

# BILLIARDS IN POLYGONS AND HOMOGENEOUS FOLIATIONS ON $\mathbb{C}^2$ .

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ABSTRACT. In this article we construct a new framework for the study of polygonal billiards. To every polygonal billiard we associate a holomorphic foliation on  $\mathbb{C}^2$ . The dynamics of the billiard ball is linked to the directional flow of the complex vector field defining the associated holomorphic foliation.

## 1. INTRODUCTION

For simplicity the main results and ideas in this article are stated, proved and developed in the context of triangular billiards. An appendix for the general case is given at the end in section §5.

Let  $P$  be a triangle having angles  $\lambda_1\pi$ ,  $\lambda_2\pi$  and  $\lambda_3\pi$ . We avoid degenerated cases, so  $0 < \lambda_j < 1$ ,  $\forall j$  and  $\sum_{j=1}^3 \lambda_j = 1$ . A frictionless point inside  $P$  describes a *billiard trajectory*, as time runs from  $-\infty$  to  $\infty$ , if it moves with constant velocity in the interior of  $P$ , reflects off edges so that speed is unchanged and the angle of incidence is equal to the angle of reflection. Motion stops if the point hits the vertices of  $P$ , for reflection is not well defined. A unique translation surface  $S = S(P)$  corresponds to such a dynamical system. It is generated by  $P$  through a standard unfolding process [12]. A *translation surface*, is a real surface whose transition functions are translations. In this article the surface  $S(P)$  is smooth and not closed, for it is supposed to be generated by a triangle with no vertices. Unless stating the contrary, *henceforth  $P$  is supposed to be with no vertices*. The billiard dynamics in  $P$  is equivalent to the geodesic flow on the unitary tangent bundle of  $S(P)$ . This surface has trivial holonomy, therefore one can consider, without ambiguity, the set of geodesics parallel to a given direction  $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ . This set defines a real foliation on  $S(P)$ , usually called *the geodesic foliation on  $S(P)$  defined by the direction  $\theta$* . We denote it by  $D_\theta$ .

The aim of this article is to describe the set of foliations  $D_\theta$  in terms of a special class of holomorphic homogeneous foliations on  $\mathbb{C}^2$  and

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their projections to  $\mathbf{RP}(3)$ .

Let  $\mathcal{F}_\lambda$  be the foliation on  $\mathbf{C}^2$  defined by the holomorphic 1-form  $\omega_\lambda$ ,

$$(1) \quad \frac{\omega_\lambda}{z_1 z_2 (z_2 - z_1)} = \lambda_1 \frac{dz_1}{z_1} + \lambda_2 \frac{dz_2}{z_2} + \lambda_3 \frac{d(z_2 - z_1)}{z_2 - z_1},$$

where  $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ . This foliation is homogeneous, *i.e.* invariant under the natural action of the homothetic transformation group  $\{T_k(z_1, z_2) := k(z_1, z_2) \mid k \in \mathbf{C}^*\}$ . The foliation  $\mathcal{F}_\lambda$  presents an isolated singularity at the origin. It leaves three complex lines invariant through the origin. This set of complex lines is commonly called the *tangent cone* of the foliation; it is described by the equation  $z_1 z_2 (z_2 - z_1) = 0$ . In the complement of the tangent cone, any two leaves,  $\mathcal{L}, \mathcal{L}' \in \mathcal{F}_\lambda$ , are diffeomorphic, for there always exists  $k \in \mathbf{C}^*$  such that  $T_{k|_{\mathcal{L}}} : \mathcal{L} \rightarrow \mathcal{L}'$  is a diffeomorphism. Every leaf of  $\mathcal{F}_\lambda$  in the complement of the tangent cone will be called a *generic leaf*. Up to diffeomorphism,  $\mathcal{F}_\lambda$  presents three kinds of leaves: a point,  $\mathbf{C}^*$  and a generic leaf.

**Theorem 1.** *If the submodule  $\text{Res}(\lambda) := \{(n_1, n_2, n_3) \in \mathbf{Z}^3 \mid \sum_j n_j \lambda_j = 0\}$  of  $\mathbf{Z}^3$  is equal to zero, then the generic leaf of  $\mathcal{F}_\lambda$  is homeomorphic to a plane with a countable set of handles. This topological surface is also known as the Loch Ness monster.*



*Loch Ness monster*

This nomenclature can be found in [5]. When the angles of the triangle are rational multiples of  $\pi$ , the generic leaf of  $\mathcal{F}_\lambda$  is homeomorphic to  $S \setminus E$ , where  $S$  is a finite genus orientable surface and  $E \subseteq S$  is a finite set of points [8].

The integral curves of the holomorphic vector field  $X_\lambda$ , solving the equation  $i_{X_\lambda} dz_1 \wedge dz_2 = \omega_\lambda$ , are the leaves of  $\mathcal{F}_\lambda$ . Let  $\mathcal{F}_{\lambda,0}$  be the real foliation on  $\mathbf{C}^2$  defined by the integral curves of the real analytic vector field  $\text{Re}(X_\lambda)$ . This vector field is quadratic and homogeneous. The foliation  $\mathcal{F}_{\lambda,0}$  has an isolated singularity at the origin and, for every non-singular leaf  $\mathcal{L} \in \mathcal{F}_\lambda$ , the restriction  $\mathcal{F}_{\lambda,0}|_{\mathcal{L}}$  is a real foliation with no singularities. Let  $k\mathcal{L}$  be the image of the homothety  $T_{k|_{\mathcal{L}}}$ ,  $k \in \mathbf{C}^*$ . Our main result is the following:

**Theorem 2.** *For every generic leaf  $\mathcal{L} \in \mathcal{F}_\lambda$  there exists a direction  $\theta \in \mathbf{R}/2\pi\mathbf{Z}$  such that the foliation  $\mathcal{F}_{\lambda,0}|_{\mathcal{L}}$  is holomorphically conjugated to the geodesic foliation defined by the direction  $\theta$  on the translation surface  $S(P)$ . Furthermore, for every  $\theta' \in \mathbf{R}/2\pi\mathbf{Z}$  and  $\rho \in \mathbf{R}^*$ ,  $\mathcal{F}_{\lambda,0}|_{\rho e^{i\theta'}\mathcal{L}}$  is holomorphically conjugated to the geodesic foliation defined by the direction  $\theta + \theta'$  on  $S(P)$ .*

This theorem sets the foundation for constructing a dictionary between the billiard game in a triangle and the holomorphic foliations  $\mathcal{F}_\lambda$  and  $\mathcal{F}_{\lambda,0}$ . For example, we obtain results on the existence of periodic orbits:

**Theorem 3.** *Let  $\lambda \in \mathbf{Q}^3$ , then periodic orbits in  $\mathcal{F}_{\lambda,0}$  are dense in  $\mathbf{C}^2$ .*

As for billiards, the main application of the dictionary contained in this article is the topological classification of the translation surfaces generated by non rational “generic” triangles *with no vertices*.

