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H^* -ALGEBRAS AND QUANTIZATION OF PARA-HERMITIAN SPACES

GERRIT VAN DIJK, MICHAEL PEVZNER

ABSTRACT. In the present note we describe a family of H^* -algebra structures on the set $L^2(X)$ of square integrable functions on a rank-one para-Hermitian symmetric space X .

INTRODUCTION

Let X be a para-Hermitian symmetric space of rank one. It is well-known that X is isomorphic (up to a covering) to the quotient space $SL(n, \mathbb{R})/GL(n-1, \mathbb{R})$, see [4] for more details. We shall thus assume throughout this note that $X = G/H$, where $G = SL(n, \mathbb{R})$ and $H = GL(n-1, \mathbb{R})$.

The space X allows the definition of a covariant symbolic calculus that generalizes the so-called convolution-first calculus on \mathbb{R}^2 , see ([2, 7, 8]) for instance. Such a calculus, or quantization map Op_σ , from the set of functions on X , called symbols, onto the set of linear operators acting on the representation space of the maximal degenerate series $\pi_{-\frac{n}{2}+i\sigma}$ of the group G , induces a non-commutative algebra structure on the set of symbols, that we suppose to be square integrable. On the other hand, the taking of the adjoint of an operator in such a calculus defines an involution on symbols. It turns out that these two data give rise to a H^* -algebra structure on $L^2(X)$.

According to the general theory, ([1, 5, 6]), every H^* -algebra is the direct orthogonal sum of its closed minimal two-sided ideals which are simple H^* -algebras. The main result of this note is the explicit description of such a decomposition for the Hilbert algebra $L^2(X)$ and its commutative subalgebra of $SO(n, \mathbb{R})$ -invariants.

1. DEFINITIONS AND BASIC FACTS

1.1. H^* -algebras.

Definition 1.1. *A set R is called a H^* -algebra (or Hilbert algebra) if*

- (1) R is a Banach algebra with involution;
- (2) R is a Hilbert space;
- (3) the norm on the algebra R coincides with the norm on the Hilbert space R ;
- (4) For all $x, y, z \in R$ one has $(xy, z) = (y, x^*z)$;
- (5) For all $x \in R$ one has $\|x^*\| = \|x\|$;
- (6) $xx^* \neq 0$ for $x \neq 0$.

An example of a Hilbert algebra is the set of Hilbert-Schmidt operators $HS(I)$ that one can identify with the set of all matrices $(a_{\alpha\beta})$, where α, β belong to a fixed set of indices I , satisfying the condition $\sum_I |a_{\alpha\beta}|^2 < \infty$.

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Theorem 1.2. [6], p. 331. *Every Hilbert algebra is the direct orthogonal sum of its closed minimal two-sided ideals, which are simple Hilbert algebras.*

Every simple Hilbert algebra is isomorphic to some algebra $HS(I)$ of Hilbert-Schmidt operators.

Definition 1.3. [5], p. 101 *An idempotent $e \in R$ is said to be irreducible if it cannot be expressed as a sum $e = e_1 + e_2$ with e_1, e_2 idempotents which annihilate each other: $e_1e_2 = e_2e_1 = 0$.*

Lemma 1.4. [5], p. 102. *A subset I of a Hilbert algebra R is a minimal left (right) ideal if and only if it is of the form $I = R \cdot e$ ($I = e \cdot R$), where e is an irreducible self-adjoint idempotent. Moreover $e \cdot R \cdot e$ is isomorphic to the set of complex numbers and R is spanned by its minimal left ideals.*

Observe that any minimal left ideal is closed, since it is of the form $R \cdot e$.

Corollary 1.5. *If R is a commutative Hilbert algebra, then any minimal left (or right) ideal is one-dimensional.*

1.2. An algebra structure on $L^2(X)$. Let $G = SL(n, \mathbb{R})$, $H = GL(n-1, \mathbb{R})$, $K = SO(n)$ and $M = SO(n-1)$. We consider H as a subgroup of G , consisting of the matrices of the form $\begin{pmatrix} (\det h)^{-1} & 0 \\ 0 & h \end{pmatrix}$ with $h \in GL(n-1, \mathbb{R})$.

