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Hardy's theorem for the q -Bessel Fourier transform

Lazhar Dhaouadi *

Abstract

In this paper we give a q -analogue of the Hardy's theorem for the q -Bessel Fourier transform. The celebrated theorem asserts that if a function f and its Fourier transform \widehat{f} satisfying $|f(x)| \leq c.e^{-\frac{1}{2}x^2}$ and $|\widehat{f}(x)| \leq c.e^{-\frac{1}{2}x^2}$ for all $x \in \mathbb{R}$ then $f(x) = \text{const}.e^{-\frac{1}{2}x^2}$.

1 The q -Fourier Bessel transform

Throughout this paper we consider $0 < q < 1$ and we adopt the standard conventional notations of [2]. We put

$$\mathbb{R}_q^+ = \{q^n, \quad n \in \mathbb{Z}\},$$

and for complex a

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i), \quad n = 1 \dots \infty.$$

Jackson's q -integral (see [3]) in the interval $[0, \infty[$ is defined by

$$\int_0^\infty f(x) d_q x = (1 - q) \sum_{n=-\infty}^{\infty} q^n f(q^n).$$

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We introduce the following functional spaces $\mathcal{L}_{q,1,\nu}$ of even functions f defined on \mathbb{R}_q^+ such that

$$\|f\|_{q,1,\nu} = \left[\int_0^\infty |f(x)|x^{2\nu+1}d_qx \right] < \infty.$$

The normalized Hahn-Exton q -Bessel function of order $\nu > -1$ (see [5]) is defined by

$$j_\nu(z, q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{n(n-1)}{2}}}{(q, q)_n (q^{\nu+1}, q)_n} z^n.$$

It's an entire analytic function in z .

Lemma 1 *For every $p \in \mathbb{N}$, there exist $\sigma_p > 0$ for which*

$$|z|^{2p} |j_\nu(z, q^2)| < \sigma_p e^{|z|}, \quad \forall z \in \mathbb{C}.$$

Proof. In fact

$$\begin{aligned} |z|^{2p} |j_\nu(z, q^2)| &\leq \frac{1}{(q^2, q^2)_\infty (q^{2\nu+2}, q^2)_\infty} \sum_{n=0}^{\infty} q^{n(n-1)} |z|^{2n+2p} \\ &\leq \frac{q^{p(p+1)}}{(q^2, q^2)_\infty (q^{2\nu+2}, q^2)_\infty} \sum_{n=p}^{\infty} q^{n(n-2p-1)} |z|^{2n}. \end{aligned}$$

Now using the Stirling's formula

$$n! \sim \sqrt{2\pi n} \frac{n^n}{e^n},$$

we see that there exist an entire $n_0 \geq p$ such that

$$q^{n(n-2p-1)} < \frac{1}{(2n)!}, \quad \forall n \geq n_0,$$

which implies

$$\sum_{n=n_0}^{\infty} q^{n(n-2p-1)} |z|^{2n} < \sum_{n=n_0}^{\infty} \frac{1}{(2n)!} |z|^{2n} < e^{|z|}.$$

Finally there exist $\sigma_p > 0$ such that

$$\frac{|z|^{2p} |j_\nu(z, q^2)|}{e^{|z|}} < \sigma_p, \quad \forall z \in \mathbb{C}.$$

This finish the proof. ■

The q -Bessel Fourier transform $\mathcal{F}_{q,\nu}$ introduced in [1,4] as follow

$$\mathcal{F}_{q,\nu}f(x) = c_{q,\nu} \int_0^\infty f(t)j_\nu(xt, q^2)t^{2\nu+1}d_qt.$$

where

$$c_{q,\nu} = \frac{1}{1-q} \frac{(q^{2\nu+2}, q^2)_\infty}{(q^2, q^2)_\infty}.$$

The following theorem was proved in [1]

Theorem 1 *Given $f \in \mathcal{L}_{q,1,\nu}$ then we have*

$$\mathcal{F}_{q,\nu}^2(f)(x) = f(x), \quad \forall x \in \mathbb{R}_q^+.$$

Proof. See [1] p 3. ■

2 Hardy's theorem

The following Lemma from complex analysis is crucial for the proof of our main theorem.

