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# The Galois lattice as a hierarchical structure for topological relations

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## Abstract

This paper presents the construction and the comparison of Galois lattices of topological relations for qualitative spatial representation and reasoning. The lattices rely on a correspondence between computational operations working on quantitative data, on the one hand, and topological relations working on qualitative knowledge units, on the other hand. After introducing the context of the present research work, i.e. the **RCC-8** model of topological relations, we present computational operations for checking topological relations on spatial regions. From these operations are derived two sets of computational conditions that can be associated to topological relations through a Galois connection. The associated Galois lattices are presented and compared. Elements on the practical use of the lattices for representing spatial knowledge and for reasoning are also introduced and discussed.

*Key words:* Galois lattice, Galois connection, lattice-based classification, topological relations, **RCC-8**, hierarchical knowledge representation.

*PACS:*

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## 1 Introduction

This paper presents the construction and the comparison of Galois lattices of topological relations for qualitative spatial representation and reasoning. This research work has been carried out in the context of the design of a knowledge-based system for agricultural landscape analysis. The main objective of this system, named LOLA, is to recognize *landscape models* on land-use maps extracted from satellite images. Landscape models are abstract models describing agricultural spatial structures as sets of spatial entities and qualitative spatial relations between these entities. They are used to classify *zones*,

i.e. collections of raster regions, extracted from the maps. A raster region is characterized by a label denoting the land-use category, e.g. crops, meadows, forest, buildings, etc. [1].

Following these needs, we have designed a hierarchical representation of topological relations based on a *Galois lattice* –or *concept lattice structure*– relying on the Galois lattice theory [2,3,4]. A Galois lattice is a "multi-faceted structure": it allows the construction of a hierarchical structure both for representing knowledge and for reasoning. In a Galois lattice, a concept is defined by an *intension* or *intent*, i.e. the set of properties shared by all instances of the concept, and by an *extension*, or *extent*, i.e. the set of individuals being instances of the concept. In our framework, the extension of concepts corresponds to *topological relations* between regions of an image, and the intension of concepts corresponds to properties computed on that image regions (also called *computational conditions*). Thus, the Galois lattice emphasizes the correspondence between qualitative models, i.e. topological relations, and quantitative data, i.e. vector or raster regions.

Actually, linking qualitative models with quantitative data, and implementing these qualitative models for real-world applications such as landscape analysis, land use planning, integrating maps and remote sensing data, etc., is an important question in spatial reasoning. Research works have been mainly carried out in two directions: first, for linking topological relations with vector or raster data (e.g. [5,6,7]), second, for implementing topological relations within logical frameworks (e.g. [8,9,10]). In these frameworks, boolean lattices have been used for automatic theorem proving and relations inferring [8]. Besides, lattices have been used to represent hierarchical spatial entities (points, lines, triangles) and then to compute topological relations between these entities [11]. But, as far as we know, Galois lattices have not been studied for spatial reasoning.

Our goal is to show that Galois lattices can be useful for both spatial representation, computation and reasoning. In particular, Galois lattices can be used for linking various sets of relations, and various sets of computational conditions. In this paper, we study two sets of computational conditions, one set of relations, and then compare the structures of the resulting Galois lattices. We then compare the two Galois lattices with a third boolean lattice, and discuss different points of view, such as the implementation, the memory occupation, the representation of concepts, and the reasoning capabilities. Practically, a Galois lattice has been implemented within an object-based knowledge representation system, equipped with a classification process. In this framework, the exploitation of land-use maps for landscape analysis is considered as an *instance classification* problem, where landscape models correspond to classes, while zones correspond to instances that have to be classified according to landscape model classes [12]. Substantial and encouraging results have been

obtained, showing the practicability of the approach.

The paper is organized as follows. The second section introduces the context of the work, i.e. topological relations, especially the **RCC-8** theory, and computational operations. We present two sets of computational conditions that are used to check topological relations on vector or raster regions. The third section focuses on the construction of two Galois lattices based on the implication relation between the base relations of **RCC-8**, and the two sets of computational conditions. The fourth section presents firstly a comparison of the two lattices, and secondly, a comparison of both Galois lattices with the boolean lattice associated with the base relations of **RCC-8**. The fifth section briefly shows how a Galois lattice can be implemented and used in practice. The present research work is then discussed and compared with related work in the sixth section. Finally, we conclude the paper and give research perspectives.

## 2 Qualitative and quantitative aspects of topological relations

### 2.1 A logical approach of topological relations: the **RCC-8** theory

In the context of qualitative spatial reasoning, topological relations are opposed to distance and orientation relations. The topological relations considered are mostly binary and are independent of the orientation of the entities. The basic ontological entity is a point or a region. Regions are generally defined as portions of space determined by concrete objects, and are not considered as sets of spatial points [13]. The usual topological relations between regions are the relations of the so-called **RCC-8** theory [14,15], that relies on a set of eight base relations, namely the  $\mathcal{B}$  set. The names and iconic representations of the  $\mathcal{B}$  relations are given in Table 1. The base relations are exhaustive and mutually exclusive, i.e. any configuration of two spatial regions can be described by the  $\mathcal{B}$  set, and, if one of these relations is true, then the others are false. Moreover, in the **RCC-8** theory, the spatial regions considered are 2-dimensional, potentially infinite in number, and with any degree of connection between them.

The topological relations of the **RCC-8** theory are based on the *connection relation* between two spatial regions [16]: two regions  $x$  and  $y$  are *connected* – denoted by  $C(x, y)$  – if they share at least a point. A point is a zero-dimensional spatial element, according to its mathematical meaning. The  $C(x, y)$  relation is defined by the following axioms:









relation	notation	icons
“ <i>x is identical with y</i> ”	$EQ(x, y)$	
“ <i>x is a tangential proper part of y</i> ”	$TPP(x, y)$	
“ <i>x is a non tangential proper part of y</i> ”	$NTPP(x, y)$	
“ <i>x tangentially contains as a proper part y</i> ”	$TPP^{-1}(x, y)$	
“ <i>x non tangentially contains as a proper part y</i> ”	$NTPP^{-1}(x, y)$	
“ <i>x partially overlaps y</i> ”	$PO(x, y)$	
“ <i>x is externally connected with y</i> ”	$EC(x, y)$	
“ <i>x is disconnected from y</i> ”	$DC(x, y)$	

Table 1

The names and the icons associated to the eight base relations of the **RCC-8** theory.

$$\forall x \quad : \quad C(x, x)$$

$$\forall x, y \quad : \quad C(x, y) \implies C(y, x)$$

$$\forall x, y \quad : \quad (\forall z : C(z, x) \iff C(z, y)) \implies x = y$$

Relying on the connection relation, five relations can be defined (see Table 2):  $DC(x, y)$  “*x is disconnected from y*”;  $P(x, y)$  “*x is a part of y*”;  $PP(x, y)$  “*x is a proper part of y*”;  $O(x, y)$  “*x overlaps y*”;  $DR(x, y)$  “*x is discrete from y*”. The theory based on these relations is interesting because it makes a difference between a point and a region, thanks to the relations  $C(x, y)$  “*x and y share a point*”, and  $O(x, y)$  “*x and y share a region*”. This difference allows the definition of three additional relations:  $EC(x, y)$  “*x is externally connected with y*”,  $TP(x, y)$  “*x is a tangential part of y*”,  $NTP(x, y)$  “*x is a non tangential part of y*” (see Table 2). Finally this difference is used to introduce topological notions –*interior, boundary*– into purely set-based notions, constituting the so-called *mereotopology*. The definitions of the base relations of **RCC-8** in terms of the mereotopological theory are given in Table 3.

## 2.2 Mathematical operations on topological relations

It is often useful to deduce new facts from a given situation, or to verify the consistency of the available knowledge on a given situation. This inferences can be drawn using the topological relations and the following mathematical operations.

