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Lazhar Dhaouadi

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Applications of the q -Fourier Analysis to the Symmetric Moment Problem

Lazhar Dhaouadi *

Abstract

Sufficient condition for the symmetric moment problem to be determinate is given using standard methods of q -Fourier analysis. This condition is not a particular case of Carleman's criterion.

1 Introduction

For a positive measure μ on \mathbb{R} , the n th moment is defined as

$$s_n = \int x^n d\mu(x),$$

provided the integral exists. Let $\mathcal{M}^*(\mathbb{R})$ be the set of positive measures on \mathbb{R} with moment of any order and with infinite support. Given $\mu \in \mathcal{M}^*(\mathbb{R})$, then we consider V_μ the set of all $\nu \in \mathcal{M}^*(\mathbb{R})$ such that

$$s_n = \int x^n d\mu(x) = \int x^n d\nu(x) \quad \text{for } n \geq 0.$$

We say that μ is determinate if $V_\mu = \{\mu\}$, otherwise μ is indeterminate. For more detail on this subject, the reader can consult the references [1] and [2], and the references therein. We recall that a moment problem is said to be symmetric if all moments of odd order are 0. It is desirable to be able to tell whether the moment problem is determinate or indeterminate just by looking

*Institut Préparatoire aux Etudes d'Ingénieur de Bizerte (Université du 7 novembre Carthage). Route Menzel Abderrahmene Bizerte, 7021 Zarzouna, Tunisia. E-mail lazhardhaouadi@yahoo.fr

at the moment sequence $(s_n)_{n \geq 0}$. We have the following classical sufficient condition for determinacy:

The Perron's criterion

$$\limsup_{n \rightarrow \infty} \left\{ \frac{s_{2n}}{(2n)!} \right\}^{1/2n} < \infty \quad (1)$$

The Riesz's criterion

$$\liminf_{n \rightarrow \infty} \left\{ \frac{s_n}{n!} \right\}^{1/n} < \infty \quad (2)$$

The Carleman's criterion

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{s_{2n}}} = \infty \quad (3)$$

It follows easily that we have

$$(1) \Rightarrow (2) \Rightarrow (3).$$

So by Carlemans theorem (see [3]), if μ satisfy (3) then μ is determinate. In the end we have the following equivalent form of the Perron condition

$$\int_0^{\infty} e^{\alpha|x|} d\mu(x) < \infty, \quad \text{for some } \alpha > 0.$$

For $\mu \in \mathcal{M}^*(\mathbb{R})$ let (P_n) be the corresponding orthonormal polynomials determined by

$$P_n(x) = k_n x^n + \dots, \quad k_n > 0; \quad \int P_n(x) P_m(x) d\mu(x) = \delta_{nm}.$$

For an indeterminate $\mu \in \mathcal{M}^*(\mathbb{R})$, it is known that the series $\sum |P_n(z)|^2$ and $\sum |Q_n(z)|^2$ converge uniformly on compact subsets of \mathbb{C} , where (Q_n) the polynomials of the second kind

$$Q_n(x) = \int \frac{P_n(x) - P_n(y)}{x - y} d\mu(y), \quad x \in \mathbb{C}.$$

Therefore, the series

$$\begin{aligned} A(z) &= z \sum_{n=0}^{\infty} Q_n(0)Q_n(z) \\ B(z) &= -1 + z \sum_{n=0}^{\infty} Q_n(0)P_n(z) \\ C(z) &= 1 + z \sum_{n=0}^{\infty} P_n(0)Q_n(z) \\ D(z) &= z \sum_{n=0}^{\infty} P_n(0)P_n(z) \end{aligned}$$

determine entire functions. For $t \in \mathbb{R} \cup \{\infty\}$, we introduce the special discrete measures so called N-extremal of the form

$$\begin{aligned} \nu_t &= \sum_{\lambda \in \Lambda_t} m_\lambda \delta_\lambda, \quad \Lambda_t = \{z \in \mathbb{C} \mid B(z)t - D(z) = 0\}, \\ m_\lambda &= \frac{A(\lambda)t - C(\lambda)}{B'(\lambda)t - D'(\lambda)} = \left(\sum_{n=0}^{\infty} P_n^2(\lambda) \right)^{-1}, \quad \text{for } \lambda \in \Lambda_t, \end{aligned}$$

with $\Lambda_\infty = \{z \in \mathbb{C} \mid B(z) = 0\}$.

