Functions of q-positive type

Lazhar Dhaouadi *

Abstract

In this paper we characterize the subspace of $\mathcal{L}_{q,1,v}$ of function which are the q-Bessel Fourier transform of positive functions in $\mathcal{L}_{q,1,v}$. As application we give a q-version of the Bochner's theorem.

1 Introduction and Preliminaries

Given a positive finite Borel measure μ on the real line $\mathbb{R}$, the Fourier transform Q of μ is the continuous function

$$Q(x) = \int_{\mathbb{R}} e^{-itx} d\mu(t).$$

The function Q is a positive definite function, i.e for any finite list of complex numbers $z_1, \ldots, z_n$ and real numbers $x_1, \ldots, x_n$

$$\sum_{r=1}^{n} \sum_{l=1}^{n} z_r \overline{z_l} Q(x_r - x_l) \geq 0.$$

Bochner's theorem says the converse is true, i.e. every positive definite function Q is the Fourier transform of a positive finite Borel measure. In q-Fourier analysis, semelar phenomenon will appear. It is the subject of our article.

In the following we consider $0 < q < 1$ and we adopt the standard conventional notations of [2]. We put

$$\mathbb{R}_q^+ = \{q^n, \quad n \in \mathbb{Z}\},$$

*Institut Préparatoire aux Etudes d'Ingénieur de Bizerte (Université du 7 novembre Carthage). Route Menzel Abderrahmene Bizerte, 7021 Zarzouna, Tunisia. E-mail lazhardhaouadi@yahoo.fr

the set of q-real numbers and for complex a

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i), \quad n = 1...\infty.$$

Jackson's q-integral (see [3]) in the interval $[0,\infty[$ is defined by

$$\int_0^\infty f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n).$$

Let $\mathcal{C}_{q,0}$ and $\mathcal{C}_{q,b}$ denote the spaces of functions defined on $\mathbb{R}_q^+$ continued at 0, which are respectively vanishing at infinity and bounded. These spaces are equipped with the topology of uniform convergence, and by $\mathcal{L}_{q,p,v}$ the space of functions f defined on $\mathbb{R}_q^+$ such that

$$\|f\|_{q,p,v} = \left[\int_0^\infty |f(x)|^p x^{2v+1} d_qx\right]^{1/p} < \infty.$$

The q-exponential function is defined by

$$e(z,q) = \sum_{n=0}^\infty \frac{z^n}{(q,q)_n} = \frac{1}{(z;q)_\infty}, \quad |z| < 1.$$

The normalized Hahn-Exton q-Bessel function of order $v > -1$ (see [5]) is defined by

$$j_v(z,q) = \sum_{n=0}^\infty (-1)^n \frac{q^{\frac{n(n-1)}{2}}}{(q,q)_n(q^{v+1},q)_n} z^n.$$

The q-Bessel Fourier transform $\mathcal{F}_{q,v}$ introduced in [1,4] as follow

$$\mathcal{F}_{q,v}f(x) = c_{q,v}\int_0^\infty f(t)j_v(xt,q^2)t^{2v+1}d_qt,$$

where

$$c_{q,v} = \frac{1}{1-q}\frac{(q^{2v+2},q^2)_\infty}{(q^2,q^2)_\infty}.$$

Define the $q-$Bessel translation operator as follows:

$$T_{q,x}^v f(y) = c_{q,v}\int_0^\infty \mathcal{F}_{q,v}(f)(t)j_v(xt,q^2)j_v(yt,q^2)t^{2v+1}d_qt, \quad \forall x,y \in \mathbb{R}_q^+, \forall f \in \mathcal{L}_{q,v,1}.$$

Recall that $T^v_{q,x}$ is said positive if $T^v_{q,x}f \geq 0$ for $f \geq 0$. In the following we tack $q \in Q_v$ where

$$Q_v = \{q \in]0,1[, \quad T^v_{q,x} \quad \text{is positive for all} \quad x \in \mathbb{R}^+_q\}.$$

The $q-$convolution product of both functions $f, g \in \mathcal{L}_{q,1,v}$ is defined by

$$f *_q g(x) = c_{q,v} \int_0^\infty T^v_{q,x} f(y) g(y) y^{2v+1} d_q y.$$

In the end we denote by $\mathcal{A}_{q,v}$ the q-Wiener algebra

$$\mathcal{A}_{q,v} = \{f \in \mathcal{L}_{q,1,v}, \quad \mathcal{F}_{q,v}(f) \in \mathcal{L}_{q,1,v}\}.$$

The followings results in this sections was proved in [1].