- The *converse* relation of a relation  $r$  is the relation  $r^{-1}$  such that:

$DC(x, y)$	$\equiv_{def} \neg C(x, y)$
$P(x, y)$	$\equiv_{def} \forall z : C(z, x) \implies C(z, y)$
$PP(x, y)$	$\equiv_{def} P(x, y) \wedge \neg P(y, x)$
$O(x, y)$	$\equiv_{def} \exists z : P(z, x) \wedge P(z, y)$
$DR(x, y)$	$\equiv_{def} \neg O(x, y)$
$EC(x, y)$	$\equiv_{def} C(x, y) \wedge \neg O(x, y)$
$TP(x, y)$	$\equiv_{def} P(x, y) \wedge \exists z : EC(z, x) \wedge EC(z, y)$
$NTP(x, y)$	$\equiv_{def} P(x, y) \wedge \neg \exists z : EC(z, x) \wedge EC(z, y)$

Table 2

The set of mereotopological relations [16].

$EQ(x, y)$	$\equiv_{def} P(x, y) \wedge P(y, x)$
$TPP(x, y)$	$\equiv_{def} PP(x, y) \wedge \exists z : EC(z, x) \wedge EC(z, y)$
$NTPP(x, y)$	$\equiv_{def} PP(x, y) \wedge \neg \exists z : EC(z, x) \wedge EC(z, y)$
$TPP^{-1}(x, y)$	$\equiv_{def} TPP(y, x)$
$NTPP^{-1}(x, y)$	$\equiv_{def} NTPP(y, x)$
$PO(x, y)$	$\equiv_{def} O(x, y) \wedge \neg P(x, y) \wedge \neg P(y, x)$
$EC(x, y)$	$\equiv_{def} C(x, y) \wedge \neg O(x, y)$
$DC(x, y)$	$\equiv_{def} \neg C(x, y)$

Table 3

The definitions of the base relations of **RCC-8** rely on the mereotopology.

$$\forall x, \forall y, r(x, y) \iff r^{-1}(y, x)$$

- Two relations  $r_1$  et  $r_2$  are said to be *disjoint* if:

$$\forall x, \forall y, (r_1(x, y) \implies \neg r_2(x, y)) \wedge (r_2(x, y) \implies \neg r_1(x, y))$$

- The *complement* of a relation  $r$  is the relation  $r^c$  such that :

$$r \text{ and } r^c \text{ are disjoint and } \forall x, \forall y, r(x, y) \vee r^c(x, y) \text{ holds.}$$

- Given three spatial regions  $x, y, z$ , and a pair of relations  $r_1$  and  $r_2$ , such that  $r_1(x, y)$  and  $r_2(y, z)$ , the *composition* of  $r_1$  and  $r_2$  is the disjunction  $r$  of all the possible relations holding between  $x$  and  $z$ .

The last operation is especially interesting: rules of composition, as proposed in [8], may be used for inferring relations holding between two regions  $x$  and  $z$ , from the knowledge of the relations holding between the two regions  $x$  and  $z$ , and a third region  $y$ . For instance, Figure 1 depicts two configurations where three regions  $x, y$  and  $z$  are drawn with respect to the relations  $TPP(x, y)$  and  $EC(y, z)$ . Thus  $DC(x, z)$  or  $EC(x, z)$  can be deduced. The two schemas of the figure correspond to the two possible relative locations of  $x$  and  $z$ .

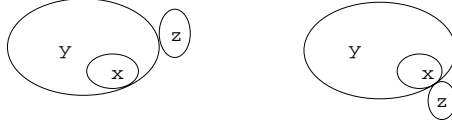


Figure 1. Composition of topological relations. The two configurations are described by  $TPP(x, y)$  and  $EC(y, z)$ . In the left case,  $DC(x, z)$  is true, and in the right case,  $EC(x, z)$  is true.

The rules of composition on the topological relations are represented in *composition tables*: such a table is defined for 9 relations in [17], and for 22 relations in [18]. A method for building large composition tables based on several theories is described in [19].

### 2.3 The computation of topological relations

In this section we present computational operations that can be used for checking **RCC-8** relations on vector or raster data (e.g. data from geographic information systems, or aerial or satellite images). We present two sets of computational operations from which we derive two sets of computational conditions, namely *CE-8* and *CM-8*. These conditions apply on a set  $\mathcal{D}$  of regular closed regions that are non empty, internally connected, and that may overlap, i.e. a point may belong to several regions. Topological problems linked to discrete representations are studied in [20].

#### 2.3.1 A first set *CE-8* of computational conditions

In [21], an  $n$ -dimensional region  $x$  is characterized by two sets, the interior (of dimension  $n$ ), denoted by  $x^\circ$ , and the boundary (of dimension  $n - 1$ ), denoted by  $\partial x$ . Two regions are characterized by four sets whose intersections define four *computational operations*:  $\partial x \cap \partial y$ ,  $x^\circ \cap y^\circ$ ,  $\partial x \cap y^\circ$  and  $x^\circ \cap \partial y$ . These computational operations can be used to *check* or to *characterize* the **RCC-8** relations in the following way. When a topological relation, say “ $x$  partially overlaps  $y$ ” or  $PO(x, y)$  holds between two regions  $x$  and  $y$ , then the four computational operations applied to regions  $x$  and  $y$  give the following results:  $\partial x \cap \partial y \neq \emptyset$ ,  $x^\circ \cap y^\circ \neq \emptyset$ ,  $\partial x \cap y^\circ \neq \emptyset$  and  $x^\circ \cap \partial y \neq \emptyset$ . Conversely, if the four computational operations give the previous results when applied to two regions  $x$  and  $y$ , then the relation  $PO(x, y)$  holds (see Figure 2). The relationship between these operations and the base relations of **RCC-8** is described in Table 4.

We call *CE-8* the set  $\{\text{NFO, FO, NOF, OF, DR, O, NA, A}\}$  of the eight *conditions* derived from these operations, according to the emptiness or non-emptiness of the result:

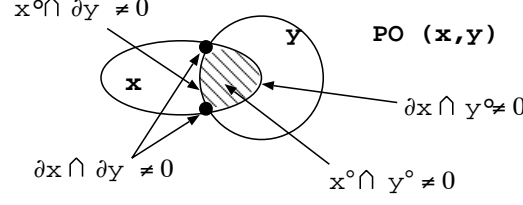


Figure 2. Relationship between the relation  $PO(x, y)$  and the computational operations defined in [21].

$x, y$	$\partial x \cap \partial y$	$x^\circ \cap y^\circ$	$\partial x \cap y^\circ$	$x^\circ \cap \partial y$
EQ	$\neq \emptyset$	$\neq \emptyset$	$\emptyset$	$\emptyset$
TPP	$\neq \emptyset$	$\neq \emptyset$	$\neq \emptyset$	$\emptyset$
NTPP	$\emptyset$	$\neq \emptyset$	$\neq \emptyset$	$\emptyset$
$TPP^{-1}$	$\neq \emptyset$	$\neq \emptyset$	$\emptyset$	$\neq \emptyset$
$NTPP^{-1}$	$\emptyset$	$\neq \emptyset$	$\emptyset$	$\neq \emptyset$
PO	$\neq \emptyset$	$\neq \emptyset$	$\neq \emptyset$	$\neq \emptyset$
EC	$\neq \emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
DC	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Table 4

Relationship between the **RCC-8** relations and the operations on interior and boundary sets proposed in [21].