Let P^- be the parabolic subgroup of G consisting of $1 \times (n-1)$ lower block matrices $P = \begin{pmatrix} a & 0 \\ c & A \end{pmatrix}$, $a \in \mathbb{R}^*$, $c \in \mathbb{R}^{n-1}$ and $A \in GL(n-1, \mathbb{R})$ such that $a \cdot \det A = 1$. Similarly,

let P^+ be the group of upper block matrices $P = \begin{pmatrix} a & b \\ 0 & A \end{pmatrix}$ $a \in \mathbb{R}^*$, $b \in \mathbb{R}^{n-1}$ and $A \in GL(n-1, \mathbb{R})$ such that $a \cdot \det A = 1$.

The group G acts on the sphere $S = \{s \in \mathbb{R}^n, \|s\|^2 = 1\}$ and acts transitively on the set $\tilde{S} = S / \sim$, where $s \sim s'$ if and only if $s = \pm s'$, by $g \cdot s = \frac{g(s)}{\|g(s)\|}$, where $g(s)$ denotes the linear action of G on \mathbb{R}^n . Clearly the stabilizer of the equivalence class of the first basis vector \tilde{e}_1 is the group P^+ , thus $\tilde{S} \simeq G/P^+$. If ds is the usual normalized surface measure on S , then $d(g \cdot s) = \|g(s)\|^{-n} ds$.

For $\mu \in \mathbb{C}$, define the character ω_μ of P^\pm by $\omega_\mu(P) = |a|^\mu$. Consider the induced representations $\pi_\mu^\pm = \text{Ind}_{P^\pm \omega_\mp \mu}^G$.

Both π_μ^+ and π_μ^- can be realized on $C^\infty(\tilde{S})$, the space of even smooth functions ϕ on S . This action is given by

$$\pi_\mu^+(g)\phi(s) = \phi(g^{-1} \cdot s) \cdot \|g^{-1}(s)\|^\mu.$$

Let θ be the Cartan involution of G given by $\theta(g) = {}^t g^{-1}$. Then

$$\pi_\mu^-(g)\phi(s) = \phi(\theta(g^{-1}) \cdot s) \cdot \|\theta(g^{-1})(s)\|^\mu.$$

Let $(,)$ denote the usual inner product on $L^2(S)$: $(\phi, \psi) = \int_S \phi(s) \bar{\psi}(s) ds$. Then this sesqui-linear form is invariant with respect to the pairs of representations $(\pi_\mu^+, \pi_{-\mu-n}^+)$ and $(\pi_\mu^-, \pi_{-\mu-n}^-)$. Therefore the representations π_μ^\pm are unitary for $\text{Re } \mu = -\frac{n}{2}$.

The group G acts also on $\tilde{S} \times \tilde{S}$ by

$$(1) \quad g(u, v) = (g \cdot u, \theta(g)v).$$

This action is not transitive: the orbit $(\tilde{S} \times \tilde{S})^\circ = G \cdot (\tilde{e}_1, \tilde{e}_1) = \{(u, v) : \langle u, v \rangle \neq 0\} / \sim$ is dense (here \langle , \rangle denotes the canonical inner product on \mathbb{R}^n). Moreover $(\tilde{S} \times \tilde{S})^\circ \simeq X$.

The map $f \mapsto f(u, v)|\langle u, v \rangle|^{-\frac{n}{2}+i\sigma}$, with $\sigma \in \mathbb{R}$ is a unitary G -isomorphism between $L^2(X)$ and $\pi_{-\frac{n}{2}+i\sigma}^+ \widehat{\otimes}_2 \pi_{-\frac{n}{2}+i\sigma}^-$ acting on $L^2(\tilde{S} \times \tilde{S})$. The latter space is provided with the usual inner product.

Define the operator A_μ on $C^\infty(\tilde{S})$ by the formula :

$$A_\mu \phi(s) = \int_S |\langle s, t \rangle|^{-\mu-n} \phi(t) dt.$$

This integral converges absolutely for $\text{Re } \mu < -1$ and can be analytically extended to the whole complex plane as a meromorphic function of μ . It is easily checked that A_μ is an intertwining operator, that is, $A_\mu \pi_\mu^\pm(g) = \pi_{-\mu-n}^\mp(g) A_\mu$.

The operator $A_{-\mu-n} \circ A_\mu$ intertwines the representation π_μ^\pm with itself and is therefore a scalar $c(\mu)\text{Id}$ depending only on μ . It can be computed using K -types.