**1.1. Billiards on  $\mathbf{RP}(3)$ .** In a canonical way, we associate to the holomorphic foliations  $\mathcal{F}_\lambda$  and  $\mathcal{F}_{\lambda,0}$  singular foliations on the projective space  $\mathbf{RP}(3)$ ,  $\mathcal{G}_\lambda$  and  $\mathcal{G}_{\lambda,0}$ , with respective real codimension 1 and 2. Also, any two leaves  $\mathcal{L}, \mathcal{L}' \in \mathcal{G}_\lambda$  in the complement of the singular locus of  $\mathcal{G}_\lambda$  are diffeomorphic. As with  $\mathcal{F}_\lambda$ , such leaves will be called *generic leaves*. Under additional hypothesis for  $\lambda$ , the generic leaves of  $\mathcal{G}_\lambda$  and  $\mathcal{F}_\lambda$  are diffeomorphic. In such a case we will say that  $\lambda$  is *not strongly resonant*. The diffeomorphism is achieved by the canonical projection  $\Pi_{\mathbf{RP}(3)} : \mathbf{C}^2 \setminus 0 \rightarrow \mathbf{RP}(3)$ . In this context, for every generic leaf  $\mathcal{L} \in \mathcal{F}_\lambda$ , the projection  $\Pi_{\mathbf{RP}(3)}$  conjugates the foliation  $\mathcal{F}_{\lambda,0}|_{\mathcal{L}}$  to the restriction of  $\mathcal{G}_{\lambda,0}$  to  $\Pi_{\mathbf{RP}(3)}(\mathcal{L})$ . For strong resonant parameters  $\lambda$ , the projection  $\Pi_{\mathbf{RP}(3)}$ , when restricted to a generic leaf of  $\mathcal{F}_\lambda$ , defines a 2-sheeted covering over its image that semiconjugates  $\mathcal{F}_{\lambda,0}|_{\mathcal{L}}$  to the restriction of  $\mathcal{G}_{\lambda,0}$  to  $\Pi_{\mathbf{RP}(3)}(\mathcal{L})$ .

For every leaf  $\Pi_{\mathbf{RP}(3)}(\mathcal{L}') = \mathcal{L} \in \mathcal{G}_\lambda$ , with  $\mathcal{L}' \in \mathcal{F}_\lambda$  generic, we denote  $\rho e^{i\theta}\mathcal{L} := \Pi_{\mathbf{RP}(3)}(\rho e^{i\theta}\mathcal{L}')$ .

**Corollary 1.** *Let  $\lambda$  be non strongly resonant. Then, for every generic leave  $\mathcal{L} \in \mathcal{G}_\lambda$  there exists a direction  $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ , such that the foliation  $\mathcal{G}_{\lambda,0}|_{\mathcal{L}}$  is holomorphically conjugated to the geodesic foliation defined by the direction  $\theta$  on the translation surface  $S(P)$ . Furthermore, for every  $\theta' \in \mathbf{R}/2\pi\mathbf{Z}$ ,  $\mathcal{G}_{\lambda,0}|_{e^{i\theta'}\mathcal{L}}$  is holomorphically conjugated to the geodesic foliation defined by the direction  $\theta + \theta'$  on  $S(P)$ .*

The result above is just a projective version of theorem 2. The advantage of working in  $\mathbf{RP}(3)$  is that this variety and the phase space of the billiard game on  $P$  have the same real dimension. We recall that the phase space of the billiard is naturally identified with  $P \times \mathbf{R}/2\pi\mathbf{Z}$ . From [7] we know that the billiard flow in a typical polygon is ergodic. Using this fact we deduce the following

**Corollary 2.** *There exist a dense  $G_\delta$  in the space of triangles for which almost every (LEBESGUE) point in  $\mathbf{RP}(3)$  is contained in a leaf of  $\mathcal{G}_{\lambda,0}$  which is dense in  $\mathbf{RP}(3)$ .*

We begin this paper with a general survey of quadratic homogeneous holomorphic foliations of  $\mathbf{C}^2$  having an isolated singularity and a reduced tangent cone. Also, in section §2.2, we describe the topology of their leaves as well as the foliations they naturally induce in the projective space  $\mathbf{RP}(3)$ . In section §3, theorem 2 is proved and later, in section §4, we show the main applications of this result to polygonal billiards and homogeneous foliations. Finally, in section §5, we prove that the main results obtained for triangles and quadratic homogeneous foliations remain valid for general polygons under certain hypotheses.

## 2. HOMOGENEOUS FOLIATIONS ON $\mathbf{C}^2$

Homogeneous holomorphic foliations on  $\mathbf{C}^2$  constitute the natural setting for the framework which we construct for studying polygonal billiards. Every holomorphic homogeneous foliation on  $\mathbf{C}^2$  having an isolated singularity is given by a PFFAFIAN form  $\omega = A_1 dz_1 + A_2 dz_2$ , where  $A_j \in \mathbf{C}[z_1, z_2]$  are homogeneous polynomials having the same degree and such that  $\gcd(A_1, A_2) = 1$ . Let  $R_{\mathbf{C}}$  be the radial vector field on  $\mathbf{C}^2$ . The holomorphic foliation presents only  $\nu + 1$  invariant complex lines through the singularity, if and only if  $z_1 A_1 + z_2 A_2$  is an homogeneous irreducible polynomial of degree  $\nu + 1$ . The zero locus of this polynomial is called the *tangent cone* of the foliation and the natural number  $\nu$  is the *degree* of the foliation. As a consequence of the decomposition of rational fractions in one variable,  $\omega$  can be written in the same form as (1) except that all  $\lambda_j \in \mathbf{C}^*$ ,  $j = 1, 2, 3$ , and  $\sum_{j=1}^3 \lambda_j$  equals to a given normalization constant [3]. Thus,  $\mathcal{F}_\lambda$  is a special type of quadratic homogeneous foliation: where  $\{\lambda_j \pi\}_{j=1}^3$  are the angles of a non degenerated triangle  $P$  and the normalization constant is equal to one. We observe that

$$(2) \quad F_\lambda(z_1, z_2) = z_1^{\lambda_1} z_2^{\lambda_2} (z_2 - z_1)^{\lambda_3}$$

is a first integral for  $\mathcal{F}_\lambda$ . This expression is usually multivaluated. Let  $\sim_\lambda$  be the equivalence relation induced by the action of  $(\mathbf{Z}^3, +)$  on  $\mathbf{C}^*$ , defined by  $\mathbf{Z} \times \mathbf{C} \ni ((n_1, n_2, n_3), z) \rightsquigarrow e^{2\pi i \sum_j n_j \lambda_j} z$ . As proved in [9], the normalization  $\sum_{j=1}^3 \lambda_j = 1$ , implies that the fibers  $F_\lambda^{-1}(k)$ ,  $k \in \mathbf{C}^* / \sim_\lambda$  are the (connected) leaves of  $\mathcal{F}_\lambda$  in the complement of the tangent cone. A straightforward computation shows that if  $k, k' \in \mathbf{C}^*$ , the homothety  $T_{k/k'}$  defines a diffeomorphism between  $F_\lambda^{-1}(k)$  and  $F_\lambda^{-1}(k')$ . In particular, each leaf in  $\mathcal{F}_\lambda$  is invariant under every homothety  $T_{e^{2\pi i \sum_j n_j \lambda_j}}$ . We will call every leaf in the complement of the tangent cone of  $\mathcal{F}_\lambda$  a *generic leaf*.

**2.1. Topology of generic leaves.** As studied in [2], the generic leaf of a generic holomorphic homogeneous foliation on  $\mathbf{C}^2$  is dense in  $\mathbf{C}^2$ . That is not the case for the foliation,  $\mathcal{F}_\lambda$ , associated to the billiard in the triangle.

Let  $\Pi : \widetilde{\mathbf{CP}(2)} \longrightarrow \mathbf{CP}(2)$  be the complex surface obtained by blowing up the origin of the affine chart  $\mathbf{C}^2 = \{(z_1, z_2)\}$  of  $\mathbf{CP}(2)$ . This surface is well known: it is HIRZEBRUCH's first surface. The foliation  $\mathcal{F}_\lambda$  extends to a foliation  $\widetilde{\mathcal{F}}_\lambda$  on  $\widetilde{\mathbf{CP}(2)}$ . Let  $\pi$  be the projection of  $\widetilde{\mathbf{CP}(2)}$  onto the exceptional divisor  $\mathbf{CP}(1)$ . In the complement of  $\Pi^{-1}(z_1 z_2 (z_2 - z_1) = 0)$ , the foliation  $\widetilde{\mathcal{F}}_\lambda$  is transversal to the fibration defined by  $\pi$ . Using the first integral,  $F_\lambda$ , one can calculate explicitly the generators of the holonomy of  $\widetilde{\mathcal{F}}_\lambda$ , relative to this fibration. They are given by the maps  $z \rightsquigarrow e^{2\pi i \lambda_j} z$ ,  $j = 1, 2, 3$ . Here  $z \in \mathbf{C}$  is the variable of the fiber  $\pi^{-1}(t)$ , with  $t \in \mathbf{CP}(1) \setminus \{0, 1, \infty\}$ .