- $\partial x \cap y^\circ = \emptyset$ : “the border of  $x$  is disjoint from the interior of  $y$ ” is denoted by  $NFO(x, y)$ ,
- $\partial x \cap y^\circ \neq \emptyset$  is denoted by  $FO(x, y)$ ,
- $x^\circ \cap \partial y = \emptyset$  is denoted by  $NOF(x, y)$ ,
- $x^\circ \cap \partial y \neq \emptyset$  is denoted by  $OF(x, y)$ ,
- $x^\circ \cap y^\circ = \emptyset$ : “ $x$  is discrete from  $y$ ” is denoted by  $DR(x, y)$ ,
- $x^\circ \cap y^\circ \neq \emptyset$ : “ $x$  overlaps  $y$ ” is denoted by  $O(x, y)$ ,
- $\partial x \cap \partial y = \emptyset$ : “ $x$  does not share a boundary with  $y$ ” is denoted by  $NA(x, y)$ ,
- $\partial x \cap \partial y \neq \emptyset$ : “ $x$  shares a boundary with  $y$ ” is denoted by  $A(x, y)$ .

### 2.3.2 A second set CM-8 of computational conditions

Following the work presented in [21], another set of computational operations based on the *set-difference* and the *intersection* of interior sets and boundaries has been proposed in [22]. The following four computational operations are taken into account: the intersection of the interior sets,  $x^\circ \cap y^\circ$ ; the intersection of the boundary sets,  $\partial x \cap \partial y$ ; the two differences of the interior sets,  $x^\circ - y^\circ$  and  $y^\circ - x^\circ$ . These four computational operations allow to check the **RCC-8** relations on the images, and the correspondence between each relation and



the results of the operations are described on Table 5.

x,y	$x^\circ - y^\circ$	$y^\circ - x^\circ$	$x^\circ \cap y^\circ$	$\partial x \cap \partial y$
EQ	$\emptyset$	$\emptyset$	$\neq \emptyset$	$\neq \emptyset$
TPP	$\emptyset$	$\neq \emptyset$	$\neq \emptyset$	$\neq \emptyset$
NTPP	$\emptyset$	$\neq \emptyset$	$\neq \emptyset$	$\emptyset$
TPP <sup>-1</sup>	$\neq \emptyset$	$\emptyset$	$\neq \emptyset$	$\neq \emptyset$
NTPP <sup>-1</sup>	$\neq \emptyset$	$\emptyset$	$\neq \emptyset$	$\emptyset$
PO	$\neq \emptyset$	$\neq \emptyset$	$\neq \emptyset$	$\neq \emptyset$
EC	$\neq \emptyset$	$\neq \emptyset$	$\emptyset$	$\neq \emptyset$
DC	$\neq \emptyset$	$\neq \emptyset$	$\emptyset$	$\emptyset$

Table 5

The correspondence between the computational operations defined in [22] for checking relations on raster regions, and the **RCC-8** relations.

From these four operations eight conditions are derived that compose the *CM-8* set, namely  $\{P, NP, P^{-1}, NP^{-1}, DR, O, NA, A\}$ . It can be noticed that four conditions in *CE-8*, namely DR, O, NA, and A, remain in *CM-8*:

- $x^\circ - y^\circ = \emptyset$ : “*x is a part of y*” is denoted by  $P(x, y)$ <sup>1</sup>,
- $x^\circ - y^\circ \neq \emptyset$ : “*x is not a part of y*” is denoted by  $NP(x, y)$ ,
- $y^\circ - x^\circ = \emptyset$ : “*x contains y*” (“*y is a part of x*”) is denoted by  $P^{-1}(x, y)$ ,
- $y^\circ - x^\circ \neq \emptyset$ : “*x does not contain y*” (“*y is not a part of x*”) is denoted by  $NP^{-1}(x, y)$ ,
- $x^\circ \cap y^\circ = \emptyset$  is denoted by  $DR(x, y)$  as indicated previously,
- $x^\circ \cap y^\circ \neq \emptyset$  is denoted by  $O(x, y)$ ,
- $\partial x \cap \partial y = \emptyset$  is denoted by  $NA(x, y)$ ,
- $\partial x \cap \partial y \neq \emptyset$  is denoted by  $A(x, y)$ .

Besides, it is important to notice that some conditions of *CM-8* (or *CE-8*) imply others because of the properties of the spatial regions (this will be discussed further in § 4.1). For example, each of the conditions  $x^\circ - y^\circ = \emptyset$  or  $y^\circ - x^\circ = \emptyset$  separately implies the condition  $x^\circ \cap y^\circ \neq \emptyset$  (the regions are non empty); the condition  $\partial x \cap \partial y \neq \emptyset$  is implied by the conjunction of the same conditions (the regions are non empty and closed).

<sup>1</sup> The regions are closed and thus  $x^\circ - y^\circ = \emptyset \iff x - y = \emptyset$ .

### 3 Two Galois lattices of topological relations

In this section, we present the construction of two Galois lattices of topological relations. The lattices have to satisfy the following requirements: firstly, to include the set of relations  $\mathcal{B}$ , and, respectively, the condition sets  $CM-8$  and  $CE-8$ , and secondly, to make explicit the relationship between the eight relations of  $\mathcal{B}$ , and the eight conditions of  $CM-8$  or  $CE-8$ . The two lattices are built thanks to a Galois connection between  $\mathcal{B}$ , and the sets of conditions  $CM-8$  and  $CE-8$  respectively.

Hereafter, we first give details on the design of the Galois lattice based on  $\mathcal{B} \times CM-8$ , that actually constitutes our own research work. The second Galois lattice construction, based on  $\mathcal{B} \times CE-8$ , can be read and understood in the same way.

#### 3.1 The Galois lattice based on $\mathcal{B} \times CM-8$

The presentation of the Galois lattice based on  $\mathcal{B} \times CM-8$  proposed hereafter relies on the classical presentation of concept lattices given in [4]. The formal context  $(\mathcal{B}, CM-8, I)$  consists of the set of “objects”  $\mathcal{B}$  and the set of “attributes”  $CM-8$ , and the incidence relation  $I$  that establishes a correspondence between the relations in  $\mathcal{B}$  and the conditions in  $CM-8$  as indicated below:

- For a set of relations  $R \subseteq \mathcal{B}$ , let us define  $R' = \{c \in CM-8 / \forall (x, y) \in \mathcal{D}^2 : \forall r \in R, r(x, y) \implies c(x, y)\}$ , i.e. for a pair  $(x, y) \in \mathcal{D}^2$ ,  $R'$  is the set of conditions  $c(x, y)$  that are implied by all the relations  $r(x, y) \in R$ .
- For a set of conditions  $C \subseteq CM-8$ , let us define  $C' = \{r \in \mathcal{B} / \forall (x, y) \in \mathcal{D}^2 : \forall c \in C, r(x, y) \implies c(x, y)\}$ , i.e. for a pair  $(x, y) \in \mathcal{D}^2$ ,  $C'$  is the set of relations  $r(x, y)$  implying all the conditions  $c(x, y) \in C$ .

The two derivation operators, i.e.  $R \longrightarrow R'$  and  $C \longrightarrow C'$ , can be combined in a pair of composite operators, denoted by  $''$ , that respectively map  $2^{\mathcal{B}}$  onto itself, and  $2^{CM-8}$  onto itself, and that are closure operators [2,4]. The mapping  $R \longrightarrow R'$  from  $2^{\mathcal{B}}$  onto  $2^{CM-8}$ , and the mapping  $C \longrightarrow C'$  from  $2^{CM-8}$  onto  $2^{\mathcal{B}}$ , define a Galois connection between  $2^{\mathcal{B}}$  and  $2^{CM-8}$  (see Table 6). The Galois (or concept) lattice  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  is built on the basis of these two closure operators. A concept in the Galois lattice is a pair  $(R, C)$  where the *extension* or *extent*  $R$  is such that  $R \subseteq \mathcal{B}$  and  $R' = C$ , and the *intension* or *intent*  $C$  is such that  $C \subseteq CM-8$  and  $C' = R$ . The algebraic definition of the Galois lattice  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  is given hereafter.

