)	0.03081(8)	0.00460(2)	0.00074(1)
b	1.1571(10)	1.316(9)	1.316(9)	0.03095(15)	0.00478(2)	0.00073(1)

TABLE II. – *Universal combinations of the critical amplitudes in the 4-state Potts model.*

$A_+/A_-$	$\Gamma_+/\Gamma_-$	$\Gamma_T/\Gamma_-$	$R_C^+$	$R_C^-$	source
1.	4.013	0.129	0.0204	–	[2, 4]
–	3.14(70)	–	0.021(5)	0.0068(9)	[5]
–	3.5(4)	0.11(4)	–	–	[6]
1.00(1)	6.7(4)	0.161(3)	0.0307(2)	0.00459(3)	fit a
1.00(1)	6.5(1)	0.153(3)	0.0304(4)	0.00470(6)	fit b

We believe that our fitting procedure is reliable, and since the disagreement with theoretical calculations can hardly be resolved, we suspect that the discrepancy might be attributed to the assumptions made in Ref. [2] in order to predict the susceptibility ratios.

Even more puzzling is the fact that Delfino and Cardy argue in favour of a higher robustness (reported in Ref. [6]) of their results for  $\Gamma_T/\Gamma_-$  than for  $\Gamma_+/\Gamma_-$ , but the disagreement is indisputable in both cases.

Finally, in favour of our results, one may mention a work in progress on the amplitude ratios in the Baxter-Wu model (in the 4-state Potts model universality class), according to which  $\Gamma_+/\Gamma_- \simeq 6.9$  and  $R_C^- \simeq 0.0044$  [23]. These results show a similar discrepancy with Delfino and Cardy's results and a further analysis still seems to be necessary.

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