**Proposition 1.** [3] *If  $\lambda_j = p_j/q \in \mathbf{Q}$ ,  $\gcd(p_1, p_2, p_3, q) = 1$ , then the closure in  $\widetilde{\mathbf{CP}(2)}$  of every generic leaf  $\mathcal{L} \in \widetilde{\mathcal{F}}_\lambda$  is an algebraic curve defined by  $z_1^{p_1} z_2^{p_2} (z_2 - z_1)^{p_3} = k$ ,  $k \in \mathbf{C}^*$ . On the other hand, if for some  $j = 1, 2, 3$  the parameter  $\lambda_j$  is irrational, then the closure in  $\mathbf{CP}(2)$  of every generic leaf  $\mathcal{L} \in \widetilde{\mathcal{F}}_\lambda$  is a real Levi-Flat singular manifold of dimension 3 given by  $\Pi^{-1}(|z_1|^{\lambda_1} |z_2|^{\lambda_2} |z_2 - z_1|^{\lambda_3} = k)$ ,  $k \in \mathbf{R}^*$ .*

*Proof of Theorem 1.* We calculate the space of ends of the generic leaf  $\mathcal{L}$ . This space is a topological invariant for noncompact surfaces [10].

Let  $\widetilde{U}$  be the complement in  $\mathbf{CP}(1)$  of three infinitesimal neighbourhoods containing the points  $\{0, 1, \infty\}$ . We define  $U$  as the complement on  $\widetilde{U}$  of the segments  $A = ]\infty, 0[$  and  $B = ]1, \infty[$  contained

in  $\mathbf{RP}(1) \subset \mathbf{CP}(1)$ . Using  $U$  and the monodromy of the covering  $\pi|_{\mathcal{L}} : \mathcal{L} \rightarrow E$ , we construct a surface having the same topological type as  $\mathcal{L}$ . Let  $\gamma_0, \gamma_1$  be the generators of  $\pi_1(\mathbf{CP}(1) \setminus \{0, 1, \infty\}, 1/2)$  given by the circles of radius  $1/2$  and centers at  $0$  and  $1$ , respectively. The monodromy action of the covering  $\pi|_{\mathcal{L}} : \mathcal{L} \rightarrow E$ , is given by the holonomy of  $\widetilde{\mathcal{F}}_\lambda$  relative to the fibration  $\pi$ . Without loss of generality, we suppose that  $\gamma_0$  and  $\gamma_1$  determine the holonomy maps  $z \rightsquigarrow e^{2\pi i \lambda_2} z$  and  $z \rightsquigarrow e^{2\pi i \lambda_3} z$ , respectively. Let  $m$  be a word in this two generators. We denote by  $U_m$  the lifting of the fundamental domain  $U$  to the leaf beginning at a fixed point in the fiber  $\pi^{-1}(1/2)$  and following the path given by the word  $m$ . Let  $[\cdot, \cdot]$  be the commutator of two loops in the fundamental group. We remark that  $U_{[\gamma_0, \gamma_1]}$  is homeomorphic to the open ring  $1 < |z| < 2$ . This implies that  $U_{[\gamma_0, \gamma_1][\gamma_0^{-1}, \gamma_1]}$  is homeomorphic to a punctured torus.

Consider the additive subgroup of  $\mathbf{Z} \times \mathbf{Z}$  given by  $\{(n_2, n_3) \in \mathbf{Z} \times \mathbf{Z} \mid n_2 \lambda_2 + n_3 \lambda_3 \in \mathbf{Z}\}$ . If  $Res(\lambda)$  is equal to zero then this additive subgroup is equal to zero as well. The base space of the covering  $\pi$  admits a retraction to the set of generators  $\{\gamma_0, \gamma_1\}$ . This retraction induces a retraction on the covering constructed from  $\widetilde{U}$  onto a subspace homeomorphic to the Cayley graph of  $\mathbf{Z} \times \mathbf{Z}$ . Clearly, this graph has only one end. Therefore, the generic leaf of  $\mathcal{F}_\lambda$  has only one end. Note that  $\{U_{([\gamma_0, \gamma_1][\gamma_0^{-1}, \gamma_1])\gamma_0^{10n}}\}_{n \in \mathbf{Z}}$  is a family of disjoint punctured tori in  $\mathcal{L}$ . We conclude that the only end of the leaf  $\mathcal{L}$  must be non planar.  $\square$

**2.2. Homogeneous foliations on  $\mathbf{RP}(3)$ .** Let  $R_{\mathbf{R}}$  be the radial vector field on  $\mathbf{R}^4$  and  $z_j = x_j + \sqrt{-1}y_j$ , for  $j = 1, 2$ . The real integrable homogeneous 1-form

$$(3) \quad i_{X_{\mathbf{R}}} i_{Re(X_\lambda)} i_{Im(X_\lambda)} dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$$

defines a real codimension 1 singular foliation  $\mathcal{G}_\lambda$  on the projective space  $\mathbf{RP}(3)$ . The projection of the real cones,  $\{\rho \mathcal{L} \mid \rho \in \mathbf{R}^*, \mathcal{L} \in \mathcal{F}_\lambda \setminus \{0\}\}$ , to the projective space  $\mathbf{RP}(3)$  are the leaves of  $\mathcal{G}_\lambda$ . A leaf of this foliation is called *generic* if and only if it originates from the projection of a real cone whose basis is a generic leaf of  $\mathcal{F}_\lambda$ . The singular locus of the foliation  $\mathcal{G}_\lambda$ , that we denote by  $Sing(\mathcal{G}_\lambda)$ , is given by three projective real lines,  $\mathbf{RP}(1)$ , corresponding to the projection of the irreducible components of tangent cone of  $\mathcal{F}_\lambda$ . Any two leaves of  $\mathcal{G}_\lambda$  in  $\mathbf{RP}(3) \setminus Sing(\mathcal{G}_\lambda)$  are diffeomorphic. The homogeneous real vector field  $Re(X_\lambda)$  defines a singular codimension 2 real foliation  $\mathcal{G}_{\lambda,0}$  on  $\mathbf{RP}(3)$ . A direct calculation on  $Re(X_\lambda)$  restricted to each complex line in the tangent cone of  $\mathcal{F}_\lambda$  shows that the singular locus of  $\mathcal{G}_{\lambda,0}$  is



given, in homogeneous coordinates  $[x_1 : y_1 : x_2 : y_2]$ , by the points  $p_1 := [0 : 0 : 1 : 0]$ ,  $p_2 := [1 : 0 : 0 : 0]$  and  $p_3 := [1 : 0 : 1 : 0]$ . Let  $C_j$  note the projective real line in  $Sing(\mathcal{G}_\lambda)$  containing the point  $p_j \in Sing(\mathcal{G}_{\lambda,0})$ . For every  $\mathcal{L} \in \mathcal{G}_\lambda \setminus Sing(\mathcal{G}_\lambda)$ , the restriction  $\mathcal{G}_{\lambda,0}|_{\mathcal{L}}$  is a real foliation with no singularities.

**Definition 1.** A point  $\lambda \in \mathbf{R}^3$  in the hyperplane  $\sum \lambda_j = 1$  is strongly resonant if and only if there exists a non trivial solution,  $(n_1, n_2, n_3) \in \mathbf{Z}^3$ , to the equation  $\sum n_j \lambda_j \cong 1/2 \pmod{\mathbf{Z}}$ .