**Definition 1** *The Galois lattice  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  based on the formal context  $(\mathcal{B}, CM-8, I)$  is the structure  $\langle \mathcal{E}_{\mathcal{G}}, \sqsubseteq, \frown, \smile, (\emptyset, \mathcal{B}), (CM-8, \emptyset) \rangle$  where:*

	P	NP	P <sup>-1</sup>	NP <sup>-1</sup>	DR	O	NA	A
EQ	1	0	1	0	0	1	0	1
NTPP	1	0	0	1	0	1	1	0
TPP	1	0	0	1	0	1	0	1
NTPP <sup>-1</sup>	0	1	1	0	0	1	1	0
TPP <sup>-1</sup>	0	1	1	0	0	1	0	1
PO	0	1	0	1	0	1	0	1
EC	0	1	0	1	1	0	0	1
DC	0	1	0	1	1	0	1	0

Table 6

This table shows the formal context  $(\mathcal{B}, CM-8, I)$ , where an element  $(r, c) = 1$  whenever  $\forall(x, y) \in \mathcal{D}^2, r(x, y) \implies c(x, y)$ , otherwise  $(r, c) = 0$ , with  $r \in \mathcal{B}$  (line) and  $c \in CM-8$  (column).

- $\mathcal{E}_{\mathcal{G}}$  is the set of all pairs  $(R, C)$  where  $R$  is a subset of  $\mathcal{B}$  closed for  $''$ , with  $R' = C$ , and where  $C$  is a subset of  $CM-8$  closed for  $''$ , with  $C' = R$ .
- The partial ordering  $\sqsubseteq$  between two elements  $(R_1, C_1)$  and  $(R_2, C_2)$  is defined as follows:  $(R_1, C_1) \sqsubseteq (R_2, C_2) \iff R_1 \subseteq R_2$  (or, equivalently  $C_2 \subseteq C_1$ ).
- The greatest lower bound (denoted by  $\frown$ ) of two elements is defined as follows:

$$(R_1, C_1) \frown (R_2, C_2) = (R_1 \cap R_2, (C_1 \cup C_2)'')$$

- The least upper bound (denoted by  $\smile$ ) of two elements is defined as follows:

$$(R_1, C_1) \smile (R_2, C_2) = ((R_1 \cup R_2)'', C_1 \cap C_2)$$

- The top element is  $(\mathcal{B}, \emptyset)$  and the bottom element is  $(\emptyset, CM-8)$ .

In the following, the concepts of the  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  lattice may be named either as a topological relation (i.e a disjunction of  $\mathcal{B}$  relations) or as a conjunction of conditions. Moreover, the writing  $\mathbf{A}$  denotes a topological relation or a condition, while  $\mathbf{A}$  denotes the associated lattice concept.

### 3.2 The $\mathcal{T}_{\mathcal{G}\mathcal{M}}$ ordering and the implication between relations

In this paragraph, we define two projection functions mapping a concept  $(R, C)$  of the  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  lattice onto either a disjunction of relations or onto a conjunction of conditions. These two projections derive from the fact that, as introduced in section 2.3, each relation is equivalent to a conjunction of conditions, and, reciprocally, a condition is equivalent to a disjunction of relations. These two projections, named  $\pi_r$  and  $\pi_c$  are defined as follows:

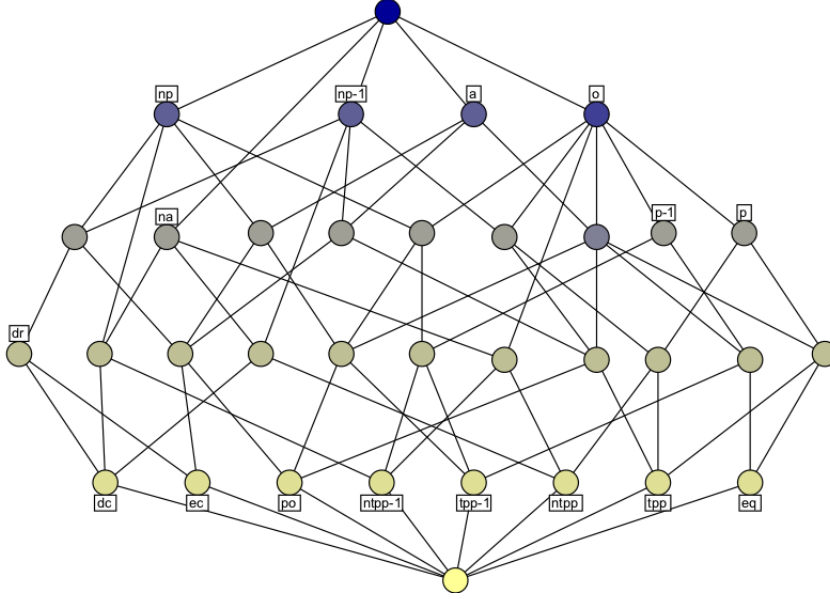


Figure 3. The Galois lattice  $\mathcal{T}_{\mathcal{GM}}$  built on the basis of Table 6.

- $\pi_r$  maps a concept  $(R, C)$  of  $\mathcal{T}_{\mathcal{GM}}$  onto the disjunction of the relations belonging to  $R$ :

$$\forall (x, y) \in \mathcal{D}^2, \pi_r(R, C)(x, y) =_{def} \bigvee_{r \in R} r(x, y)$$

- $\pi_c$  maps a concept  $(R, C)$  of  $\mathcal{T}_{\mathcal{GM}}$  onto the conjunction of the conditions belonging to  $C$ :

$$\forall (x, y) \in \mathcal{D}^2, \pi_c(R, C)(x, y) =_{def} \bigwedge_{c \in C} c(x, y)$$

These projections are introduced for fulfilling our original and concrete purpose, namely spatial computation and reasoning. Indeed, based on the construction of the  $\mathcal{T}_{\mathcal{GM}}$  lattice, the relation  $\pi_r(R, C)$  is logically equivalent to the condition  $\pi_c(R, C)$ . In other words:

$$(R, C) \in \mathcal{T}_{\mathcal{GM}} \implies (\forall (x, y) \in \mathcal{D}^2, \bigvee_{r \in R} r(x, y) \iff \bigwedge_{c \in C} c(x, y))$$

This equivalence can be interpreted as follows. Each topological relation represented in the lattice can be computed on image data, thanks to computational conditions. Conversely, each conjunction of conditions computed on spatial regions can be associated to a topological relation. For example, the topological relation  $PP(x, y)$ , “ $x$  is a proper part of  $y$ ”, is equivalent to the disjunction of the two relations  $TPP(x, y)$  and  $NTPP(x, y)$ , and is represented by the concept  $(\{TPP, NTPP\}, \{O, NP^{-1}, P\})$ . When the conditions  $O$ ,  $NP^{-1}$ , and  $P$ , on two regions  $x$  and  $y$  hold, then  $TPP \vee NTPP$  holds, and thus  $PP$  holds on the two regions.