Proposition 1 *Let $n, m \in \mathbb{Z}$ and $n \neq m$, then we have*

$$c^2_{q,v} \int_0^\infty j_v(q^n x, q^2) j_v(q^m x, q^2) x^{2v+1} d_q x = \frac{q^{-2n(v+1)}}{1-q} \delta_{nm}.$$

Proposition 2

$$|j_v(q^n, q^2)| \leq \frac{(-q^2;q^2)_\infty (-q^{2v+2};q^2)_\infty}{(q^{2v+2};q^2)_\infty} \begin{cases} 1 & \text{if} \quad n \geq 0 \\ q^{n^2+(2v+1)n} & \text{if} \quad n < 0 \end{cases}.$$

Proposition 3 *The $q-$Bessel Fourier transform*

$$\mathcal{F}_{q,v} : \mathcal{L}_{q,1,v} \to \mathcal{C}_{q,0},$$

satisfying

$$\|\mathcal{F}_{q,v}(f)\|_{\mathcal{C}_{q,0}} \leq B_{q,v} \|f\|_{q,1,v},$$

where

$$B_{q,v} = \frac{1}{1-q} \frac{(-q^2;q^2)_\infty (-q^{2v+2};q^2)_\infty}{(q^2;q^2)_\infty}.$$

Theorem 1 *Given $f \in \mathcal{L}_{q,1,v}$ then we have*

$$\mathcal{F}^2_{q,v}(f)(x) = f(x), \quad \forall x \in \mathbb{R}^+_q.$$

If $f \in \mathcal{L}_{q,1,v}$ and $\mathcal{F}_{q,v}(f) \in \mathcal{L}_{q,1,v}$ then

$$\|\mathcal{F}_{q,v}(f)\|_{q,2,v} = \|f\|_{q,2,v}.$$

Proposition 4 *Let $f \in \mathcal{L}_{q,1,v}$ then*

$$T^v_{q,x} f(y) = \int_0^\infty f(z) D_v(x,y,z) z^{2v+1} d_q z,$$

where

$$D_v(x,y,z) = c^2_{q,v} \int_0^\infty j_v(xt,q^2) j_v(yt,q^2) j_v(zt,q^2) t^{2v+1} d_q t.$$

Proposition 5 *Given two functions $f, g \in \mathcal{L}_{q,v,1}$ then*

$$f *_q g \in \mathcal{L}_{q,v,1},$$

and

$$\mathcal{F}_{q,v}(f *_q g) = \mathcal{F}_{q,v}(f) \times \mathcal{F}_{q,v}(g).$$

Proposition 6 *The q-Gauss kernel*

$$G^v(x,t,q^2) = \frac{(-q^{2v+2}t, -q^{-2v}/t; q^2)_\infty}{(-t, -q^2/t; q^2)_\infty} e(-\frac{q^{-2v}}{t} x^2, q^2),$$

satisfying

$$\mathcal{F}_{q,v}\left\{ e(-ty^2, q^2) \right\}(x) = G^v(x,t,q^2),$$

and for all function $f \in \mathcal{C}_{q,b}$

$$\lim_{a \to 0} c_{q,v} \int_0^\infty f(x) G^v(x,a^2,q^2) x^{2v+1} d_q x = f(0).$$

Theorem 2 *Given $1 < p, p', r \leq 2$ and*

$$\frac{1}{p} + \frac{1}{p'} - 1 = \frac{1}{r}.$$

If $f \in \mathcal{L}_{q,p,v}$ and $g \in \mathcal{L}_{q,p',v}$ then

$$f *_q g \in \mathcal{L}_{q,r,v}.$$

2 Functions of q-positive type

Definition 1 *A function ϕ is of q-positive type if*

$$\phi \in \mathcal{C}_{q,b} \cap \mathcal{L}_{q,1,v}$$

and for any finite list of complex numbers $z_1, \ldots, z_n$ and q-real numbers $x_1, \ldots, x_n$

$$\sum_{r=1}^{n} \sum_{l=1}^{n} z_r \overline{z_l} T_{q,x_r}^{v} \phi(x_l) \geq 0. \tag{1}$$