When the parameter defining the 1-holomorphic form  $\omega_\lambda$  is strongly resonant, the holonomy of the foliation  $\widetilde{\mathcal{F}}_\lambda$  relative to the fibration defined by the projection  $\pi$  contains the involution  $z \rightsquigarrow -z$ . This implies that, for strongly resonant parameters, the restriction of the projection,  $\Pi_{\mathbf{RP}(3)}$ , to a generic leaf of  $\mathcal{F}_\lambda$  defines a two-sheeted covering of its image.

**Lemma 1.** If  $\lambda$  is not strongly resonant, the generic leaves of the foliations  $\mathcal{F}_\lambda$  and  $\mathcal{G}_\lambda$  are diffeomorphic. The diffeomorphism is achieved by the restriction of the projection  $\Pi_{\mathbf{RP}(3)}$  to the leaves of  $\mathcal{F}_\lambda$ . Moreover, for every generic leaf  $\mathcal{L} \in \mathcal{F}_\lambda$ , this projection conjugates the restriction  $\mathcal{F}_{\lambda,0}|_{\mathcal{L}}$  to the restriction of  $\mathcal{G}_{\lambda,0}|_{\Pi_{\mathbf{RP}(3)}(\mathcal{L})}$ .

This lemma implies the projective version of theorem 2, which we will prove in the following section.

Let  $\mathbf{RP}(1) \longrightarrow \mathbf{RP}(3) \longrightarrow \mathbf{S}^2$  be the fibration naturally defined by HOPF's fibration. The action of the holonomy of  $\mathcal{G}_{\lambda,0}$  relative to this fibration is generated by the maps (in local coordinates)  $\mathbf{RP}(1) \ni [1 : s] \rightsquigarrow [1 : s + 2\pi\lambda_j]$ ,  $j = 1, 2, 3$ .

**Proposition 2.** The point  $\lambda = (\lambda_1, \lambda_2, \lambda_3)$  is not rational if and only if every generic leaf of  $\mathcal{G}_\lambda$  is dense in  $\mathbf{RP}(3)$ .

In the homogeneous coordinates of  $\mathbf{RP}(3)$ , the analytic multivaluated function  $J := F_\lambda/\overline{F}_\lambda$  is a first integral for the foliation  $\mathcal{G}_\lambda$ . In particular, the differential 1-form  $dJ/J$  defines this foliation. POINCARÉ's linearization theorem implies that in a small neighbourhood of every point in the singular locus of  $\mathcal{G}_\lambda$  this differential form is conjugated to  $ydx - xdy$ . This is just a manifestation of the KUPKA phenomenon. In other words, the local model for the foliation,  $\mathcal{G}_\lambda$ , in a point of its singular locus is given by an *open book*.

In order to understand better the global geometry of this open book, let  $\Pi_C : \widetilde{\mathbf{RP}(3)} \longrightarrow \mathbf{RP}(3)$  be the projection map from the algebraic variety  $\widetilde{\mathbf{RP}(3)}$  that results from blowing-up the singular locus

of the foliation  $\mathcal{G}_\lambda$ . Each exceptional divisor  $\mathbf{T}_j := \Pi_C^{-1}(C_j)$  is homeomorphic to a torus  $\mathbf{RP}(1) \times \mathbf{RP}(1)$ . The closed 1-form  $\Pi_C^* dJ/J$ , defines a real codimension 2 non singular foliation,  $\widetilde{\mathcal{G}}_\lambda$ . Up to diffeomorphism, this foliation has only one leaf which we also call *generic*, such generic leaf is not homeomorphic to the generic leaf of the foliation  $\mathcal{G}_\lambda$ .

**Proposition 3.** *The restriction of  $\widetilde{\mathcal{G}}_\lambda$  to each torus  $\mathbf{T}_j$  defines a linear foliation.*

This proposition follows from the fact that, in homogeneous coordinates,  $\lambda_1 \arctan(y_1/x_1) + \lambda_2 \arctan(y_2/x_2) + \lambda_3 \arctan((y_2 - y_1)/(x_2 - x_1))$ , is also a first integral for the foliation  $\mathcal{G}_\lambda$ . The normalization,  $\sum_j \lambda_j = 1$ , implies that the slope of the lines in  $\widetilde{\mathcal{G}}_\lambda$ , restricted to  $\mathbf{T}_j$ , depends only on  $\lambda_j$ . A direct computation in local coordinates shows that, up to a permutation of indexes, this slope is given by  $(1 - \lambda_j)/\lambda_j$ .

The real foliation,  $\mathcal{G}_{\lambda,0}$ , defines a real singular foliation,  $\widetilde{\mathcal{G}}_{\lambda,0}$ , on  $\mathbf{RP}(3)$ . This two foliations are isomorphic in the complement of  $\cup_{j=1}^3 \mathbf{T}_j$ . The singular locus of  $\widetilde{\mathcal{G}}_{\lambda,0}$  is  $\{\Pi_C^{-1}(p_j)\}_{j=1}^3$ . Each connected component of this singular locus is homeomorphic to a circle,  $\mathbf{RP}(1)$ , contained in an exceptional divisor,  $\mathbf{T}_j$ . From the preceding proposition, we conclude that the singular locus of  $\widetilde{\mathcal{G}}_{\lambda,0}$ , restricted to the torus, is formed by apparent singularities. Therefore  $\widetilde{\mathcal{G}}_{\lambda,0}|_{\mathbf{T}_j}$  is a line foliation whose leaves are (1) dense if and only if  $\lambda_j$  is irrational or, (2) periodic if and only if  $\lambda_j$  is rational.

**Proposition 4.** *In a small neighbourhood of each singular point  $p_j \in \text{Sing}(\mathcal{G}_{\lambda,0})$ , the real foliation  $\mathcal{G}_{\lambda,0}$  is analytically conjugated to the foliation defined by the integral curves of the linear vector field,  $\lambda_j x \partial/\partial x - y \partial/\partial y - z \partial/\partial z$ .*

*Proof.* In the coordinates  $z_2 = tz_1$ , of  $\widetilde{\mathbf{CP}}(2)$ , the foliation  $\widetilde{\mathcal{F}}_\lambda$  is given by the integral curves of  $z_1(\lambda_2 z_1 - (\lambda_2 + \lambda_3)tz_1 \partial/\partial z_1 + t(t-1)\partial/\partial t)$ . For  $t \sim 0$ , this vector field is analytically conjugated to  $\widetilde{Y} := z_1(\lambda_2 z_1 \partial/\partial z_1 + t \partial/\partial t)$ . This vector field is the pullback via  $\Pi : \widetilde{\mathbf{CP}}(2) \rightarrow \mathbf{CP}(2)$  of the holomorphic vector field defined, in the affine chart  $\mathbf{C}^2 \hookrightarrow \mathbf{CP}(2)$ , by  $Y := \lambda_2 z_1^2 \partial/\partial z_1 + (1 + \lambda_2)z_1 z_2 \partial/\partial z_2$ . Let  $Re(Y) = A_1 \partial/\partial x_1 + B_1 \partial/\partial x_1 + A_2 \partial/\partial x_2 + B_2 \partial/\partial y_2$ , this real homogeneous vector field defines a real, codimension 2, singular foliation



on  $\mathbf{RP}(3)$ . In the local coordinates  $x_1 = 1$ , this foliation is given by the integral curves of

$$(4) \quad \begin{aligned} Re(Y) - A_1 R_{\mathbf{R}} &= \lambda_2(y_1 + y_1^3)\partial/\partial y_1 \\ &-x_2 + y_1((1 - \lambda_2)y_2 + \lambda_2 y_1 x_2)\partial/\partial x_2 \\ &-y_2 + y_1((\lambda_2 - 1)x_2 + \lambda_2 y_2 y_1)\partial/\partial y_2. \end{aligned}$$

The real foliation defined by  $Re(Y) - A_1 R_{\mathbf{R}}$  is analytically conjugated to  $\mathcal{G}_{\lambda,0}$  in a neighbourhood of  $x_2 = y_2 = 0$ ; in particular, in a neighbourhood of the point  $p_2$ , corresponding to the origin  $x_2 = y_2 = y_1 = 0$ .  $\square$

From the preceding proposition we notice that, for each singular point  $p_j$ , for  $j = 1, 2, 3$ , the existence of an invariant, 2-dimensional, real manifold given in local coordinates by  $x = 0$ ; and we denote it  $\mathcal{V}_j$ . The restriction of  $\mathcal{G}_{\lambda,0}$  to this invariant manifold is conjugated to a *radial foliation*.