Lemma 2 *Let h be an entire function on \mathbb{C} such that*

$$|h(z)| \leq Ce^{a|z|^2}, \quad z \in \mathbb{C},$$

$$|h(x)| \leq Ce^{-ax^2}, \quad x \in \mathbb{R},$$

for some positive constants a and C . Then $h(z) = \text{Const.}e^{-ax^2}$.

Proof. See [6] p 4. ■

Now we are in a position to state and prove the q -analogue of the Hardy's theorem

Theorem 2 Suppose $f \in \mathcal{L}_{q,1,\nu}$ satisfying the following estimates

$$|f(x)| \leq C e^{-\frac{1}{2}x^2}, \quad \forall x \in \mathbb{R}_q^+,$$

$$|\mathcal{F}_{q,\nu}f(x)| \leq C e^{-\frac{1}{2}x^2}, \quad \forall x \in \mathbb{R},$$

where C is a positive constant. Then there exist $A \in \mathbb{R}$ such that

$$f(z) = A.c_{q,\nu}\mathcal{F}_{q,\nu}\left(e^{-\frac{1}{2}x^2}\right)(z), \quad \forall z \in \mathbb{C}.$$

Proof. We claim that $\mathcal{F}_{q,\nu}f$ is an analytic function and there exist $C' > 0$ such that

$$|\mathcal{F}_{q,\nu}f(z)| \leq C' e^{\frac{1}{2}|z|^2}, \quad \forall z \in \mathbb{C}.$$

We have

$$|\mathcal{F}_{q,\nu}f(z)| \leq c_{q,\nu} \int_0^\infty |f(x)| |j_\nu(zx, q^2)| x^{2\nu+1} d_q x.$$

From the Lemma 1, if $|z| > 1$ then there exist $\sigma_1 > 0$ such that

$$x^{2\nu+1} |j_\nu(zx, q^2)| = \frac{1}{|z|^{2\nu+1}} (|z|x)^{2\nu+1} |j_\nu(zx, q^2)| < \frac{\sigma_1}{1 + |z|^2 x^2} e^{x|z|}, \quad \forall x \in \mathbb{R}_q^+.$$

Then we obtain

$$|\mathcal{F}_{q,\nu}f(z)| \leq C\sigma_1 c_{q,\nu} \left[\int_0^\infty \frac{e^{-\frac{1}{2}(x-|z|)^2}}{1 + |z|^2 x^2} d_q x \right] e^{\frac{1}{2}|z|^2} < C\sigma_1 c_{q,\nu} \left[\int_0^\infty \frac{1}{1 + x^2} d_q x \right] e^{\frac{1}{2}|z|^2}.$$

Now, if $|z| \leq 1$ then there exist $\sigma_2 > 0$ such that

$$x^{2\nu+1} |j_\nu(zx, q^2)| \leq \sigma_2 e^x, \quad \forall x \in \mathbb{R}_q^+.$$

Therefore

$$|\mathcal{F}_{q,\nu}f(z)| \leq C\sigma_2 c_{q,\nu} \left[\int_0^\infty e^{-\frac{1}{2}x^2+x} d_q x \right] \leq C\sigma_2 c_{q,\nu} \left[\int_0^\infty e^{-\frac{1}{2}x^2+x} d_q x \right] e^{\frac{1}{2}|z|^2}.$$

Which leads to the estimate (1). Using Lemma 2, we obtain

$$\mathcal{F}_{q,\nu}f(z) = \text{const.} e^{-\frac{1}{2}z^2}, \quad \forall z \in \mathbb{C},$$

and by theorem 1, we conclude that

$$f(z) = \text{const.} \mathcal{F}_{q,\nu}\left(e^{-\frac{1}{2}t^2}\right)(z), \quad \forall z \in \mathbb{C}.$$