We recall the following result of M. Riesz [6]:

Let $\mu \in \mathcal{M}^*(\mathbb{R})$

1. If μ is indeterminate and $\nu \in V_\mu$ then (P_n) form an orthonormal basis of $L^2(\nu)$ if and only if ν is N-extremal.
2. If μ is determinate then (P_n) form an orthonormal basis of $L^2(\mu)$.

2 Main Results

We consider $0 < q < 1$ and we design by μ the positive measure defined by

$$d\mu(x) = \omega(x)^2 d\sigma(x) = \omega(x)^2 |x|^{2v+1} d_q x, \quad v > -1,$$

where $d_q x$ a discrete measure defined by

$$d_q x = (1 - q) \sum_{n \in \mathbb{Z}} q^n \delta_{q^n}.$$

and ω an even real function satisfies $\omega(q^n) \neq 0$ for all $n \in \mathbb{Z}$.

Let (P_n) be the corresponding orthonormal polynomials for μ , and we consider \mathbb{R}_q^+ the support of μ

$$\mathbb{R}_q^+ = \{q^n, \quad n \in \mathbb{Z}\}.$$

Theorem 1 *The sequence (P_n) form an orthonormal basis of $L^2(\mu)$ if and only if μ is determinate.*

Proof. If the sequence (P_n) form an orthonormal basis of $L^2(\mu)$ then μ is determinate or N-extremal. Suppose that μ is N-extremal, then there exist $t \in \mathbb{R} \cup \{\infty\}$ such that

$$\mu = \nu_t = \sum_{\lambda \in \Lambda_t} m_\lambda \delta_\lambda.$$

This implies $\Lambda_t = \mathbb{R}_q^+$. In the other hand Λ_t is the set of zero of entire function and $\{0\}$ is an accumulation point of Λ_t , then $\Lambda_t = \mathbb{C}$, which is absurd. ■

Proposition 1 *The sequence (P_n) form an orthonormal basis of $L^2(\mu)$ if and only if*

$$\psi_n(x) = \omega(x) P_n(x),$$

form an orthonormal basis of $L^2(\sigma)$.

Proof. It is clear that

$$\int P_n(x) P_m(x) d\mu(x) = \delta_{nm} \Leftrightarrow \int \psi_n(x) \psi_m(x) d\sigma(x) = \delta_{nm}.$$

Now, let f be a function belongs to $L^2(\sigma)$ and put $g(x) = \frac{f(x)}{\omega(x)}$, then $g \in L^2(\mu)$ and we have

$$\int g(x) P_n(x) d\mu(x) = 0 \Leftrightarrow \int f(x) \psi_n(x) d\sigma(x) = 0,$$

which lead to the result. ■

Theorem 2 Let s_n be the n th moment of σ . If

$$\lim_{n \rightarrow \infty} q^{n/4} \sqrt[2n]{s_{2n}} = 0, \quad (4)$$

then μ is determinate.

Proof. Using theorem 2, it suffice to prove that (P_n) form an orthonormal basis of $L^2(\mu)$ and by the proposition 1, this is equivalent to prove that (ψ_n) form an orthonormal basis of $L^2(\sigma)$. Let f be a function belongs to $L^2(\sigma)$ such that

$$\int f(x) \psi_n(x) d\sigma(x) = 0, \quad \forall n \in \mathbb{N}.$$

We can write $f = f^+ + f^-$, where f^+ is even and f^- is odd. This implies

$$\int f^+(x) x^{2n} \omega(x) d\sigma(x) = \int f^-(x) x^{2n+1} \omega(x) d\sigma(x) = 0, \quad \forall n \in \mathbb{N}.$$