### 3. PROOF, MAIN RESULT.

First we prove that the generic leaf  $\mathcal{L}$  of  $\mathcal{F}_\lambda$  and the translation surface,  $S(P)$ , are biholomorphic. The invariant complex lines through the origin in  $\mathcal{F}_\lambda$  determine three points in the exceptional divisor in  $\widetilde{\mathbf{CP}(2)}$ . In local coordinates  $z_2 = tz_1$ , this points are given by  $t = 0, 1, \infty$ . Let  $E$  be the complement of the points in the exceptional divisor, then  $E$  is homeomorphic to the sphere with three punctures. Every generic leaf  $\mathcal{L} \in \widetilde{\mathcal{F}}_\lambda$  defines a covering space

$$(5) \quad \pi_{|\mathcal{L}} : \mathcal{L} \longrightarrow E.$$

The identification of two copies  $P \times \{*\}$ ,  $P \times \{*\}'$  by the relation  $(\eta, *) \sim (\eta, *')$ , if and only if  $\eta \in \partial P$  defines a locally euclidean surface  $\Sigma_\lambda$ . Observe that  $\Sigma_\lambda$  is homeomorphic to the sphere with three punctures. As a consequence of the unfolding process, we have a natural covering  $\Pi_P : S(P) \longrightarrow \Sigma_\lambda$ . We prove that for every generic leaf,  $\mathcal{L} \in \widetilde{\mathcal{F}}_\lambda$ , the coverings whose projections are  $\pi_{|\mathcal{L}}$  and  $\Pi_P$ , belong to the same *conformal* class. Let  $z_2 = tz_1$  be local coordinates for  $\widetilde{\mathbf{CP}(2)}$ . The SCHWARZ-CHRISTOFFEL transformation

$$(6) \quad t \rightsquigarrow \int_0^t \xi^{\lambda_2-1}(\xi - 1)^{\lambda_3-1} d\xi,$$

defined on  $Im(t) > 0$ , admits an analytic continuation  $f : E \rightarrow \Sigma_\lambda$ . This map sends conformally each half-plane in the complement of  $Im(t) = 0$  onto the interior of each copy of  $P$  in  $\Sigma_\lambda$ . For every  $* \in \Sigma_\lambda$ , the conformal map  $f$ , defines a natural isomorphism  $\phi : \pi_1(E, f^{-1}(*)) \rightarrow \pi_1(\Sigma_\lambda, *)$ . Let  $h : \pi_1(\Sigma_\lambda, *) \rightarrow Isom_+(\mathbb{R}^2)$  be the *holonomy homomorphism*, defined by developing a loop using analytic continuation. For every  $\gamma \in \pi_1(\Sigma_\lambda, *)$  we define  $dh(\gamma) \in \mathbb{C}^*$  to be the derivative of  $h(\gamma)$ . Let  $\gamma_1, \gamma_2, \gamma_3$  be simple loops generating  $\pi_1(E, f^{-1}(*))$ . We suppose that the action of  $\gamma_j$  in the fiber  $\pi^{-1}(*)$  is given by the holonomy generator  $z \rightsquigarrow e^{2\pi i \lambda_j} z$ . The SCHWARZ reflection principle implies that  $d(h \circ \phi)(\gamma_j) = e^{2\pi i \lambda_j} Id_{\mathbb{R}^2}$ . The stabilizer subgroups of the monodromy actions of both coverings are isomorphic; and the isomorphism is given by the restriction of  $\phi$ . This implies that the map  $f$  lifts to a biholomorphism  $\tilde{f} : \mathcal{L} \rightarrow S(P)$ .

We claim that  $\tilde{f}$  conjugates  $\mathcal{F}_{\lambda,0|_{\mathcal{L}}}$  to a geodesic foliation on  $S(P)$ . Let  $\widetilde{X}_\lambda$  and  $\widetilde{Re}(X_\lambda)$  be the vector fields naturally defined on  $\widetilde{\mathbb{C}P}(2)$  by  $X_\lambda$  and  $Re(X_\lambda)$ . A straight computation shows that, in the local coordinates  $z_2 = tz_1$ , the projection of  $\widetilde{X}_\lambda|_{\mathcal{L}}$  to  $E$  is given by the branches of  $kt^{1-\lambda_2}(t-1)^{1-\lambda_3}\partial/\partial t$ , where  $k \in \mathbb{C}^*$  is a constant given by the first integral  $F(z_1, z_2)$ , (2), and depending on the leaf  $\mathcal{L}$ . From (6), we conclude that,  $\tilde{f}$ , locally rectifies the real vector field  $\widetilde{Re}(X_\lambda)$ . Using the monodromy action of the covering given in (5), one can extend the rectification of  $\widetilde{Re}(X_\lambda)$  by  $\tilde{f}$  to the hole leaf. This proves our claim.

In a slight abuse of notation, let  $\rho e^{i\theta'} \mathcal{L}$  be the image in  $\widetilde{\mathbb{C}P}(2)$  of the homothetic transformation  $T_{\rho e^{i\theta'}|_{\mathcal{L}}}$ . The projection of  $\widetilde{X}_\lambda|_{\rho e^{i\theta'} \mathcal{L}}$  to  $E$  is given by the branches of  $\rho e^{i\theta'} kt^{1-\lambda_2}(t-1)^{1-\lambda_3}\partial/\partial t$ . Thus, if the lifting  $\tilde{f} : \mathcal{L} \rightarrow S(P)$  conjugates  $\mathcal{F}_{\lambda,0|_{\mathcal{L}}}$  to the geodesic foliation defined by  $\theta$ , the lifting  $\tilde{f} : \rho e^{i\theta'} \mathcal{L} \rightarrow S(P)$  conjugates  $\mathcal{F}_{\lambda,0|_{\rho e^{i\theta'} \mathcal{L}}}$  to the geodesic foliation defined by  $\theta + \theta'$ .  $\square$

*Remark.* Let  $\lambda$  be a strong resonant parameter. Let  $\gamma \in \pi_1(E, f^{-1}(*))$  define the involution  $z \rightsquigarrow -z$  in the holonomy of the foliation  $\widetilde{\mathcal{F}}_\lambda$ , relative to the fibration  $\Pi$ . For such loops,  $\phi(\gamma) = \gamma'$  satisfies  $dh\gamma(0) = -Id_{\mathbb{R}^2}$ . Conversely, if  $\gamma' \in \pi_1(\Sigma_\lambda, *)$  satisfies  $dh\gamma' = -Id_{\mathbb{R}^2}$  then  $\phi^{-1}(\gamma')$  is a loop, in  $\pi_1(E, f^{-1}(*))$ , defining the involution  $z \rightsquigarrow -z$  in the holonomy of the foliation  $\widetilde{\mathcal{F}}_\lambda$  relative to the fibration  $\Pi$ . This implies that, for strong resonant parameters, the generic leaf of the

foliation  $\mathcal{G}_\lambda$  is analytically conjugated to the *half-translation surface* generated by the triangle  $P$ . We denote this surface by  $S_{1/2}(P)$ . The surface is the result of identifying any two points in  $S(P)$  which differ by a *half-translation*. *Half-translations* are isometries of the plane given by  $z \rightsquigarrow -z + k$ , where  $k \in \mathbf{C}^*$ . By definition, the translation surface  $S(P)$  is a double cover of  $S_{1/2}(P)$ .

