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## To cite this version:

Yves de Cornulier, Yves Stalder, Alain Valette. Proper actions of lamplighter groups associated with free groups. 2007. hal-00162561v1

HAL Id: hal-00162561<br>https://hal.science/hal-00162561v1<br>Preprint submitted on 13 Jul 2007 (v1), last revised 9 Nov 2007 (v2)

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# Proper actions of lamplighter groups associated with free groups 

Yves de Cornulier, Yves Stalder and Alain Valette

July 13, 2007


#### Abstract

Given a finite group $H$ and a free group $\mathbf{F}_{n}$, we prove that the wreath product $H\left\langle\mathbf{F}_{n}\right.$ has the Haagerup property and we estimate its equivariant compression in $L^{p}$-spaces.


## 1 Introduction

Following HP98], a space with walls is a pair $(X, \mathcal{W})$ where $X$ is a set and $\mathcal{W}$ is a set of partitions of $X$ into two classes, called walls, such that for any two distinct points $x, y \in X$, the number $w(x, y)$ of walls separating $x$ from $y$ is finite. A group $G$ is a group of automorphisms of $(X, \mathcal{W})$ if $G$ acts on $X$, preserving the wall structure. The group $G$ acts properly on $X$ if, for every $x_{0} \in X$ :

$$
\lim _{g \rightarrow \infty} w\left(g x_{0}, x_{0}\right)=+\infty
$$

It is known (see CCJ 01 , Corollary 7.4.2) that, if $G$ acts properly on a non-empty space with walls, then $G$ has the Haagerup property (or is $a$ - $T$ menable), i.e. $G$ admits a metrically proper, isometric action on a Hilbert space.

An important example of a space with walls is the vertex set of a tree, where walls are given by edges. A group acting properly on a tree (e.g. a free group) is therefore a-T-menable.

Let us define the class $\mathcal{P W}$ as the class of countable groups $G$ carrying a structure of space with walls on which $G$ acts properly by left multiplication. This class contains for instance $\mathbf{Z}^{n}$, free groups, surface groups (see 1.2.7 in

CCJ ${ }^{+} 01$ for this fact）．More generally it contains all groups acting properly on some finite product of trees．

Recall that the（standard，restricted）wreath product of two groups $H$ and $G$ is the semidirect product $H^{(G)} \rtimes G$ ，where $G$ acts by shifting the direct sum $H^{(G)}$ of copies of $H$ ．Up to now，the problem of stability of the Haagerup Property under wreath products was open．Indeed，the Haagerup Prop－ erty is closed under direct sums，but not under general semidirect products． However it is known to be closed under extension with amenable quotients $\mathrm{CCJ}^{+}$01，Example 6．1．6］，so that in particular $H$ 亿 $G$ is Haagerup whenever $H$ is Haagerup and $G$ is amenable．These were however the only known examples of wreath products with the Haagerup Property．This paper con－ tains the first examples of a－T－menable wreath products $H$ 亿 $G$ with $H \neq 1$ and $G$ non－amenable，including the first natural such example，namely the ＂lamplighter group＂$(\mathbf{Z} / 2 \mathbf{Z})$ ） $\mathbf{F}_{2}$ over the free group $\mathbf{F}_{2}$ of rank two．

Theorem 1 Let $H$ be a finite group．If $G$ is a group in $\mathcal{P W}$ ，then so is $H$ 亿 $G$ ．In particular，$H$ 亿 $G$ has the Haagerup property．

In fact，the proof we present below works in more general cases．We restrict to Theorem 1 because of its interesting connections with operator algebras and harmonic analysis（see below），while its proof avoids most of technical problems．The general result，to appear in a forthcoming paper，is：

Theorem The Haagerup property and the property of existence of a metri－ cally proper action on some space with walls，are closed under taking wreath products．

Equivariant Hilbert space compression was defined by Guentner and Kam－ inker［GK04］as a way to quantify the Haagerup property for finitely gen－ erated groups $G$ ．We define $B(G)$ as the supremum of those $t \in[0,1]$ such that there exists an affine isometric action $\alpha$ of $G$ on a Hilbert space，with

$$
\|\alpha(g)(0)\| \succeq|g|_{S}^{t},
$$

where $|g|_{S}$ denotes word length on $G$ ．We prove：
Theorem 2 Let $\mathbf{F}_{n}$ be the free group on $n$ generators $(2 \leq n<\infty)$ ，and $H$ be a finite group．Then，the following hold：

1. One has $B\left(H \backslash \mathbf{F}_{n}\right)=\frac{1}{2}$;
2. $H \succ \mathbf{F}_{n}$ admits an affine isometric action $\beta$ on some $L^{1}$-space, such that the map $\gamma \mapsto \beta(\gamma)(0)$ is a quasi-isometric embedding (with respect to the word length).