Proposition 7 *Let $\phi \in \mathcal{A}_{q,v}$ of q-positive type then $\mathcal{F}_{q,v}\phi$ is of q-positive type.*

Proof. From Proposition 3 and the definition of the q-Wiener algebra

$$\mathcal{F}_{q,v}(\phi) \in \mathcal{C}_{q,b} \cap \mathcal{L}_{q,1,v}.$$

On the other hand, with the inversion formula in Theorem 1 we get

$$T_{q,x}^{v} \mathcal{F}_{q,v}(\xi)(y) = \int_0^\infty j_v(tx, q^2) j_v(ty, q^2) t^{2v+1} \phi(t) d_q t,$$

then

$$\begin{aligned}
\sum_{r=1}^{n} \sum_{l=1}^{n} z_r \overline{z_l} T_{q,x_r}^{v} \mathcal{F}_{q,v}\xi(x_l) &= c_{q,v} \int_0^\infty \left[\sum_{r=1}^{n} \sum_{l=1}^{n} z_r \overline{z_l} j_v(x_r t, q^2) j_v(x_l t, q^2) \right] t^{2v+1} \phi(t) d_q t \\
&= c_{q,v} \int_0^\infty \left[\sum_{r=1}^{n} z_r j_v(x_r t, q^2) \right] \overline{\left[\sum_{l=1}^{n} z_l j_v(x_l t, q^2) \right]} t^{2v+1} \phi(t) d_q t \\
&= c_{q,v} \int_0^\infty \left| \sum_{r=1}^{n} z_r j_v(x_r t, q^2) \right|^2 t^{2v+1} \phi(t) d_q t \geq 0.
\end{aligned}$$

This finish the proof. ■

Proposition 8 *If ϕ is of q-positive type and $f \in \mathcal{L}_{q,2,v}$ then*

$$\phi *_q f \in \mathcal{L}_{q,2,v},$$

and

$$\langle \phi *_q f, f \rangle \geq 0.$$

Proof. From Theorem 2 we see that $\phi *_q f \in \mathcal{L}_{q,2,v}$. On the other hand

$$\begin{aligned}\langle \phi *_q f, f\rangle &= c_{q,v}^2 \int_0^\infty \left[\int_0^\infty T_{q,x}^v \phi(y) f(y) y^{2v+1} d_q y\right] f(x) x^{2v+1} d_q x \\ &= (1-q)^2 c_{q,v}^2 \sum_{r=1}^{\infty}\sum_{l=1}^{\infty} q^{(2v+2)r} f(q^r) q^{(2v+2)l} f(q^l) T_{q,q^r}^v \phi(q^l) \geq 0.\end{aligned}$$

This finish the proof. ■

Corollary 1 *If ϕ is of q-positive type then*

$$\mathcal{F}_{q,v}\phi(x) \geq 0, \quad \forall x \in \mathbb{R}_q^+.$$

Proof. Given $x \in \mathbb{R}_q^+$ and let

$$f_x : t \mapsto c_{q,v} j_v(xt, q^2),$$

then with Proposition 2 we see that $f_x \in \mathcal{L}_{q,2,v}$ and by Proposition 1

$$\mathcal{F}_{q,v} f_x(y) = \delta_{q,v}(x,y),$$

which implies (see[1])

$$\langle \mathcal{F}_{q,v}\phi \times \mathcal{F}_{q,v} f_x, \mathcal{F}_{q,v} f_x\rangle = \mathcal{F}_{q,v}\phi(x)\delta_{q,v}(x,x) = \frac{1}{(1-q)x^{2v+2}}\mathcal{F}_{q,v}\phi(x).$$