#### 4. APPLICATIONS

We recall some facts concerning the phase space of the billiard game. The state of any point in the triangular billiard  $P$  is completely determined by its position and the direction of its velocity. Therefore, the phase space of this dynamical system is naturally identified with  $P \times \mathbf{R}/2\pi\mathbf{Z}$ , and this set will henceforth be endowed with the natural product topology. We assume that the point in the interior of  $P \times \{\theta\}$  describes a trajectory parallel to the direction  $\theta \in \mathbf{R}/2\pi\mathbf{Z}$  until it reaches the boundary at a point  $(\xi, \theta)$ . Recall that  $P$  is supposed to have no vertices. The point  $(\xi, \theta)$  “jumps” to the point  $(\xi, \theta')$ , where  $\theta$  and  $\theta'$  are related by the law “angle of reflection equal to the angle of incidence”. In this way, the billiard trajectories define a “flow” in the phase space usually called *the billiard flow*.

The linear parts of the three reflections of the plane in respect of the lines containing the sides of the triangle  $P$ , generate a subgroup of  $\mathbf{O}(2)$  which we denote  $\mathbf{G}$ . For every direction  $\theta$  we let  $\mathbf{G}\theta$  be the orbit of  $\theta$  under the natural action of  $\mathbf{G}$  over  $\mathbf{R}/2\pi\mathbf{Z}$ . Clearly, for every  $\theta \in \mathbf{R}/2\pi\mathbf{Z}$  the set  $\bigcup_{\beta \in \mathbf{G}\theta} P \times \{\beta\}$  is invariant under the billiard flow. The identification of any two points  $(\xi, \beta) \sim (\xi, \beta')$  in this invariant set for which the directions  $\beta$  and  $\beta'$  are related by the reflection law defines an invariant surface  $\Sigma_\theta$ . The billiard flow defines a real non singular foliation  $B_\theta$  on this surface. This foliation is formed by all billiard trajectories for which the velocity of the ball is, eventually, parallel to the direction  $\theta$ .

**Definition 2.** Let  $\theta_1, \theta_2$  and  $\theta_3$  be the directions of the sides of the triangle  $P$ . We call every element in  $\bigcup_{j=1}^3 \mathbf{G}\theta_j$  a singular direction of the billiard in  $P$ .

Let  $X$  be a topological space and  $\text{int}(X)$  its interior. For every  $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ , there exists a natural embedding  $\text{int}(\Sigma_\theta) \hookrightarrow S(P)$ , conjugating  $B_\theta$  to the geodesic foliation  $D_\theta$ . In particular, closed billiard trajectories are sent into closed geodesics. For a non singular direction, this embedding extends to the whole surface  $\Sigma_\theta$  and defines a

conjugation between  $B_\theta$  and the geodesic foliation  $D_\theta$ , in the whole surface  $S(P)$ .

**Lemma 2.** *Let  $M_1 := \{e^{i\theta} F_\lambda^{-1}(1)\}_{\theta \in \mathbf{R}/2\pi\mathbf{Z}}$ . Then the points in  $M_1$  defining a periodic orbit in  $\mathcal{F}_0$  form a dense set if and only if the points in the phase space  $P \times \mathbf{R}/2\pi\mathbf{Z}$  defining a periodic trajectory of the billiard flow form a dense set.*

If periodic orbits of  $\mathcal{F}_{\lambda,0}$ , restricted to  $M_1$ , are dense in  $M_1$ , then periodic orbits of  $\mathcal{F}_{\lambda,0}$  are dense in  $\mathbf{C}^2$ . This follows from the fact that, for every generic leaf  $\mathcal{L} \in \mathcal{F}_\lambda$  and  $\rho \in \mathbf{R}^*$ , the foliations  $\mathcal{F}_{\lambda,0|_{\mathcal{L}}}$  and  $\mathcal{F}_{\lambda,0|\rho\mathcal{L}}$  are conjugated. We remark that  $\Pi_{\mathbf{RP}(3)} : M_1 \rightarrow \mathbf{RP}(3) \setminus \text{Sing}(\mathcal{G}_\lambda)$  is a double covering. This implies that if periodic orbits of  $\mathcal{F}_{\lambda,0}$ , restricted to  $M_1$ , are dense in  $M_1$  then periodic orbits of  $\mathcal{G}_{\lambda,0}$  are dense in  $\mathbf{RP}(3)$ , even if  $\lambda$  is a strongly resonant. Then, from

**Theorem 4.** [1],[11] *Let  $P$  be a right or rational triangle. Then periodic orbits of the billiard flow in the phase space  $\mathbf{R}/2\pi\mathbf{Z}$  are dense.*

we deduce, using lemma 2,

**Corollary 3.** *Let  $P$  be a right or rational triangle. Then periodic orbits of  $\mathcal{G}_{\lambda,0}$  and  $\mathcal{F}_{\lambda,0}$  are dense in  $\mathbf{RP}(3)$  and  $\mathbf{C}^2$  respectively.*

*Proof lemma 2.* Let  $\widetilde{M}_1 := \Pi^{-1}(M_1)$  be the image of  $M_1$  in  $\widetilde{\mathbf{CP}(2)}$ . We define, in local coordinates  $z_2 = tz_1$ , the set  $M_1^+ := \{(t, z_1) \in \widetilde{M}_1 \mid \text{Im}(t) > 0\}$ . The ‘‘parametrization’’  $\{\text{Im}(t) > 0\} \times \mathbf{R}/2\pi\mathbf{Z} \ni (t, \theta) \rightsquigarrow (t, e^{i\theta} t^{-\lambda_2} (t-1)^{-\lambda_3})$  induces a differentiable structure of 3-real manifold on  $M_1$ . Analogously,  $\text{Im}(t) < 0$  defines  $M_1^-$ . The restriction of the foliations  $\widetilde{\mathcal{F}}_\lambda$  and  $\widetilde{\mathcal{F}}_{0,\lambda}$  to  $M_1$  define two real foliations that we note  $\mathcal{F}_+$  and  $\mathcal{F}_{+,0}$ .

Let  $f_+$  denote the restriction of the SCHWARZ-CHRISTOFFEL transformation, (6), to  $\text{Im}(t) > 0$ . Let  $\mathcal{L} \in \mathcal{F}_+$  be a leaf and  $\pi$  the projection of  $\widetilde{\mathbf{CP}(2)}$  onto the exceptional divisor. Then, the restriction of the composition  $f_+ \circ \pi$  conjugates  $\mathcal{F}_{+,0|_{\mathcal{L}}}$  to the foliation defined by the billiard flow on  $P \times \{\theta\} \subset P \times \mathbf{R}/2\pi\mathbf{Z}$ , for some direction  $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ . In this way, we obtain a diffeomorphism  $\widetilde{f}_+ : M_1^+ \rightarrow \text{int}(P) \times \mathbf{R}/2\pi\mathbf{Z}$ . A point  $\eta \in M_1^+$  is contained in a periodic orbit of  $\widetilde{\mathcal{F}}_0$  if and only if  $\widetilde{f}_+(\eta)$  is contained in a trajectory of the billiard flow. Reasoning in a similar way, we obtain that the density of points in the phase space that define a periodic trajectory, implies the density of points in  $M_1^-$  that define a periodic orbit and *vice versa*.  $\square$

For every generic leaf  $\mathcal{L} \in \mathcal{F}_\lambda$ , periodic orbits in  $\mathcal{F}_{\lambda,0| \mathcal{L}}$  are never isolated. This is just a consequence of the commutativity, when considering the usual LIE bracket, of the real vector fields  $Re(X_\lambda)$  and  $Im(X_\lambda)$ . Such is also the case for billiard trajectories: a periodic trajectory parallel to a direction  $\theta \in \mathbf{R}/2\pi\mathbf{Z}$  is always included into a strip of periodic parallel trajectories [8]. The following example illustrates the general case.