Then

$$\int f^+(x) \omega(x) j_v(\lambda x, q^2) d\sigma(x) = \int x f^-(x) \omega(x) j_v(\lambda x, q^2) d\sigma(x) = 0, \quad \forall \lambda \in \mathbb{R}_q^+,$$

where $j_v(x, q^2)$ is the the normalized q -Bessel function of third kind defined by

$$j_v(x, q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^2, q^2)_n (q^{2v+2}, q^2)_n} x^{2n}.$$

We can exchange integral and sum because

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2, q^2)_n (q^{2v+2}, q^2)_n} \lambda^{2n} \int |f^+(x)| x^{2n} \omega(x) d\sigma(x) \\ & \leq \|f^+\|_2 \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2, q^2)_n (q^{2v+2}, q^2)_n} \lambda^{2n} \left[\int x^{4n} \omega^2(x) d\sigma(x) \right]^{1/2} \\ & \leq \|f^+\|_2 \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2, q^2)_n (q^{2v+2}, q^2)_n} \sqrt{s_{4n}} \lambda^{2n} < \infty. \end{aligned}$$

and

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2, q^2)_n (q^{2v+2}, q^2)_n} \lambda^{2n} \int |x f^-(x)| x^{2n} \omega(x) d\sigma(x) \\
& \leq \|f^-\|_2 \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2, q^2)_n (q^{2v+2}, q^2)_n} \lambda^{2n} \left[\int x^{2(2n+1)} \omega^2(x) d\sigma(x) \right]^{1/2} \\
& \leq \|f^-\|_2 \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2, q^2)_n (q^{2v+2}, q^2)_n} \sqrt{s_{2(2n+1)}} \lambda^{2n} < \infty.
\end{aligned}$$

The convergence of the above series is a simple consequence of the Cauchy root test (4).

In [4] the authors introduce the q -Bessel Fourier transform as follows

$$\mathcal{F}_{q,v} f(\lambda) = c_{q,v} \int_0^\infty f(x) j_v(\lambda x, q^2) x^{2v+1} d_q x, \quad \forall \lambda \in \mathbb{R}_q,$$

where f is an even function belongs to $L^1(\sigma)$ and he have proved that if $\mathcal{F}_{q,v} f = 0$ then $f = 0$. Using the Cauchy-Schwartz inequality we prove that $f^+ \omega$ and $x f^- \omega$ belongs to $L^1(\sigma)$, which implies that $f^+ = f^- = 0$. ■

3 Application

In [5], the Ramanujan identity was proved

$$\sum_{k \in \mathbb{Z}} \frac{z^k}{(bq^k, q)_\infty} = \frac{(bz, q/bz, q, q)_\infty}{(b, z, q/b, q)_\infty},$$

where

$$(x, q)_\infty = \prod_{i=0}^{\infty} (1 - q^i x).$$

Let $\alpha > 0$ and replace in the above sum

$$b \rightarrow -1, z \rightarrow q^\alpha z,$$

we obtain

$$\sum_{k \in \mathbb{Z}} q^{\alpha k} \frac{z^k}{(-q^k, q)_\infty} = \frac{(-q^\alpha z, -q^{-\alpha} q/z, q, q)_\infty}{(-1, q^\alpha z, -q, q)_\infty} = \frac{(-q^\alpha z, q, q)_\infty}{(-1, q^\alpha z, -q, q)_\infty} (-q^{-\alpha} q/z, q)_\infty.$$

On the other hand

$$\begin{aligned}
(-q^{-\alpha}q/z, q)_\infty &= \prod_{i=0}^{\infty} (1 + q^{i-\alpha}q/z) = \prod_{i=0}^{[\alpha]} (1 + q^{i-\alpha}q/z) \prod_{i=[\alpha]+1}^{\infty} (1 + q^{i-\alpha}q/z) \\
&= \left(\frac{q}{z}\right)^\alpha \prod_{i=0}^{[\alpha]} q^{i-\alpha} \prod_{i=0}^{[\alpha]} (1 + q^{\alpha-i}z/q) \prod_{i=[\alpha]+1}^{\infty} (1 + q^{i-\alpha}z/q) \\
&= \left(\frac{q}{z}\right)^\alpha \prod_{i=0}^{[\alpha]} (1 + q^i q^{\alpha-[\alpha]}z/q) \prod_{i=[\alpha]+1}^{\infty} (1 + q^{i-\alpha}z/q) q^{\left(\frac{[\alpha]}{2}-\alpha\right)([\alpha]+1)}.
\end{aligned}$$