Let $e(\mu) = \int_S |\langle s, t \rangle|^{-\mu-n} dt$, then $c(\mu) = e(\mu)e(-\mu-n)$. But on the other hand side $e(\mu) = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}} \frac{\Gamma(\frac{-\mu-n+1}{2})}{\Gamma(-\frac{n}{2})}$. One also shows that $A_\mu^* = A_{\bar{\mu}}$. So that, for $\mu = -\frac{n}{2} + i\sigma$ we get (by abuse of notations):

$$c(\sigma) = \frac{\Gamma(\frac{n}{2})^2}{\pi} \cdot \frac{\Gamma\left(\frac{-n/2-i\sigma+1}{2}\right) \Gamma\left(\frac{-n/2+i\sigma+1}{2}\right)}{\Gamma\left(\frac{n/2+i\sigma}{2}\right) \Gamma\left(\frac{-n/2-i\sigma}{2}\right)},$$

and moreover $A_{-\frac{n}{2}+i\sigma} \circ A_{-\frac{n}{2}+i\sigma}^* = c(\sigma)\text{Id}$, so that the operator $d(\sigma)A_{-\frac{n}{2}+i\sigma}$, where $d(\sigma) = \frac{\sqrt{\pi}}{\Gamma(\frac{n}{2})} \frac{\Gamma(\frac{n/2+i\sigma}{2})}{\Gamma(\frac{-n/2+i\sigma+1}{2})}$ is a unitary intertwiner between $\pi_{-\frac{n}{2}+i\sigma}^-$ and $\pi_{-\frac{n}{2}-i\sigma}^+$.

We thus get a $\pi_{-\frac{n}{2}+i\sigma}^+ \widehat{\otimes}_2 \pi_{-\frac{n}{2}+i\sigma}^+$ invariant map from $L^2(X)$ onto $L^2(\tilde{S} \times \tilde{S})$ given by

$$f \mapsto d(\sigma) \int_S f(u, w)|\langle u, w \rangle|^{-\frac{n}{2}+i\sigma} |\langle v, w \rangle|^{-\frac{n}{2}-i\sigma} dw =: (T_\sigma f)(u, v), \forall \sigma \neq 0.$$

This integral does not converge absolutely, it must be considered as obtained by analytic continuation.

Definition 1.6. *A symbolic calculus on X is a linear map $Op_\sigma : L^2(X) \rightarrow \mathcal{L}(L^2(\tilde{S}))$ such that for every $f \in L^2(X)$ the function $(T_\sigma f)(u, v)$ is the kernel of the Hilbert-Schmidt operator $Op_\sigma(f)$ acting on $L^2(\tilde{S})$.*

Definition 1.7. *The product $\#_\sigma$ on $L^2(X)$ is defined by*

$$Op_\sigma(f \#_\sigma g) = Op_\sigma(f) \circ Op_\sigma(g), \forall f, g \in L^2(X).$$

We thus have

- The product $\#_\sigma$ is associative.
- $\|f \#_\sigma g\|_2 \leq \|f\|_2 \cdot \|g\|_2$, for all $f, g \in L^2(X)$.
- $Op_\sigma(L_x f) = \pi_{-\frac{n}{2}+i\sigma}^+(x) Op_\sigma(f) \pi_{-\frac{n}{2}+i\sigma}^+(x^{-1})$, so $L_x(f \#_\sigma g) = (L_x f) \#_\sigma (L_x g)$, for all $x \in G$, where L_x denotes the left translation by $x \in G$ on $L^2(X)$.

This non-commutative product can be described explicitly:

$$(2) \quad (f \#_\sigma g)(u, v) = d(\sigma) \int_S \int_S f(u, x)g(y, v)|[u, y, x, v]|^{-\frac{n}{2}+i\sigma} d\mu(x, y),$$

where $d\mu(x, y) = |\langle x, y \rangle|^{-n} dx dy$ is a G -invariant measure on $\tilde{S} \times \tilde{S}$ for the G -action (1), and $[u, y, x, v] = \frac{\langle u, x \rangle \langle y, v \rangle}{\langle u, v \rangle \langle x, y \rangle}$.

On the space $L^2(X)$ there exists an (family of) involution $f \rightarrow f^*$ given by : $Op_\sigma(f^*) =: Op_\sigma(f)^*$. Notice that the correspondance $f \rightarrow Op_\sigma(f^*)$ is what one calls in pseudo-differential analysis "anti-standard symbolic calculus". The link between symbols of standard and anti-standard calculus in the setting of the para-Hermitian symmetric space X has been made explicit in [7] Corollary 1.4, see also Section 3.