*FAGNANO'S triangle.* In every acute triangle  $\triangle ABC$ , the triangle whose vertices are the endpoints of the altitudes from the vertices  $A$ ,  $B$  and  $C$ , defines a closed billiard trajectory. This trajectory is usually called FAGNANO'S trajectory in honor to GIULIO C. FAGNANO (1682-1766).

**Corollary 4.** (FAGNANO) *Let  $P$  be an acute triangle, then  $\mathcal{F}_{\lambda,0}$  presents at least one (non isolated) periodic orbit.*

A simple drawing shows that the obstruction for enlarging the strip containing FAGNANO'S trajectory is composed by singular billiard trajectories "joining" two vertices of the triangle. Such billiard trajectories are usually called *generalized diagonals*. Later, we will describe the role of these singular trajectories when considered as leaves of  $\mathcal{G}_{\lambda,0}$ .

A foliation on a manifold  $M$  is called *minimal* if every leaf is dense in  $M$ . If, for a given direction  $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ , the billiard on a rational triangle  $P$  presents no generalized diagonal, then the foliation  $B_\theta$ , on the invariant surface  $\Sigma_\theta$ , is minimal [8].

**Corollary 5.** *If  $(\lambda_1, \lambda_2, \lambda_3) \in \mathbf{Q}$ , then the set of leaves  $\mathcal{L} \subset \mathbf{C}^2$  in  $\mathcal{F}_\lambda$  where  $\mathcal{F}_{\lambda,0| \mathcal{L}}$  is not minimal has LEBESGUE measure zero.*

For rational triangles, the LEBESGUE measure of the plane induces a natural probability measure on the invariant surfaces  $\Sigma_\theta$  of the billiard flow. In [7], KERCKHOFF, MASUR and SMILLIE proof the following milestone theorem.

**Theorem 5.** [7] (*Triangular case*) *Let  $P$  be rational. Then for almost every direction  $\theta \in \mathbf{R}/2\pi\mathbf{Z}$  the restriction of the billiard flow to  $\Sigma_\theta$  is uniquely ergodic.*

From a remark, made independently by A. KATOK and M. BOSHERNITZAN, one can deduce the following,

**Corollary 6.** [7] *There is a dense  $G_\delta$  in the space of triangles consisting of polygons for which the billiard flow is ergodic.*

This corollary leads to

**Theorem 6.** *There is a dense  $G_\delta$  in the space of triangles for which almost every (LEBESGUE) leaf  $l \in \mathcal{G}_{\lambda,0}$  is dense in  $\mathbf{RP}(3)$ .*

*Proof.* Consider a trajectory  $l'$  of the billiard flow dense in the phase space passing through a point  $(p, \theta) \in P \times \mathbf{R}/2\pi\mathbf{Z}$ . Let  $\widetilde{f}_+ : M_1^+ \rightarrow \text{int}(P) \times \mathbf{R}/2\pi\mathbf{Z}$  be the diffeomorphism defined in the proof of lemma 2 and  $l \in \widetilde{\mathcal{F}}_0$  the leaf passing through the point  $\widetilde{f}_+^{-1}(p, \theta)$ . Suppose that  $l$  avoids an open set  $U$  in  $\widetilde{M}_1$ , and that this set, without loss of generality, is contained in either  $M_1^+$  or  $M_1^-$ . Then, using either  $\widetilde{f}_+$  or  $\widetilde{f}_-$ , we deduce that  $l'$  must avoid an open set in the phase space. This is a contradiction.  $\square$

*Generalized diagonals in  $\mathcal{G}_{\lambda,0}$ .* In the following paragraphs, we describe the set of leaves in  $\mathcal{G}_{\lambda,0}$  intersecting the local invariant manifolds  $\mathcal{V}_j$ , for  $j = 1, 2, 3$ , in terms of billiard trajectories in  $P$ .

**Definition 3.** *Let  $l$  be a non compact leaf of a foliation in a (real or complex) manifold  $M$ . A point  $p \in M$  is called an extremity of  $l$  if and only if there exists a parametrization,  $\gamma : ]0, +\infty[ \rightarrow l$ , such that  $\lim_{s \rightarrow +\infty} \gamma(s) = p$ . We say that  $p$  is an analytic extremity if and only if  $p$  is an analytic point of the closure of  $l$ .*

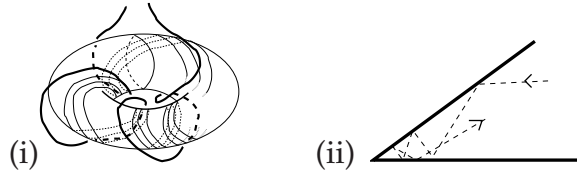
Let  $l \in \mathcal{G}_{\lambda,0} \setminus \text{Sing}(\mathcal{G}_\lambda)$  be a leaf having the singular point  $p_j$  as an extremity. Then, it intersects a local invariant manifold  $\mathcal{V}_j$ . Let  $\widetilde{l \cap \mathcal{V}_j}$  be the image of  $\Pi_{\mathbf{RP}(3)}^{-1}(l \cap \mathcal{V}_j)$  in  $\widetilde{\mathbf{CP}(2)}$  via  $\Pi^{-1}$ . The projection of  $\widetilde{l \cap \mathcal{V}_j}$  to the exceptional divisor in  $\widetilde{\mathbf{CP}(2)}$  has an extremity  $q_j$  in  $t = 0, 1, \infty$ . A direct calculation, in local coordinates, say  $z_2 = tz_1$ , shows that the SCHWARZ-CHRISTOFFEL transformation defined by formula (6) sends  $q_j$  over the vertex in  $P$ . Thus, if  $l \subset \mathcal{L} \in \mathcal{G}_\lambda$  and  $\mathcal{G}_{\lambda,0} |_{\mathcal{L}}$  is analytically conjugated (or semiconjugated for a strong resonant  $\lambda$ ) to the geodesic foliation  $D_\theta$  in the translation surface  $S(P)$ , then the geodesic determined by the leaf  $l$  is sent into the polygon by the projection  $S(P) \rightarrow P$  over a singular billiard trajectory, that is, a trajectory meeting a vertex. Conversely, every such singular trajectory in the billiard lifts up to a geodesic in  $S(P)$ , forming part of a geodesic foliation  $D_\theta$ . If this foliation on the translation surface is conjugated (or semiconjugated for a strong resonant  $\lambda$ ) to  $\mathcal{G}_{\lambda,0} |_{\mathcal{L}}$ , for a certain  $\mathcal{L} \in \mathcal{G}_\lambda$ , then the leaf  $l \in \mathcal{G}_{\lambda,0}$ , determined by the singular billiard trajectory, has an extremity in the singular locus of  $\mathcal{G}_{\lambda,0}$ .



**Theorem 7.** *Every non compact leaf of  $\mathcal{G}_{\lambda,0}$  has either 0, 1 or 2 analytic extremities in the singular locus. The set of leaves in  $\mathcal{G}_{\lambda,0}$  having two extremities is countable and each singularity in  $\mathcal{G}_{\lambda,0}$  is an analytic extremity for a countable set of this kind of leaves.*

From proposition 4, we deduce that extremities of leaves in  $\mathcal{G}_{\lambda,0}$  are analytic. The preceding paragraph implies that every leaf of  $\mathcal{G}_{\lambda,0} \setminus \text{Sing}(\mathcal{G}_\lambda)$  having two extremities corresponds to a *generalized diagonal* of the billiard on the triangle  $P$ . The set of generalized diagonals having an extremity in a fixed vertex is countable [8]. As a consequence, the reunion of all leaves in  $\mathcal{G}_{\lambda,0} \setminus \text{Sing}(\mathcal{G}_\lambda)$  intersecting  $\cup_{j=1}^3 \mathcal{V}_j$  do not form a global invariant variety.