This gives

$$\begin{aligned}
&\sum_{k \in \mathbb{Z}} q^{\alpha k} \frac{z^k}{(-q^k, q)_\infty} \\
&= \left[\frac{(-q^\alpha z, q, q)_\infty}{(-1, q^\alpha z, -q, q)_\infty} \prod_{i=0}^{[\alpha]} (1 + q^i q^{\alpha-[\alpha]}z/q) \prod_{i=[\alpha]+1}^{\infty} (1 + q^{i-\alpha}z/q) \right] \left(\frac{q}{z}\right)^\alpha q^{\left(\frac{[\alpha]}{2}-\alpha\right)([\alpha]+1)}.
\end{aligned}$$

Put

$$C(\alpha, q, z) = \left[\frac{(-q^\alpha z, q, q)_\infty}{(-1, q^\alpha z, -q, q)_\infty} \prod_{i=0}^{[\alpha]} (1 + q^i q^{\alpha-[\alpha]}z/q) \prod_{i=0}^{\infty} (1 + q^{i+[\alpha]-\alpha}z) \right] \left(\frac{q}{z}\right)^\alpha,$$

and

$$\sigma(\alpha) = \left(\frac{[\alpha]}{2} - \alpha\right)([\alpha] + 1)$$

Now, let $p > 0$ and replace

$$\alpha \rightarrow \frac{n}{p}, q \rightarrow q^{2p}, z \rightarrow q^{2v+2},$$

we get

$$(1 - q) \sum_{k \in \mathbb{Z}} q^{2nk} \frac{q^{2k(v+1)}}{(-q^{2pk}, q^{2p})_\infty} = (1 - q) C\left(\frac{n}{p}, q^{2p}, q^{2v+2}\right) q^{2p\sigma\left(\frac{n}{p}\right)},$$

We recall the definition of the q -exponential function

$$e(x, q) = \frac{1}{(x, q)_\infty} = \prod_{i=0}^{\infty} \frac{1}{1 - q^i x},$$

and we can write

$$s_{2n} = \int_0^\infty x^{2n} e(-x^{2p}, q^{2p}) x^{2v+1} d_q x = (1-q) C\left(\frac{n}{p}, q^{2p}, q^{2v+2}\right) q^{2p\sigma(\frac{n}{p})},$$

and then

$$\sqrt[2n]{s_{2n}} = \left[(1-q) C\left(\frac{n}{p}, q^{2p}, q^{2v+2}\right) \right]^{\frac{1}{2n}} q^{\frac{p}{n}\sigma(\frac{n}{p})}.$$

It is easy to prove that

$$\lim_{n \rightarrow \infty} \left[(1-q) C\left(\frac{n}{p}, q^{2p}, q^{2v+2}\right) \right]^{\frac{1}{2n}} = 1.$$

For the particular case

$$p = 3,$$

we have

$$\lim_{n \rightarrow \infty} \frac{3}{n} \sigma\left(\frac{n}{3}\right) + \frac{n}{4} = +\infty \Rightarrow \lim_{n \rightarrow \infty} q^{n/4} \sqrt[2n]{s_{2n}} = 0$$

and there exist $n_0 > 0$ such that for all $n > n_0$

$$\frac{3}{n} \sigma\left(\frac{n}{3}\right) + \frac{n}{4} > 0 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{\sqrt[2n]{s_{2n}}} < \infty.$$

In the end, if we denote by

$$d\mu(x) = e(-x^6, q^6) x^{2v+1} d_q x,$$

then by the Theorem 2 we see that μ is determinate, but not satisfying the Carleman's criterion.

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