Obviously we have $(f \#_\sigma g)^* = g^* \#_\sigma f^*$ and with the above product and involution, the Hilbert space $L^2(X)$ becomes a Hilbert algebra.

2. THE STRUCTURE OF THE SUBALGEBRA OF K -INVARIANT FUNCTIONS IN $L^2(X)$

Let \mathcal{A} be the subspace of all K -invariant functions in $L^2(X)$.

Theorem 2.1. *The subset \mathcal{A} is a closed subalgebra of $L^2(X)$ with respect to the product $\#_\sigma$.*

This statement clearly follows from the covariance of the symbolic calculus Op_σ , namely: $L_x(f \#_\sigma g) = (L_x f) \#_\sigma (L_x g)$, for all $x \in G, f, g \in L^2(X)$.

Theorem 2.2. *Let $n > 2$, then the subalgebra \mathcal{A} is commutative.*

Proof. For a function $f \in L^2(X)$ we set $\check{f}(u, v) = f(v, u)$. The map $f \rightarrow \check{f}$ is a linear involution. Indeed,

$$(f \#_\sigma g)(u, v) = d(\sigma) \int_S \int_S \check{f}(x, u) \check{g}(v, y) |[u, y, x, v]|^{-\frac{n}{2} + i\sigma} d\mu(x, y).$$

Permuting x and y and u and v respectively, we get

$$(f \#_\sigma g)(v, u) = d(\sigma) \int_S \int_S \check{g}(u, x) \check{f}(y, v) |[v, x, y, u]|^{-\frac{n}{2} + i\sigma} d\mu(x, y).$$

But $|[v, x, y, u]| = |[u, y, x, v]|$, therefore $(f \#_\sigma g)^\check{ } = \check{g} \#_\sigma \check{f}$.

On the other hand, given a couple $(u, v) \in \tilde{S} \times \tilde{S}$ there exists an element $k \in K$ such that $k.(u, v) = (v, u)$. Geometrically k can be seen as a rotation of angle $\pi[2\pi]$ around the axis defined by the bisectrix of vectors u and v in the plane they generate. Of course, such a k exists for an arbitrary couple (u, v) only if $n > 2$.

Hence for every $f \in \mathcal{A}$ we have $f = \check{f}$ and therefore $f \#_\sigma g = g \#_\sigma f$, for $f, g \in \mathcal{A}$. \square

3. IRREDUCIBLE SELF-ADJOINT IDEMPOTENTS OF \mathcal{A}

We begin with a **reduction theorem** for the multiplication and involution in $L^2(X)$.

As usual, we shall identify $L^2(X)$ with $L^2(\tilde{S} \times \tilde{S}; |\langle x, y \rangle|^{-n} dx dy)$. If $\phi \in L^2(X)$ we shall write $\phi(u, v) = |\langle u, v \rangle|^{n/2 - i\sigma} \phi_o(u, v)$. Then $\phi_o \in L^2(\tilde{S} \times \tilde{S}; ds dt) = L^2(\tilde{S} \times \tilde{S})$, and therefore the map $\phi \rightarrow \phi_o$ is an isomorphism.

Theorem 3.1. *Under the isomorphism $\phi \rightarrow \phi_o$ the product $\#_\sigma$ translates into*

$$\phi_o \#_\sigma \psi_o(u, v) = d(\sigma) \int_S \int_S \phi_o(u, x) \psi_o(y, v) |\langle x, y \rangle|^{-n/2 - i\sigma} dx dy$$

and the involution becomes:

$$\phi_o^*(u, v) = \overline{d(\sigma)}^2 \int_S \int_S \bar{\phi}_o(x, y) (|\langle x, v \rangle| |\langle u, y \rangle|)^{-n/2 + i\sigma} dx dy.$$

The proof is straightforward. So we have translated the algebra structure of $L^2(X)$ to $L^2(\tilde{S} \times \tilde{S})$.

Let ϕ be an irreducible self-adjoint idempotent in \mathcal{A} . We shall give an explicit formula for the ϕ_σ -component of ϕ .

Consider the decomposition of the space $L^2(\tilde{S}) = \bigoplus_{\ell \in 2\mathbb{N}} V_\ell$, where V_ℓ is the space of harmonic polynomials on \mathbb{R}^n , homogeneous of even degree ℓ .