From Proposition 8

$$\langle \mathcal{F}_{q,v}\phi \times \mathcal{F}_{q,v} f, \mathcal{F}_{q,v} f\rangle = \langle \phi *_q f, f\rangle \geq 0,$$

this leads to the result. ■

Proposition 9 *If ϕ is of q-positive type then $\mathcal{F}_{q,v}\phi \in \mathcal{L}_{q,1,v}$.*

Proof. From Proposition 6

$$\lim_{a\to 0}\int_0^\infty e(-a^2 x, q^2)\mathcal{F}_{q,v}\phi(x) x^{2v+1} d_q x = \lim_{a\to 0} c_{q,v}\int_0^\infty G^v(x, a^2, q^2)\phi(x) x^{2v+1} d_q x = \phi(0).$$

By the monotone convergence theorem and the preview corollary we see that

$$\int_0^\infty |\mathcal{F}_{q,v}\phi(x)| x^{2v+1} d_q x = \int_0^\infty \mathcal{F}_{q,v}\phi(x) x^{2v+1} d_q x = \phi(0).$$

This finish the proof. ■

Corollary 2 *If ϕ is of q-positive type then there exist a positive function $\xi \in \mathcal{A}_{q,v}$ such that*

$$\phi(x) = \mathcal{F}_{q,v}\xi(x), \quad \forall x \in \mathbb{R}_q^+.$$

Proof. From the inversion formula in theorem 1

$$\phi(x) = \mathcal{F}_{q,v}^2\phi(x), \quad \forall x \in \mathbb{R}_q^+.$$

Define the function ξ as follows

$$\xi(x) = \mathcal{F}_{q,v}\phi(x).$$

By the use of Corollary 1 and Proposition 9 we see that ξ is a positive function of $\mathcal{A}_{q,v}$. ■

Proposition 10 *Suppose ϕ is of q-positive type. If $f \in \mathcal{L}_{q,1,v}$ is positive function then the product $\phi\mathcal{F}_{q,v}f$ is of q-positive type.*

Proof. Proposition 5 and Proposition 9 give

$$\mathcal{F}_{q,v}(\phi\mathcal{F}_{q,v}f)(t) = \mathcal{F}_{q,v}\phi *_q f(t), \quad \forall t \in \mathbb{R}_q^+,$$

then

$$\begin{aligned}
&\sum_{r=1}^{n}\sum_{l=1}^{n} z_r\overline{z_l}T_{q,x_r}^{v}\left(\phi\mathcal{F}_{q,v}f\right)(x_l)\\
&= c_{q,v}\int_0^\infty \left[\sum_{r=1}^{n}\sum_{l=1}^{n} z_r j_v(x_r t, q^2)\overline{z_l j_v(x_l t, q^2)}\right]\mathcal{F}_{q,v}(\phi\mathcal{F}_{q,v}f)(t)t^{2v+1}d_qt\\
&= c_{q,v}\int_0^\infty \left[\sum_{r=1}^{n}\sum_{l=1}^{n} z_r j_v(x_r t, q^2)\overline{z_l j_v(x_l t, q^2)}\right]\mathcal{F}_{q,v}\phi *_q f(t)t^{2v+1}d_qt\\
&= c_{q,v}\int_0^\infty \left|\sum_{r=1}^{n} z_r j_v(x_r t, q^2)\right|^2 \mathcal{F}_{q,v}\phi *_q f(t)t^{2v+1}d_qt.
\end{aligned}$$

From the definition of the q-convolution product we write

$$\mathcal{F}_{q,v}\phi *_q f(t) = c_{q,v}\int_0^\infty \mathcal{F}_{q,v}\phi(z)T_{q,t}f(z)z^{2v+1}d_qz.$$

Proposition 4 give

$$T_{q,t}f(z) = c_{q,v}\int_0^\infty D_v(t,z,s)f(s)s^{2v+1}d_qs \geq 0.$$

This implies with Corollary 1

$$\mathcal{F}_{q,v}\phi *_q f(t) \geq 0,$$

which leads to the result. ■

Corollary 3 *Given two functions ϕ_1, ϕ_2 which are of q-positive type then the product $\phi_1 \times \phi_2$ is also of q-positive type.*