The  $\pi_c$  and  $\pi_r$  projections make explicit the links between the lattice ordering and the implication relations between topological relations. Actually the following property may be proved for all pairs of elements of  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$ , due to the definition of partial ordering between the lattice concepts [23]:

$$(R_1, C_1) \sqsubseteq (R_2, C_2) \iff \begin{cases} \forall(x, y) \in \mathcal{D}^2, \bigvee_{r \in R_1} r(x, y) \implies \bigvee_{r \in R_2} r(x, y) \\ \forall(x, y) \in \mathcal{D}^2, \bigwedge_{c \in C_1} c(x, y) \implies \bigwedge_{c \in C_2} c(x, y) \end{cases}$$

Thus, if an element  $\mathbf{E}_1 = (R_1, C_1)$ , e.g.  $\mathbf{EQ} = (\{\mathbf{EQ}\}, \{\mathbf{O}, \mathbf{A}, \mathbf{P}, \mathbf{P}^{-1}\})$ , is less in  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  than another element  $\mathbf{E}_2 = (R_2, C_2)$ , e.g.  $\mathbf{TP} = (\{\mathbf{TPP}, \mathbf{EQ}\}, \{\mathbf{O}, \mathbf{A}, \mathbf{P}\})$ , then the disjunction of the relations in  $R_1$  implies the disjunction of the relations in  $R_2$ , and the conjunction of the conditions in  $C_1$  implies the conjunction of the conditions in  $C_2$ .

$$\begin{aligned} \mathbf{EQ} \sqsubseteq \mathbf{TP} \iff \forall(x, y) \in \mathcal{D}^2, \mathbf{EQ}(x, y) \implies (\mathbf{EQ} \vee \mathbf{TPP})(x, y) \\ \text{and } (\mathbf{O} \wedge \mathbf{A} \wedge \mathbf{P} \wedge \mathbf{P}^{-1})(x, y) \implies (\mathbf{O} \wedge \mathbf{A} \wedge \mathbf{P})(x, y) \end{aligned}$$

The following properties characterize the *glb* ( $\frown$ ) and the *lub* ( $\smile$ ) in the Galois lattice  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  (proofs are given in [23]).

- The *glb* operator  $\frown$  in the lattice is equivalent to the conjunction operator ( $\wedge$ ) on relations, i.e.  $\mathbf{E} = \mathbf{E}_1 \frown \mathbf{E}_2$  in  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  if and only if  $\pi_r(\mathbf{E})$  is equivalent to  $\pi_r(\mathbf{E}_1) \wedge \pi_r(\mathbf{E}_2)$ , or, dually,  $\mathbf{E} = \mathbf{E}_1 \frown \mathbf{E}_2$  if and only if  $\pi_c(\mathbf{E})$  is equivalent to the conjunction  $\pi_c(\mathbf{E}_1) \wedge \pi_c(\mathbf{E}_2)$ .
- By contrast, the *lub* operator is not equivalent to the disjunction operator ( $\vee$ ) on relations, because the *lub* of two closed sets is generally not a closed set, whereas the *glb* is always a closed set [2,3]. However, for all pairs of lattice concepts  $(\mathbf{E}_1, \mathbf{E}_2)$ , the *lub*  $\mathbf{F} = \mathbf{E}_1 \smile \mathbf{E}_2$  is such that the disjunction  $\pi_r(\mathbf{E}_1) \vee \pi_r(\mathbf{E}_2)$  implies the relation  $\pi_r(\mathbf{F})$ .

### 3.3 The Galois lattice based on $\mathcal{B} \times \mathcal{CE}\text{-}8$

The second Galois lattice, named  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$ , is based on the computational operations proposed in [21], and is built in the same way as  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$ . The Galois connection between the sets  $\mathcal{B}$  and  $\mathcal{CE}\text{-}8$  is given in the Table 7, and the resulting Galois lattice  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$  is displayed on Figure 4. The concepts in  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$  are

pairs  $(R, C)$ , where  $R$  is a subset of  $\mathcal{B}$ , and  $C$  is a subset of  $CE\text{-}\delta$ , e.g.  $(\{PO, TPP\}, \{FO, O, A\})$  and  $(\{EQ\}, \{NFO, NOF, O, A\})$ .

	FO	NFO	OF	NOF	DR	O	NA	A
EQ	0	1	0	1	0	1	0	1
NTPP	1	0	0	1	0	1	1	0
TPP	1	0	0	1	0	1	0	1
NTPP <sup>-1</sup>	0	1	1	0	0	1	1	0
TPP <sup>-1</sup>	0	1	1	0	0	1	0	1
PO	1	0	1	0	0	1	0	1
EC	0	1	0	1	1	0	0	1
DC	0	1	0	1	1	0	1	0

Table 7  
The Galois connection between the  $\mathcal{B}$  relations and the  $CE\text{-}\delta$  conditions.

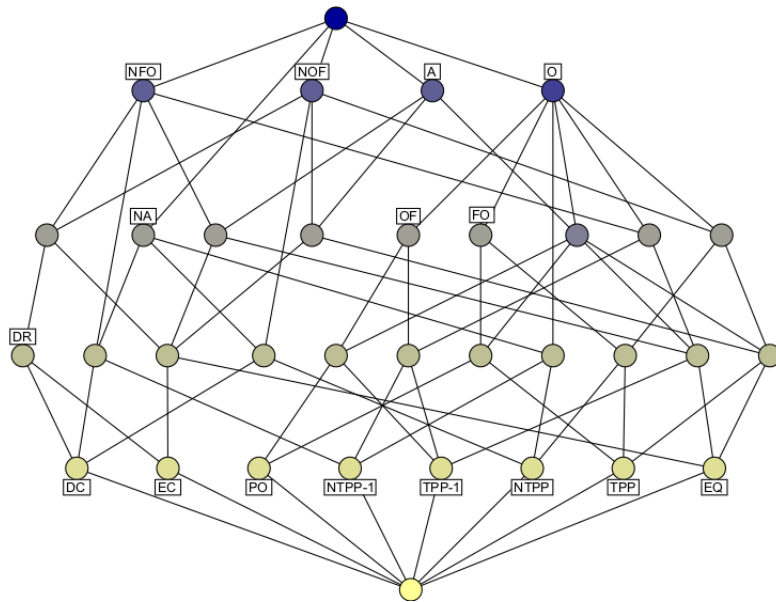


Figure 4. The Galois lattice  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$  based on  $CE\text{-}\delta$  and  $\mathcal{B}$ .

## 4 Elements for a comparison between lattices of topological relations

### 4.1 Comparing the concepts in the two $\mathcal{B}$ -based Galois lattices

Since they rely on the same set  $\mathcal{B}$  of relations, the distinction between the  $\mathcal{T}_{\mathcal{GM}}$  and  $\mathcal{T}_{\mathcal{GE}}$  Galois lattices is mainly due to the different sets of conditions, i.e.  $CM-8$  and  $CE-8$ , on which  $\mathcal{T}_{\mathcal{GM}}$  and  $\mathcal{T}_{\mathcal{GE}}$  are built.

Examining the  $\mathcal{T}_{\mathcal{GM}}$  lattice, it can be remarked first that the eight atoms of the lattice, i.e. the immediate ancestors of  $\perp$ , are the eight  $\mathcal{B}$  relations (see Figure 3). The atoms are also join-irreducible elements [2,4,24]. Then, it can be remarked that five conditions of the  $CM-8$  set are expressed within coatoms, i.e. concepts that are immediate descendants of  $\top$  in the  $\mathcal{T}_{\mathcal{GM}}$  lattice. These concepts are  $\mathbf{NA}$ ,  $\mathbf{A}$ ,  $\mathbf{NP}$ ,  $\mathbf{NP}^{-1}$ , and  $\mathbf{O}$ , whose sets of conditions are closed:  $\forall c \in \{\mathbf{NA}, \mathbf{A}, \mathbf{NP}, \mathbf{NP}^{-1}, \mathbf{O}\}, \{c\}'' = \{c\}$ . The coatoms are also meet-irreducible elements in the lattice. The other meet-irreducible elements are  $\mathbf{P}$  and  $\mathbf{P}^{-1}$  that are less than  $\mathbf{O}$ , and  $\mathbf{DR}$  that is less than the *glb* of  $\mathbf{NP}$  and  $\mathbf{NP}^{-1}$ :

$$\begin{aligned} \{\mathbf{P}\}'' &= \{\mathbf{P}, \mathbf{O}\} \\ \{\mathbf{P}^{-1}\}'' &= \{\mathbf{P}^{-1}, \mathbf{O}\} \\ \{\mathbf{DR}\}'' &= \{\mathbf{DR}, \mathbf{NP}, \mathbf{NP}^{-1}\} \end{aligned}$$