Then the space $L^2(\tilde{S} \times \tilde{S})$ decomposes into a direct sum of tensor products $\bigoplus_{\ell, m \in 2\mathbb{N}} V_\ell \otimes \bar{V}_m$ and consequently $L^2_K(\tilde{S} \times \tilde{S}) = \bigoplus_{\ell \in 2\mathbb{N}} (V_\ell \otimes \bar{V}_\ell)^K$, where the sub(super-)script K means: “the K -invariants in”.

Let $\dim V_\ell = d$ and f_1, \dots, f_d be an orthonormal basis of V_ℓ . Then the function $\theta_\ell(u, v) = \sum_{i=1}^d f_i(u) \bar{f}_i(v)$, that is the reproducing kernel of V_ℓ , is, up to a constant, the K -invariant element of $V_\ell \otimes \bar{V}_\ell$.

Theorem 3.2. *Let $\phi(u, v) = |\langle u, v \rangle|^{n/2-i\sigma} \phi_\sigma(u, v)$ be an irreducible self-adjoint idempotent in \mathcal{A} . Then there exist complex numbers $c(\sigma, \ell)$ such that for any $\ell \in 2\mathbb{N}$ one has*

$$\phi_\sigma(u, v) = c(\sigma, \ell) \theta_\ell(u, v).$$

For different ℓ and ℓ' the idempotents annihilate each other. Moreover they span \mathcal{A} .

Proof. Firstly we shall show that θ_ℓ satisfies the condition

$$\theta_\ell \#_\sigma \theta_\ell = a(\sigma, \ell) \theta_\ell$$

for some constant $a(\sigma, \ell)$. Indeed,

$$\begin{aligned} d(\sigma) \int_S \int_S \theta_\ell(u, x) \theta_\ell(y, v) |\langle x, y \rangle|^{-\frac{n}{2}-i\sigma} dx dy \\ = d(\sigma) e_\ell(\sigma) \int_S \theta_\ell(u, y) \theta_\ell(y, v) dy = d(\sigma) e_\ell(\sigma) \theta_\ell(u, v) \end{aligned}$$

by the intertwining relation (apply $A_{-\frac{n}{2}+i\sigma}$ to $\theta_\ell(\cdot, x)$):

$$\int_S \theta_\ell(u, x) |\langle x, y \rangle|^{-\frac{n}{2}-i\sigma} dx = e_\ell(\sigma) \theta_\ell(u, y)$$

where $e_\ell(\sigma) = \int_S \frac{\theta_\ell(e_1, x)}{\theta_\ell(e_1, e_1)} |x_1|^{-\frac{n}{2}-i\sigma} dx$.

Observe that $\frac{\theta_\ell(e_1, x)}{\theta_\ell(e_1, e_1)}$ is a spherical function on \tilde{S} with respect to M of the form $a_\ell C_\ell^{\frac{n-2}{2}}(|x_1|)$ where $C_\ell^{\frac{n-2}{2}}(u)$ is a Gegenbauer polynomial and

$$a_\ell^{-1} = C_\ell^{\frac{n-2}{2}}(1) = 2^\ell \frac{\Gamma(\frac{n-2}{2} + \ell)}{\Gamma(\frac{n-2}{2}) \ell!}.$$

See for instance [9], Chapter IX, §3. Notice that $\theta_\ell(e_1, e_1) = \dim V_\ell = \frac{(n + \ell - 1)!}{(n - 1)! \ell!} \neq 0$.

The integral defining $e_\ell(\sigma)$ does not converge absolutely, but has to be considered as the meromorphic extension of an analytic function. Poles only occur in half-integer points on the real axis. So we have to restrict (and we do) to $\sigma \neq 0$.

So we have $\theta_\ell \#_\sigma \theta_\ell = d(\sigma) e_\ell(\sigma) \theta_\ell$ and hence $\varphi_\ell = [d(\sigma) e_\ell(\sigma)]^{-1} \theta_\ell$ is the ϕ_σ -component of an idempotent in \mathcal{A} . Furthermore $\theta_\ell \#_\sigma \theta_{\ell'} = 0$ if $\ell \neq \ell'$. Clearly φ_ℓ is self-adjoint, since $|d(\sigma)|^{-2} = |e_\ell(\sigma)|^2$, being equal to the constant $c(\sigma)$ from Section 1.