Theorem 1 has interesting consequences in view of a recent result of Popa and Ozawa PO]. Recall from CH89 that a countable group $G$ has the complete metric approximation property (also called "weak amenability with constant 1 ") if there exists a sequence $\left(f_{n}\right)_{n>0}$ of functions with finite support on $G$, converging pointwise to 1 , and such that $\limsup _{n \rightarrow \infty}\left\|f_{n}\right\|_{c b}=1$, where $\|f\|_{c b}$ is the Herz-Schur multiplier norm of the function $f$.

In (PQ, Corollary 2.11], it is proved that, if $H$ is non-trivial and $G$ is non-amenable, then $H \imath G$ does not have the complete metric approximation approximation property. Combining with Theorem 1, we get:

Corollary 3 For $H$ a non-trivial finite group, $H 乙 \mathbf{F}_{2}$ is an a-T-menable group without the complete metric approximation property.

This disproves a conjecture of Cowling (see page 7 in [CCJ+01]), stating that the class of a-T-menable groups coincides with the class of groups with the complete metric approximation property. Whether every such group is a-T-menable, is still an open question.

It was recently proved by Guentner and Higson (GH) that a group acting properly isometrically on a finite-dimensional $C A T(0)$ cube complex, has the complete metric approximation property. In view of the dictionary between spaces with walls and $C A T(0)$ cube complexes, set up in CN05 and independently in Nic04, we get from Theorem 1:

Corollary 4 Let $H$ be a non-trivial finite group. The wreath product $H \backslash \mathbf{F}_{2}$ admits a proper isometric action on a $C A T(0)$ cube complex, but does not have the complete metric approximation property.

In other words, the finite-dimensionality assumption cannot be omitted in the Guentner-Higson result.

Ackowledgements: Thanks are due to I. Chatterji, C. Drutu and S. Popa for useful conversations and correspondence. The second-named author is especially grateful to Frédéric Haglund for very fruitful discussions and hints about the construction of families of walls for wreath products.

## 2 Proof of Theorem 1

We first fix some notation. Write $\Lambda=H^{(G)}$ and $\Gamma=H \imath G=\Lambda \rtimes G$. Elements of $\Gamma$ are denoted

$$
\gamma=\lambda g(\lambda \in \Lambda, g \in G)
$$

The support of $\lambda$ is

$$
\operatorname{supp}(\lambda)=\left\{g \in G: \lambda(g) \neq 1_{H}\right\}
$$

If $(X, \mathcal{W})$ is a space with walls, a half-space in $X$ is a class of some wall in $\mathcal{W}$. By assumption, $G$ carries a structure of space with walls on which it acts properly, and we denote by $\mathcal{H}$ the set of its half-spaces. For $a \in \mathcal{H}$, we denote by $a^{c}$ the other half-space in the same wall, i.e. the complement of $a$ in $G$. For $a \in \mathcal{H}$ and $\mu: a^{c} \rightarrow H$ a function with finite support, we set

$$
E(a, \mu)=:\left\{\gamma=\lambda g \in \Gamma: g \in a,\left.\lambda\right|_{a^{c}}=\mu\right\} .
$$

We define walls in $\Gamma$ as partitions $\left\{E(a, \mu), E(a, \mu)^{c}\right\}$, for $a \in \mathcal{H}$ and $\mu: a^{c} \rightarrow H$ finitely supported. We check in three steps that this equips $\Gamma$ with a structure of spaces with walls on which $\Gamma$ acts properly.

1 st step: $\Gamma$ is a space with walls.
Let $\gamma_{1}=\lambda_{1} g_{1}$ and $\gamma_{2}=\lambda_{2} g_{2}$ be two elements of $\Gamma$. Let us show that there are finitely many $E(a, \mu)$ 's such that $\gamma_{1} \in E(a, \mu)$ and $\gamma_{2} \notin E(a, \mu)$.