Examining the  $\mathcal{T}_{\mathcal{GE}}$  lattice, it can be remarked first, as this is the case for  $\mathcal{T}_{\mathcal{GM}}$ , that the eight atoms of the lattice are still the eight  $\mathcal{B}$  relations (see Figure 4). Then, it can be remarked that five conditions of the  $CE-8$  set are expressed within coatoms, namely  $\mathbf{NOF}$ ,  $\mathbf{NFO}$ ,  $\mathbf{A}$ ,  $\mathbf{NA}$ ,  $\mathbf{O}$ , whose sets of conditions are closed:  $\forall c \in \{\mathbf{NOF}, \mathbf{NFO}, \mathbf{A}, \mathbf{NA}, \mathbf{O}\}, \{c\}'' = \{c\}$ . The coatoms are also meet-irreducible elements in the  $\mathcal{T}_{\mathcal{GE}}$  lattice. The three other meet-irreducible elements behave as follows:

$$\begin{aligned} \{\mathbf{OF}\}'' &= \{\mathbf{OF}, \mathbf{O}\} \\ \{\mathbf{FO}\}'' &= \{\mathbf{FO}, \mathbf{O}\} \\ \{\mathbf{DR}\}'' &= \{\mathbf{DR}, \mathbf{NOF}, \mathbf{NFO}\} \end{aligned}$$

Furthermore, the Galois lattices can be compared on the basis of their sets of meet and join-irreducible elements [3], that are respectively  $\mathcal{B}$  and  $CM-8$  for  $\mathcal{T}_{\mathcal{GM}}$ , and  $\mathcal{B}$  and  $CE-8$  for  $\mathcal{T}_{\mathcal{GE}}$ . Thus, the two lattices can be directly compared with the help of the tables 6 and 7: when the two lines of the relations EQ and

PO are switched, we obtain two identical tables, wrt the following conditions correspondences:  $FO \leftrightarrow P$ ,  $OF \leftrightarrow P^{-1}$ ,  $NFO \leftrightarrow NP$ , and  $NOF \leftrightarrow NP^{-1}$ . Thus, the comparison of meet and join-irreducible elements allows to conclude that the two lattices are isomorphic.

Besides, both lattices have 34 concepts, among which 28 are identical wrt the extensions of the concepts, i.e. the sets of relations  $R$  associated to each concept  $(R, C)$  (the intensions  $C$  within  $(R, C)$  are not necessarily the same as the sets  $CM-8$  and  $CE-8$  are different). Only 6 elements of the lattices are different wrt both the extensions and the intensions. They belong to the filter of  $EC \smile EQ$  in  $\mathcal{T}_{G\mathcal{E}}$ , and to the filter of  $EC \smile PO$  in  $\mathcal{T}_{G\mathcal{M}}$  (see Figure 5). The intents of these concepts respectively are:

$$\begin{aligned} &\{NFO\} \quad \{NOF\} \quad \{NFO, NOF\} \quad \{NFO, A\} \quad \{NOF, A\} \quad \{NFO, NOF, A\} \\ &\{NP\} \quad \{NP^{-1}\} \quad \{NP, NP^{-1}\} \quad \{NP, A\} \quad \{NP^{-1}, A\} \quad \{NP, NP^{-1}, A\} \end{aligned}$$

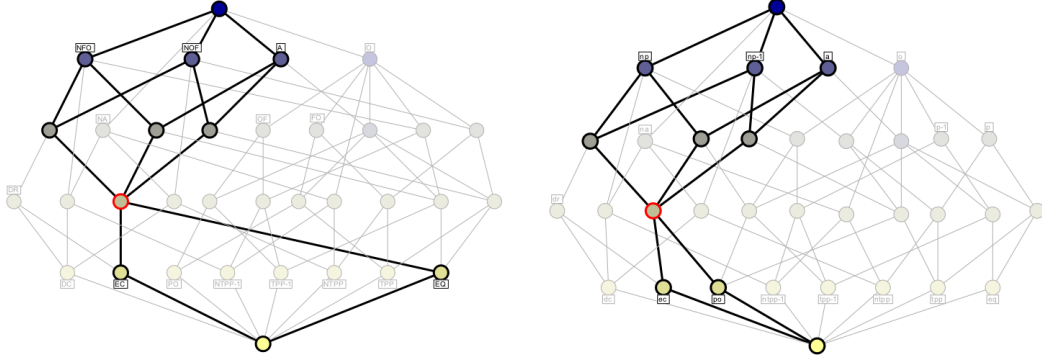


Figure 5. Comparing the concepts of  $\mathcal{T}_{G\mathcal{E}}$  (left) and  $\mathcal{T}_{G\mathcal{M}}$ (right): the differences are highlighted.

Based on these observations, we can define a bijective map  $\Phi_{ME}$  between the two lattices:

$$\begin{aligned} \Phi_{ME} : \mathcal{T}_{G\mathcal{M}} &\quad \mapsto \quad \mathcal{T}_{G\mathcal{E}} \\ (R_M, C_M) &\quad \mapsto \quad (R_E, C_E) \end{aligned}$$

with the following mappings:  $P \in C_M \mapsto FO \in C_E$ ,  $P^{-1} \in C_M \mapsto OF \in C_E$ ,  $NP \in C_M \mapsto NFO \in C_E$  and  $NP^{-1} \in C_M \mapsto NOF \in C_E$  for the intents, and  $EQ \in R_M \mapsto PO \in R_E$ ,  $PO \in R_M \mapsto EQ \in R_E$  for the extents; the other conditions and relations map onto themselves.

$\Phi_{ME}$  is order-preserving and also join- and meet-preserving according to the properties of lattice-isomorphisms [3].



#### 4.2 Comparing the computation skills of the two $\mathcal{B}$ -based Galois lattices

In a general way, the higher is a condition  $c$  in the lattice, the higher is the number of relations that can be tested using  $c$ . More precisely, a relation, say  $R$ , expressed in the Galois lattices  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  or  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$ , is associated to a set  $C$  of conditions that can be interpreted as a “necessary and sufficient” set of conditions to be tested for checking  $R$ . For example, the relation EQ in  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  is associated to the set of conditions  $C = \{O, A, P, P^{-1}\}$  (see Table 6). Thus,  $\forall(x, y) \in \mathcal{D}^2$ , if EQ( $x, y$ ) holds then  $(O \wedge A \wedge P \wedge P^{-1})(x, y)$  holds, and the converse is also true. Actually, for checking EQ( $x, y$ ) in  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$ , it is sufficient to check only a subset of conditions of  $C = \{O, A, P, P^{-1}\}$ , namely  $C_1 = \{P, P^{-1}\}$ . Indeed, EQ is the *glb* in  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  of elements P and  $P^{-1}$ , that respectively represent the conditions P and  $P^{-1}$ . By contrast, EQ is in  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$  the *glb* of O, NFO, and NOF, meaning that the three corresponding conditions have to be checked.