So the φ_ℓ are mutually orthogonal idempotents in the algebra $L^2_K((\tilde{S} \times \tilde{S}); dsdt)$ and span this space. The theorem now follows easily. \square

Remark The constant $e_\ell(\sigma)$ can of course be computed. Apply e.g. [3], Section 7.31, we get, by meromorphic continuation:

$$\begin{aligned} e_\ell(\sigma) &= a_\ell \int_S C_\ell^{\frac{n-2}{2}}(|x_1|) |x_1|^{-\frac{n}{2}-i\sigma} dx \\ &= 2a_\ell \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\sqrt{\pi}} \int_0^1 u^{-\frac{n}{2}-i\sigma} (1-u^2)^{\frac{n-2}{2}} C_\ell^{\frac{n-2}{2}}(u) du \\ &= 2^{-2\ell} \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}} \cdot \frac{\Gamma(n-2+\ell)}{\Gamma(n-2)} \cdot \frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-2}{2}+\ell)} \cdot \frac{\Gamma(-\frac{n}{2}-i\sigma+1)\Gamma(\frac{-\frac{n}{2}-i\sigma-\ell+1}{2})}{\Gamma(-\frac{n}{2}-i\sigma-\ell+1)\Gamma(\frac{\frac{n}{2}-i\sigma+\ell}{2})}. \end{aligned}$$

4. THE STRUCTURE OF THE HILBERT ALGEBRA $L^2(X)$

We now turn to the full algebra $L^2(X)$. We again reduce the computations to $L^2(\tilde{S} \times \tilde{S})$. In a similar way as for \mathcal{A} we get:

Lemma 4.1. *If $\phi_o \in V_\ell \otimes \bar{V}_m$, $\psi_o \in V_{\ell'} \otimes \bar{V}_{m'}$ then*

$$\phi_o \#_\sigma \psi_o = \begin{cases} 0 & \text{if } m \neq m' \\ \text{in } V_\ell \otimes \bar{V}_{m'} & \text{if } m = m'. \end{cases}$$

More precisely we have the following result. Let $(f_i), (g_j), (k_l)$ be orthonormal bases of V_ℓ, V_m and $V_{m'}$ respectively, and $\phi_o(u, v) = f_i(u)\bar{g}_j(v)$, $\psi_o(u, v) = g_{j'}(u)\bar{k}_l(v)$, then

$$\phi_o \#_\sigma \psi_o = \begin{cases} 0 & \text{if } j \neq j' \\ d(\sigma) e_m(\sigma) f_i(u)\bar{k}_l(v) & \text{if } j = j'. \end{cases}$$

The proof is again straightforward and uses the intertwining relation:

$$\int_S |\langle x, y \rangle|^{-n/2-i\sigma} g_{j'}(y) dy = e_m(\sigma) g_{j'}(x).$$

Theorem 4.2. *The irreducible self-adjoint idempotents of $L^2(\tilde{S} \times \tilde{S})$ are given by*

$$e_f^\ell(u, v) = \{d(\sigma) e_\ell(\sigma)\}^{-1} \cdot f(u)\bar{f}(v)$$

with $f \in V_\ell$, $\|f\|_{L^2(\tilde{S})} = 1$ and ℓ even. The left ideal generated by e_f^ℓ is equal to $L^2(\tilde{S}) \otimes \bar{f}$.

The proof is by application of Lemma (4.1)

Remarks

- (1) The minimal right ideals are obtained in a similar way.
- (2) The minimal two-sided ideal generated by $L^2(\tilde{S} \times \tilde{S}) \cdot e_f^\ell$ is the full algebra $L^2(\tilde{S} \times \tilde{S})$.
- (3) The closure of $\bigoplus_{\ell \in 2\mathbb{N}} V_\ell \otimes \bar{V}_\ell$ is a H^* -subalgebra of $L^2(\tilde{S} \times \tilde{S})$. The minimal left ideals are here $V_\ell \otimes \bar{f}$ ($f \in V_\ell$, $\|f\|_{L^2(\tilde{S})} = 1$); they are generated by the e_f^ℓ as above. The minimal two-sided ideal generated by $V_\ell \otimes \bar{f}$ is equal to $V_\ell \otimes \bar{V}_\ell$.

5. THE CASE OF A GENERAL PARA-HERMITIAN SPACE

It is not necessary to assume $\text{rank } X = 1$ in order to show that \mathcal{A} is commutative. Theorem 3.2 is also valid mutatis mutandis in the general case since $(K, K \cap H)$ is a Gelfand pair, and it clearly implies the commutativity of \mathcal{A} . To the general construction of the product and the involution we shall return in another paper.

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