Indeed $\gamma_{1} \in E(a, \mu)$ means $g_{1} \in a$ and $\left.\lambda_{1}\right|_{a^{c}}=\mu$ (so that $\mu$ is determined once $a$ is given). And $\gamma_{2} \notin E(a, \mu)$ means that either $g_{2} \notin a$, or $\left.\lambda_{2}\right|_{a^{c}} \neq \mu$; since $\mu=\left.\lambda_{1}\right|_{a^{c}}$, this can be re-written:

$$
a^{c} \cap\left(\left\{g_{2}\right\} \cup \operatorname{supp}\left(\lambda_{1}^{-1} \lambda_{2}\right)\right) \neq \emptyset .
$$

So $a$ must separate $g_{1}$ from the finite set $\left\{g_{2}\right\} \cup \operatorname{supp}\left(\lambda_{1}^{-1} \lambda_{2}\right)$. Since $G$ is a space with walls, this singles out finitely many possibilities for $a$.

2nd step: $\Gamma$ preserves the above wall structure.
This follows immediately from the formulae:

$$
\begin{gathered}
g E(a, \mu)=E(g a, g \mu) \quad(g \in G) \\
\lambda E(a, \mu)=E\left(a,\left.\lambda\right|_{a^{c}} \mu\right)(\lambda \in \Lambda)
\end{gathered}
$$

3rd step: $\Gamma$ acts properly on its wall structure.
Let $w_{\Gamma}(\gamma)$ be the number of walls separating the unit $1_{\Gamma}$ from $\gamma \in \Gamma$. We must prove that, for every $N \in \mathbf{N}$, there are finitely many $\gamma$ 's with $w_{\Gamma}(\gamma) \leq N$.

Define analogously $w_{G}(g)$ as the number of walls separating $1_{G}$ from $g$ in $G$, and set $B_{G}(N)=\left\{g \in G: w_{G}(g) \leq N\right\}$; by our assumption $B_{G}(N)$ is a finite set.

Claim: If $\gamma=\lambda g$ satisfies $w_{\Gamma}(\gamma) \leq N$, then $\{g\} \cup \operatorname{supp}(\lambda) \subset B_{G}(N)$.
Theorem 1 then follows from the claim together with the fact that $H$ is a finite group.

Proof of the claim: Contraposing, suppose that there exists $g^{\prime} \in\{g\} \cup$ $\operatorname{supp}(\lambda)$ with $w_{G}\left(g^{\prime}\right)>N$. So we find $N+1$ distinct half-spaces $a_{0}, \ldots, a_{N}$ in $G$ with $1_{G} \in a_{i}$ and $g^{\prime} \notin a_{i}(i=0, \ldots, N)$. Then the $E\left(a_{i}, 1_{H}\right)$ 's are $N+1$ distinct half-spaces in $\Gamma$ separating $1_{\Gamma}$ from $\gamma=\lambda g$, so $w_{\Gamma}(\gamma)>N$.

## 3 Equivariant $L^{p}$-compression

For $G$ a finitely generated group, we define the equivariant $L^{p}$-compression of $G$ (where $1 \leq p<\infty$ ) as

$$
B_{p}(G)=\sup \left\{t \in[0,1]: \exists \alpha:\|\alpha(g)(0)\| \succeq|g|_{S}^{t}\right\}
$$

where $\alpha$ is an affine isometric action of $G$ on some $L^{p}$-space. Clearly $B_{2}(G)$ coincides with the $B(G)$ previously introduced.

Proposition 5 Let $H$ be a finite group. Then $B_{p}\left(H\left\langle\mathbf{F}_{n}\right) \geq \frac{1}{p}\right.$ for $1 \leq p<\infty$.
Proof of Theorem 2, part 1: By the above Proposition, $B\left(H / \mathbf{F}_{n}\right) \geq \frac{1}{2}$. Equality follows from the result by Guentner and Kaminker GK04 that groups with $B>1 / 2$ must be amenable.

Proof of Proposition 1: We may assume $H \neq\{1\}$. For $h \in H \backslash\left\{1_{H}\right\}$, define $\delta(h) \in \Lambda$ as

$$
(\delta(h))(g)=\left\{\begin{array}{ccc}
1_{H} & \text { if } & g \neq 1_{\mathbf{F}_{n}} \\
h & \text { if } & g=1_{\mathbf{F}_{n}}
\end{array}\right.
$$

As a finite generating set of $\Gamma=H \succ \mathbf{F}_{n}$, let us take the set $S$ consisting in the union of the standard generators of $\mathbf{F}_{n}$ and the $\delta(h)$ 's, $h \in H \backslash\left\{1_{h}\right\}$.