These properties can be linked to the Duquenne-Guigues (DG) basis associated to each lattice [25,26]. Actually, for  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$ , the rules of the DG basis are the following:

- (1) P  $\implies$  O
- (2)  $P^{-1}$   $\implies$  O
- (3) DR  $\implies$  NP  $\wedge$  NP $^{-1}$
- (4) NA  $\wedge$  O  $\wedge$  P  $\implies$  NP $^{-1}$
- (5) NA  $\wedge$  O  $\wedge$  NP  $\implies$  P $^{-1}$
- (6) NA  $\wedge$  O  $\wedge$  P $^{-1}$   $\implies$  NP
- (7) NA  $\wedge$  O  $\wedge$  NP $^{-1}$   $\implies$  P
- (8) NA  $\wedge$  NP  $\wedge$  NP $^{-1}$   $\implies$  DR
- (9) O  $\wedge$  P  $\wedge$  P $^{-1}$   $\implies$  A
- (10) O  $\wedge$  NP  $\wedge$  NP $^{-1}$   $\implies$  A

For example, these rules lead to the following implications:

$$P \wedge P^{-1} \implies^{(1,2)} O \wedge P \wedge P^{-1} \implies^{(9)} O \wedge P \wedge P^{-1} \wedge A \iff \text{EQ}$$

Depending on the Galois lattice  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  and  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$ , there exists *minimal sets of conditions* for checking a  $\mathcal{B}$  relation. Thus, it can be worth comparing the cardinality of these minimal sets of conditions. Table 8 summarizes the subsets of relations, the associated minimal subsets, and their cardinalities. It appears that, except for EQ and PO, checking a base relation requires to check the same number of conditions in the two lattices. For example, checking the relations

NTPP or DC requires two conditions both in  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  and in  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$ :  $\{\underline{P}, \underline{NA}\}$  vs.  $\{\underline{FO}, \underline{NA}\}$ , and  $\{\underline{DR}, \underline{NA}\}$  vs.  $\{\underline{DR}, \underline{NA}\}$  respectively. By contrast, checking EQ requires two conditions in  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$ , and three conditions in  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$  as discussed here-above, checking PO requires three conditions in  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$ , namely  $\{\underline{NP}, \underline{NP}^{-1}, \underline{O}\}$ , and only two conditions in  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$ , namely  $\{\underline{FO}, \underline{OF}\}$ .

$\mathbf{E}_E/\mathbf{E}_M$	$C_E$	$C_M$	$n_E$	$n_M$
EQ	$\{\underline{NFO}, \underline{NOF}, \underline{O}, \underline{A}\}$	$\{\underline{P}, \underline{P}^{-1}, \underline{O}, \underline{A}\}$	3	2
TPP	$\{\underline{FO}, \underline{O}, \underline{A}, \underline{NOF}\}$	$\{\underline{P}, \underline{O}, \underline{A}, \underline{NP}^{-1}\}$	3	3
NTPP	$\{\underline{FO}, \underline{O}, \underline{NA}, \underline{NOF}\}$	$\{\underline{P}, \underline{O}, \underline{NA}, \underline{NP}^{-1}\}$	2	2
TPP <sup>-1</sup>	$\{\underline{OF}, \underline{O}, \underline{A}, \underline{NFO}\}$	$\{\underline{P}^{-1}, \underline{O}, \underline{A}, \underline{NP}\}$	3	3
NTPP <sup>-1</sup>	$\{\underline{OF}, \underline{O}, \underline{NA}, \underline{NFO}\}$	$\{\underline{P}^{-1}, \underline{O}, \underline{NA}, \underline{NP}\}$	2	2
PO	$\{\underline{FO}, \underline{OF}, \underline{O}, \underline{A}\}$	$\{\underline{NP}, \underline{NP}^{-1}, \underline{O}, \underline{A}\}$	2	3
EC	$\{\underline{DR}, \underline{NOF}, \underline{NFO}, \underline{A}\}$	$\{\underline{DR}, \underline{NP}, \underline{NP}^{-1}, \underline{A}\}$	2	2
DC	$\{\underline{DR}, \underline{NOF}, \underline{NFO}, \underline{NA}\}$	$\{\underline{DR}, \underline{NP}, \underline{NP}^{-1}, \underline{NA}\}$	2	2

Table 8

Sets of conditions associated to the base relations in  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$  and  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$ . The first column gives the names of the corresponding elements in the two lattices. The two following columns give the sets of conditions  $C_E$  (in  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$ ) and  $C_M$  (in  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$ ). The two last columns give the minimal numbers  $n_E, n_M$  of conditions to be used for checking the relations. The conditions to be checked are underlined.

From the point of view of minimal sets of conditions to be checked, the  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  and  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$  lattices have also similar characteristics: the same number of conditions has to be checked in both lattices  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$  and  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$ . More precisely, among the 26 sets of relations common to both lattices ( $\emptyset$  and  $\mathcal{B}$  excluded), 16 require the same number of conditions, 5 require one additional condition in  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$ , and 5 require one additional condition in  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$ . Thus, the choice of using one Galois lattice rather than another cannot be based on the average number of necessary tests to be performed for checking the base relations. This choice depends more on the adequacy of the lattice with the needs of the application: it can be preferable to compute either the intersection of a boundary set and an interior set ( $\partial x \cap y^\circ, x^\circ \cap \partial y$ ), or the difference of two interior sets ( $x^\circ - y^\circ, y^\circ - x^\circ$ ), as discussed for raster data in [20].

#### 4.3 The Galois lattices $\mathcal{T}_{\mathcal{G}\mathcal{M}}$ and $\mathcal{T}_{\mathcal{G}\mathcal{E}}$ wrt the boolean lattice $\mathcal{T}_{\mathcal{P}}$

Let us now introduce the boolean lattice  $\mathcal{T}_{\mathcal{P}} = \langle 2^{\mathcal{B}}, \subseteq, \cap, \cup, \mathcal{B}, \emptyset \rangle$ , whose cardinality is  $2^8 = 256$ . Being boolean, this lattice is distributive and complemented, and thus includes the representation of all the conjunctions, the disjunctions, and the negations of the relations of  $\mathcal{B}$ . In the following, we compare

the Galois lattices  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  and  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$  with the boolean lattice  $\mathcal{T}_{\mathcal{P}}$  from three points of view: implementation, concept representation, and reasoning.

Considering the implementation and memory occupation point of view, the Galois lattices  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  and  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$  are much easier to implement and to manage than the  $\mathcal{T}_{\mathcal{P}}$  lattice. Actually, the implementation of the  $\mathcal{T}_{\mathcal{P}}$  lattice leads to the design of 256 classes of relations in the framework of an object-based knowledge representation system. By contrast, the two Galois lattices  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  and  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$  have a smaller size of 34 concepts. Thus, if the memory occupation may be a problem, e.g. an object-based knowledge representation system has to be reasonably sized in terms of memory occupation, the choice of one of these Galois lattices seems to be obvious.

In terms of representation, by construction, the Galois lattices  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  and  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$  include the representation of the conditions used to check the relations on spatial regions, and the representation of the implication relations holding between the conditions and the relations. By contrast, the duality relations – conditions (or extent – intent) is not explicitly present in the  $\mathcal{T}_{\mathcal{P}}$  lattice. Actually, the conditions could be included in  $\mathcal{T}_{\mathcal{P}}$  as formal attributes because they are equivalent to disjunctions of relations in  $\mathcal{B}$ . Moreover, for the main part, the relations represented in  $\mathcal{T}_{\mathcal{P}}$  cannot be associated with the usual conditions. This shows that the  $\mathcal{T}_{\mathcal{P}}$  lattice is not very useful for taking into account quantitative data.

The last comparison point holds on reasoning within the lattices. Reasoning is complete –wrt conjunction, disjunction, negation, and composition– in  $\mathcal{T}_{\mathcal{P}}$  whereas it is not in the Galois lattices, except for conjunction. The reasons of the incompleteness are the following:

- The whole set of compositions of the base relations is not present in the Galois lattices  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  and  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$ . The  $\mathcal{T}_{\mathcal{P}}$  lattice includes all the disjunctions of  $\mathcal{B}$  relations, and thus it contains all the relations of the composition table of  $\mathcal{B}$ . The  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  and  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$  lattices lack respectively three and five relation compositions.
- The Galois lattices are not complemented, i.e. each concept in  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  and  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$  does not necessarily have a complement and thus reasoning is not complete wrt negation.
- The *lub* of two concepts is not equivalent to the disjunction of the corresponding relations (see paragraph 3.2), and thus reasoning is not complete wrt disjunction.