Proof. Let $\xi = \mathcal{F}_{q,v}\phi_2$ then with the inversion formula in theorem 1 we see that $\mathcal{F}_{q,v}\xi = \phi_2$. Proposition 9 give

$$\xi \in \mathcal{L}_{q,1,v},$$

and by Proposition 10 we achieved the proof. ■

3 q-Bochner's Theorem

We consider the set $\mathcal{M}_q^+$ of positives and bonded measures on $\mathbb{R}_q^+$. The q-Bessel Fourier transform of $\xi \in \mathcal{M}_q^+$ is defined by

$$\mathcal{F}_{q,v}(\xi)(x) = \int_0^\infty j_v(tx, q^2)t^{2v+1}d_q\xi(t).$$

The $q-$convolution product of two measures $\xi, \rho \in \mathcal{M}_q^+$ is given by

$$\xi *_q \rho(f) = \int_0^\infty T_{q,x}^v f(t)t^{2v+1}d_q\xi(x)d_q\rho(t),$$

and we have

$$\mathcal{F}_{q,v}(\xi *_q \rho) = \mathcal{F}_{q,v}(\xi)\mathcal{F}_{q,v}(\rho).$$

The following Theorem (see[1]) is crucial for the proof of our main result.

Theorem 3 *Let $(\xi_n)_{n\geq 0}$ be a sequences of probability measures of $\mathcal{M}_q^+$ such that*

$$\lim_{n\to\infty} \mathcal{F}_{q,v}(\xi_n)(x) = \psi(x),$$

then there exists $\xi \in \mathcal{M}_q^+$ such that the sequence ξ_n converge strongly toward ξ and

$$\mathcal{F}_{q,v}(\xi) = \psi.$$

In the following we consider the function ψ defined by

$$\psi(x) = \begin{cases} 1-x & \text{if} \quad x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Now we are in a position to state and prove the q-analogue of the Bochner's theorem

Theorem 4 *Let ϕ be a function defined on $\mathbb{R}_q^+$ continued at 0. Assume that the following function*

$$\phi_n : x \mapsto \phi(x)\psi(q^n x),$$

satisfy (1) for all $n \in \mathbb{N}$ then there exist $\xi \in \mathcal{M}_q^+$ such that

$$\mathcal{F}_{q,v}(\xi) = \phi.$$

Proof. The function ϕ_n is of q-positive type. From Corollary 2 there exist ϱ_n a positive function of $\mathcal{A}_{q,v}$ such that

$$\mathcal{F}_{q,v}(\varrho_n) = \phi_n.$$

The measure ξ_n defined by

$$d_q\xi_n(x) = \varrho_n(x) d_q x,$$

belong to $\mathcal{M}_q^+$ and

$$\int_0^\infty x^{2v+1} d_q\xi_n(x) = \mathcal{F}_{q,v}(\varrho_n)(0) = \phi_n(0) = \phi(0).$$

Assume that $\phi(0) = 1$. On the other hand

$$\lim_{n\to\infty} \mathcal{F}_{q,v}(\xi_n)(x) = \lim_{n\to\infty} \phi_n(x) = \phi(x).$$

From Theorem 3 there exists $\xi \in \mathcal{M}_q^+$ such that the sequence ξ_n converge strongly toward ξ, and

$$\mathcal{F}_{q,v}(\xi) = \phi,$$

which leads to the result. ■

References

[1] L. Dhaouadi, A. Fitouhi and J. El Kamel, Inequalities in q-Fourier Analysis, Journal of Inequalities in Pure and Applied Mathematics,Volume 7, Issue 5, Article 171, 2006.

[2] G. Gasper and M. Rahman, Basic hypergeometric series, Encycopedia of mathematics and its applications 35, Cambridge university press, 1990.

[3] F. H. Jackson, On a q-Definite Integrals, Quarterly Journal of Pure and Application Mathematics 41, 1910, 193-203.

[4] T. H. Koornwinder and R. F. Swarttouw, On q-Analogues of the Hankel and Fourier Transform, Trans. A. M. S. 1992, 333, 445-461.

[5] R. F. Swarttouw, The Hahn-Exton q-Bessel functions PhD Thesis The Technical University of Delft (1992).