By Proposition 3.1 in [CTV], from the wall structure on $\Gamma$, one deduces an affine isometric action $\alpha_{p}$ of $\Gamma$ on some $L^{p}$-space, with $w_{\Gamma}(\gamma)=\left\|\alpha_{p}(\gamma)(0)\right\|_{p}^{p}$. So the Proposition follows from the

Claim: For $\gamma \in \Gamma$ with $w_{\Gamma}(\gamma) \geq 2$, one has $\frac{1}{2} w_{\Gamma}(\gamma) \leq|\gamma|_{S} \leq 2 w_{\Gamma}(\gamma)$.
To prove this claim, we appeal to a result of Parry Par92, Theorem 1.2]: if $\gamma=\lambda g$, then

$$
|\gamma|_{S}=L_{\mathbf{F}_{n}}(g, \lambda)+\sharp(\operatorname{supp}(\lambda))
$$

where $L_{\mathbf{F}_{n}}(g, \lambda)$ is the length of the shortest path from $1_{\mathbf{F}_{n}}$ to $g$ in the Cayley tree of $\mathbf{F}_{n}$, passing through all points in $\operatorname{supp}(\lambda)$.

Lemma 6 Let $T$ be a finite tree with $m$ vertices. For every two vertices $x, y$, the length of the shortest path from $x$ to $y$, passing through all vertices of $T$, is $2 m-2-d_{T}(x, y)$.

We apply this lemma to the subtree $T$ of $\mathbf{F}_{n}$ generated by $1_{\mathbf{F}_{n}}, g$, and $\operatorname{supp}(\lambda)$. Let $m$ denote its number of vertices. Then by the lemma

$$
|\gamma|_{S}=2 m-2-|g|_{\mathbf{F}_{n}}+\sharp(\operatorname{supp}(\lambda))
$$

while $w_{\Gamma}(\gamma)=2(m-1)$, each edge of $T$ giving rise to two walls in $\Gamma$ (since $H \neq\{1\})$. As $|g|_{\mathbf{F}_{n}} \leq \operatorname{diam}(T) \leq m-1$, we already get the first inequality $\frac{1}{2} w_{\Gamma}(\gamma) \leq|\gamma|_{S}$. On the other hand $\sharp(\operatorname{supp}(\lambda)) \leq m$; since $w_{\Gamma}(\gamma)=2(m-1) \geq$ 2 , we get the second inequality

$$
|\gamma|_{S} \leq 3 m-2 \leq 4(m-1)=2 w_{\Gamma}(\gamma) .
$$

Postponing the proof of Lemma 6, we show how to finish the proof of Theorem 2.

Proof of Theorem 2, part 2: It suffices to set $\beta=\alpha_{1}$, where $\alpha_{1}$ is defined as in the proof of Proposition 1. This proof shows that $\gamma \mapsto \beta(\gamma)(0)$ is a quasi-isometric embedding.

Proof of Lemma 6: The tree $T$ has $m-1$ edges. In every path from $x$ to $y$, passing through all vertices of $T$, every edge not on the unique geodesic from $x$ to $y$ must be crossed at least twice, since $T$ has no circuit. So such a path has length at least $2\left(m-1-d_{T}(x, y)\right)+d_{T}(x, y)=2 m-2-d_{T}(x, y)$.

To prove that a path with this length always exists, we work by induction over $m$, the case $m=1$ being obvious. In general, we distinguish two cases.

- The degree of $x$ is at least 2. Remove then from $T$ some branch $T^{\prime}$ hanging from $x$, not containing $y$. Let $k$ be the number of vertices of $T^{\prime}$. By induction assumption we find a path from $x$ to $x$, of length $2+(2 k-2)$, through all the vertices of $T^{\prime}$; and we find a path of length $2(m-k)-2-d_{T}(x, y)$, from $x$ to $y$, through all the vertices of $T \backslash T^{\prime}$. Composing these two paths we get the desired path.
- The degree of $x$ is 1 , i.e. $x$ is a terminal vertex. Let $x^{\prime}$ be the unique neighbor of $x$. If $y \neq x$, we find by induction assumption a path from $x^{\prime}$ to $y$ in $T \backslash\{x\}$, of length $2 m-4-d_{T}\left(x^{\prime} y\right)$, through all vertices in $T \backslash\{x\}$. Adding the edge $x x^{\prime}$ at the beginning, we get a path of the desired length. The case $x=y$ is similar.


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