Thus, if completeness of reasoning is mandatory, the  $\mathcal{T}_{\mathcal{P}}$  lattice has to be preferred. For combining the advantages of the Galois lattices, i.e. memory occupation and concept representation, and the advantages of the  $\mathcal{T}_{\mathcal{P}}$  lattice, i.e. completeness of reasoning, it is possible to extend the Galois lattices with

the lacking relation compositions. However, this cannot be done straightforwardly, because the relation sets associated to the concepts to be added in the Galois lattices do not correspond to any condition set (details are given in [20]).

## 5 Using lattices for qualitative spatial representation and reasoning in practice

From a knowledge representation and reasoning point of view, a Galois lattice can be viewed as a hierarchical conceptual clustering of individuals, and as a representation of all implications between the attributes of the concepts [27]. Moreover, Galois lattices are well adapted to the classification of topological relations, because they combine computational capabilities involving computational conditions, and reasoning capabilities involving relations. In addition, the  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  and  $\mathcal{T}_{\mathcal{G}\mathcal{E}}$  lattices have the great advantage of being minimal in terms of memory occupation. Thus, it can be practical to implement a Galois lattice for applications dealing with spatial data. In our framework, the  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  lattice has been implemented within an object-based knowledge representation system, namely the Y3 system [28], and used for analyzing land-use maps.

The Y3 system is based on a frame language where classes and instances are implemented as *frames*, composed of *slots*, representing both attributes and methods. Attributes are characterized by declarative and procedural *facets*: the former are used to represent the range and the value of the attributes while the latter are used to specify local behaviors. Binary relations are special kinds of attributes characterized by a domain and a range classes, and possibly other information such as the inverse relation. Reasoning is mainly based on (i) the subsumption relation between classes, that can be likened to the partial ordering  $\sqsubseteq$  of a lattice, (ii) on the inheritance mechanism, i.e. property sharing between classes, (iii) on classification (for classes), and on identification or recognition (for individuals). The classification process is used for inserting a (new defined) class in the class hierarchy by searching for its most specific subsumer classes, and its most general subsumee classes. The identification process consists in searching for the classes an individual may be instance of. The classification and inheritance mechanisms are based on attribute unification. When classifying a class or identifying an individual, say  $\mathbf{x}$ , the system checks whether the pairs (`attribute`, `range`) or (`attribute`, `value`) of  $\mathbf{x}$  are *conform* to the pairs in the reference classes: if this is the case, the classification or identification process succeeds and goes on until all attributes in a class have been checked, and then until the most specific class has been checked within the class hierarchy.

The concepts of  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  have been represented within relation classes organized

according to the lattice ordering. The classes are described by attributes representing the properties of the lattice concepts, such as `complement`, `converse`, `condition`, etc. The composition of relations is implemented as a method defined for the base relations. Furthermore, methods for checking the conditions on the image regions are defined in the classes representing computational conditions, e.g. `P P-1`, `A`, etc. These methods are inherited by more specialized relation classes, according to the lattice ordering.

The representation of the  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  lattice has been used in the context of the LOLA system for landscape analysis. The main objective of this system is to recognize *landscape models* on land-use maps extracted from satellite images. With respect to this task, models describing landscape structures in the Lorraine region (East of France), e.g. territories and fields, have been represented as classes and organized within several class hierarchies. The regions of the land-use maps are then classified according to the model class hierarchies: actually, landscape structures are nested structures, and the classification of a particular image region requires the classification of a set of interrelated regions and the corresponding relations. In this way, the classification mechanism manages interrelated class hierarchies: one is the lattice of relation classes based on the  $\mathcal{T}_{\mathcal{G}\mathcal{M}}$  lattice, and the others are the hierarchies of models classes. A complete description of the LOLA system, its implementation and the results that have been obtained, are detailed in [12,20].

## 6 Related work

Our proposition is related to several research domains. From a theoretical point of view, it is related to qualitative spatial reasoning and relation computation. It aims at taking advantage of the structure and properties of Galois lattices for representing and managing topological relations, their mathematical and computational properties. Most of the work in qualitative spatial reasoning focuses on the definition of logical formalisms, whose concern is automated reasoning [16,8,9,10]. In this research work, boolean lattices have been used for representing spatial relations, for automatic theorem proving and for relation inferring: the composition of two arbitrary relations may be inferred from the lattice structure and a composition table [8]. Fewer works deal with the computation of topological relations on spatial region, with respect to quantitative data. In these works, topological relations are defined as the result of computational operations on the regions, their interiors and boundaries. Vector or raster regions are considered [5,6,7,29]. Our work combines both aspects through concept lattices, that emphasize the links between qualitative models and computational operations, and thus quantitative data. Furthermore, we show that the specific properties of Galois lattices are useful for managing the properties of qualitative relations, and thus for spatial reasoning. Finally,

this work is original because it considers Galois lattices as an actual basis for object-based representation systems, contrasting with their usual applications in formal concept analysis [4], or in knowledge discovery [30,31,32].

From the application point of view, our approach can be compared to works in the fields of geographic information or image understanding. Actually, the problem of representing and recognizing spatial structures has been studied both for images and cartographic maps. In particular, lattices have been used in geographic information systems to represent hierarchical spatial entities (points, lines, triangles), and then to compute topological relations between these entities [11]. For example, the lattice in Figure 6 describes two regions A and B, and four sub-regions 1, 2, 3, 4, where A includes the sub-regions 3, 4, 1, and B the sub-regions 2, 3, 4. This last approach is different from ours, since we do not model data but we use the lattice structure to represent concepts, and especially spatial relations and their properties.

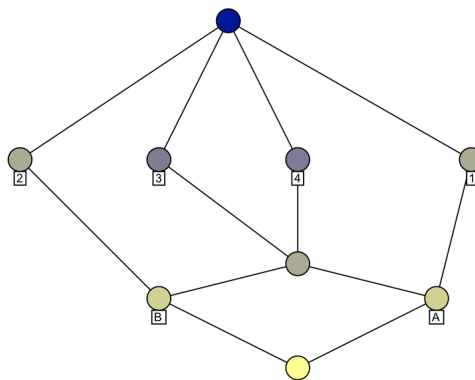


Figure 6. Two regions including sub-regions [11].

Our approach also presents similarities with the approach used in the VEIL system for the recognition of objects in an image [33,34]: the domain model is described within hierarchies of concepts implemented within the LOOM system [35,36]. The LOOM classifier is used to classify the image objects into concepts according to the available information, but there does not exist a specific representation of spatial relations as in our research work.

Finally, let us mention related work holding on qualitative spatial representation for managing map databases and spatial query processing. In [37], spatial reasoning is performed within a description logic framework, and relies on two main operations, namely consistency checking and classification. In [10], spatio-temporal default reasoning is introduced: a specific query completion problem is studied and default knowledge is used for completing and making queries more precise. These two research works have same objectives as ours, but they do not rely on an explicit lattice-based representation of spatial relations.

## 7 Conclusion and Future Work

In this paper, we have presented and compared two Galois lattices representing topological relations for spatial reasoning. These Galois lattices provide an efficient framework for classifying topological relations, and for designing a connection between computational conditions –linked with numerical data– and topological relations of the **RCC-8** theory –used for qualitative spatial reasoning. This connection is of main importance and general interest. Furthermore, Galois lattices are not only theoretical tools: they provide a practical framework for implementing hierarchical knowledge-based systems, and give adequate guidelines for implementing object-based knowledge representation systems.

Following the present research work, there is a number of research perspectives among which the comparison with Galois lattices organizing other sets of topological relations or conditions (e.g. [29]), the study of the complexity of qualitative spatial reasoning in Galois lattices, and the possible extensions of Galois lattices to ensure a complete reasoning. Indeed, there is a trade-off between completeness of reasoning and the size of the lattice organizing topological relations. From the application point of view, this work can be extended to hierarchical case-based reasoning for landscape analysis; first elements in this research direction are proposed in [38].

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