This finish the proof. ■

Corollary 1 Suppose $f \in \mathcal{L}_{q,1,\nu}$ satisfying the following estimates

$$|f(x)| \leq C e^{-px^2}, \quad \forall x \in \mathbb{R}_q^+,$$

$$|\mathcal{F}_{q,\nu} f(x)| \leq C e^{-\sigma x^2}, \quad \forall x \in \mathbb{R},$$

where C, p, σ are a positive constant and $p\sigma = \frac{1}{4}$. We suppose that there exist $a \in \mathbb{R}_q^+$ such that $a^2 p = \frac{1}{2}$. Then there exist $A \in \mathbb{R}$ such that

$$f(z) = A c_{q,\nu} \mathcal{F}_{q,\nu} \left(e^{-\sigma t^2} \right) (z), \quad \forall z \in \mathbb{C}.$$

Proof. Let $a \in \mathbb{R}_q^+$, and put

$$f_a(x) = f(ax),$$

then

$$\mathcal{F}_{q,\nu} f_a(x) = \frac{1}{a^{2\nu+2}} \mathcal{F}_{q,\nu} f(x/a).$$

In the end, applying Theorem 2 to the function f_a . ■

Remark 1 Using the q -Central limit Theorem (see [1]) we give a probability interpretation of the function $c_{q,\nu} \mathcal{F}_{q,\nu} \left(e^{-\sigma t^2} \right)$. In fact if $(\xi_n)_{n \geq 0}$ be a sequence of positive probability measures of \mathbb{R}_q^+ , satisfying

$$\lim_{n \rightarrow \infty} n\sigma_n = \frac{(q^2, q^2)_1 (q^{2\nu+2}, q^2)_1}{q^2} \sigma, \quad \text{where } \sigma_n = \int_0^\infty t^2 t^{2\nu+1} d_q \xi_n(t),$$

and

$$\lim_{n \rightarrow \infty} n\tilde{\sigma}_n = 0, \quad \text{where } \tilde{\sigma}_n = \int_0^\infty \frac{t^4}{1+t^2} t^{2\nu+1} d_q \xi_n(t),$$

then the n th q -convolution product ξ_n^{*n} converge strongly toward a measure ξ defined by

$$d_q \xi(x) = c_{q,\nu} \mathcal{F}_{q,\nu} \left(e^{-\sigma t^2} \right) (x) d_q x.$$

Corollary 2 Suppose $f \in \mathcal{L}_{q,1,\nu}$ satisfying the following estimates

$$|f(x)| \leq Ce^{-px^2}, \quad \forall x \in \mathbb{R}_q^+,$$

$$|\mathcal{F}_{q,\nu}f(x)| \leq Ce^{-\sigma x^2}, \quad \forall x \in \mathbb{R},$$

where C, p, σ are a positive constant and $p\sigma > \frac{1}{4}$. We suppose that there exist $a \in \mathbb{R}_q^+$ such that $a^2p = \frac{1}{2}$. Then $f \equiv 0$.

Proof. In fact there exist $\sigma' < \sigma$ such that $p\sigma' = \frac{1}{4}$. Then the function f satisfying the estimates of Corollary 1, if we replacing σ by σ' . Which implies

$$\mathcal{F}_{q,\nu}f(x) = \text{const.}e^{-\sigma'x^2}, \quad \forall x \in \mathbb{R}.$$

On the other hand, f satisfying the estimates of Corollary 2, then

$$|\text{const.}e^{-\sigma'x^2}| \leq Ce^{-\sigma x^2}, \quad \forall x \in \mathbb{R}.$$

This implies $\mathcal{F}_{q,\nu}f \equiv 0$, and by Theorem 1 we conclude that $f \equiv 0$ ■

Remark 2 Hardy's theorem asserts the impossibility of a function and its q -Fourier Bessel transform to be simultaneously "very rapidly decreasing". Hardy's theorem can also be viewed as a sort of "Qualitative uncertainty principles". One such example can be the fact that a function and its q -Bessel Fourier transform cannot both have compact support.

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