*Remark.* Let  $l \in \widetilde{\mathcal{G}}_{\lambda,0}$  be a leaf visiting a neighbourhood  $U_j \setminus \mathbb{T}_j$  of the torus  $\mathbb{T}_j$ , but not having an extremity in  $\Pi_C^{-1}(p_j)$ . We recall that  $\widetilde{\mathcal{G}}_{\lambda,0|_{\mathbb{T}_j}}$  is a line foliation. Then, the leaf  $l$  describes a finite number of “turns” around the torus  $\mathbb{T}_j$  before leaving the neighbourhood  $U_j$ . As we have seen before, this number of turns is bounded by a constant depending only on the “slope”,  $(1 - \lambda_j)/\lambda_j$ . In terms of billiards, this phenomenon corresponds to the fact that every billiard trajectory on  $P$ , visiting a neighbourhood of a vertex *and not having the vertex as an extremity*, makes a finite number of reflections before leaving the neighbourhood. This number of reflections is bounded by  $\lceil \lambda_j \rceil$ , where  $\lceil \cdot \rceil$  notes the ceiling function. Therefore, one can think of the line foliation  $\widetilde{\mathcal{G}}_{\lambda,0|_{\mathbb{T}_j}}$  as the foliation representing the billiard dynamics at a vertex of the triangle.



The line foliation  $\widetilde{\mathcal{G}}_{\lambda,0|_{\mathbb{T}_j}}$  (i) and the billiard dynamics near a vertex (ii).

## 5. GENERAL POLYGONS

The aim of this section is to set the basis for constructing a dictionary between the billiard game in a “general polygon” and a special class of holomorphic homogeneous foliations on  $\mathbb{C}^2$ .

Polygons are plane figures whose boundaries are formed by a finite number of sequential line segments. We say that a polygon  $P$  is *simple* if the only points belonging to two edges are the vertices of  $P$ . Henceforth *general polygon* will mean *bounded simple polygon*.

*Remark.* Henceforward the general polygon  $P$  is supposed to have no vertices.

As in the triangular case, the billiard dynamics in  $P$  is equivalent to the geodesic flow on the unitary tangent bundle of a translation surface  $S = S(P)$  “generated” by  $P$  through a standard unfolding process [12]. We keep the notation  $D_\theta$  for the geodesic foliation defined on  $S(P)$  by the choice of a direction  $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ .

*General Polygons and SCHWARZ-CHRISTOFFEL transformations.* Suppose that the general polygon  $P \subset \mathbf{C}$  has  $\nu + 1$  sides and interior angle  $\lambda_j\pi$  at a vertex  $w_j$ ,  $j = 1, \dots, \nu + 1$ . Then, there are constants  $c$ ,  $C \in \mathbf{C}$  and real numbers  $\{a_j\}_{j=1}^{\nu+1}$ , with  $a_j < a_{j+1}$ , such that

$$(7) \quad f(t) = c + C \int \prod_{j=1}^{\nu+1} (\xi - a_j)^{\lambda_j - 1} d\xi$$

is a conformal representation of  $Im(t) > 0$  into the interior of  $P$  [4]. This kind of conformal representations are usually called SCHWARZ-CHRISTOFFEL transformations. The points  $a_j$  are called the *prevertices* of the map (7). Their determination is known as the SCHWARZ-CHRISTOFFEL parameter problem.

*Homogeneous foliations.* Consider a homogeneous holomorphic foliation on  $\mathbf{C}^2$  with an isolated singularity and leaving  $\nu + 1$ ,  $\nu \geq 2$ , different complex lines invariant through the singularity. We suppose that this set of invariant lines is given by the zeros of the polynomial  $P_{\nu+1} = \prod_{j=1}^{\nu+1} (z_2 - b_j z_1)$ ,  $b_j \in \mathbf{C}^*$ , and set  $b = (b_1, \dots, b_{\nu+1})$ . From [3], this foliation is given by the holomorphic 1-form:

$$(8) \quad \frac{\omega_{b,\mu}}{P_{\nu+1}} = \sum_{j=1}^{\nu+1} \mu_j \frac{d(z_2 - b_j z_1)}{z_2 - b_j z_1},$$

where  $\mu = (\mu_1, \dots, \mu_{\nu+1})$  is a point of the affine hyperplane  $\{(\mu_1, \dots, \mu_{\nu+1}) \in \mathbf{C}^n \mid \sum_{j=1}^{\nu+1} \mu_j = K, \mu_j \neq 0 \forall j\}$  and  $K \in \mathbf{C}^*$  is an arbitrary normalization constant whose choice does not affect the

foliation defined by (8). We remark that

$$(9) \quad F_{b,\lambda}(z_1, z_2) = \prod_{j=1}^{\nu+1} (z_2 - b_j z_1)^{\lambda_j}$$

is a first integral for the foliation defined by  $\omega_{b,\lambda}$ . We denote this foliation  $\mathcal{F}_{b,\lambda}$ . Let  $X_{b,\lambda}$  be the holomorphic vector field given by  $i_{X_{b,\lambda}} dz_1 \wedge dz_2 = \omega_{b,\lambda}$ . The integral curves of  $X_{b,\lambda}$  are the leaves of  $\mathcal{F}_{b,\lambda}$ . Let  $\mathcal{F}_{b,\lambda,0}$  be the real foliation on  $\mathbf{C}^2$  defined by the integral curves of the vector field  $Re(X_{b,\lambda})$ .

*Dictionary, general version.* Let  $P$  be a polygon having  $\nu + 1$  sides and interior angle  $\lambda_j \pi$  at a vertex  $w_j$ ,  $j = 1, \dots, \nu + 1$ . Suppose that (7) is a conformal representation of  $Im(t) > 0$  into the interior of  $P$ . We associate the holomorphic foliation on  $\mathbf{C}^2$  defined by the holomorphic 1-form  $\omega_{b,\mu}$  whose parameters are  $\mu_j = \lambda_j$  and  $b_j = a_j$  for  $j = 1, \dots, \nu + 1$  (8), to the billiard game on  $P$ . We call  $\mathcal{F}_{a,\lambda}$  the *polygonal foliation associated to the billiard game on the polygon  $P$* .

*Remarks.* The SCHWARZ-CHRISTOFFEL parameter problem has three degrees of freedom. Henceforward we decide that  $a_1 = \infty$ ,  $a_2 = 0$  and  $a_3 = 1$  in (7). In the triangular case, this choice leads to the expression in (1).

If the polygon  $P$  has  $\nu + 1$  sides, then the parameter  $\lambda$  of the foliation  $\mathcal{F}_{a,\lambda}$  satisfies  $\sum_{j=1}^{\nu+1} \lambda_j = \nu - 1$ . In the coordinates  $z_2 = tz_1$ , the fibers of (9) are given by  $z_1^{\nu-1} = kt^{-\lambda_2}(t-1)^{-\lambda_3} \prod_{j=4}^{\nu+1} (t-a_j)^{-\lambda_j}$ ,  $k \in \mathbf{C}^*$ . Because of this, each fiber of the first integral (9) has at most  $\nu - 1$  connected components. The subgroup of the homothecy group of  $\mathbf{C}^2$ ,

$$(10) \quad H_{\nu-1} := \{T_\eta(z_1, z_2) = \eta(z_1, z_2) \mid \eta^{\nu-1} = 1\},$$

acts transitively on the set of connected components of each fiber  $F_{a,\lambda}^{-1}(k)$ ,  $k \in \mathbf{C}^*$ . We observe that  $H_{\nu-1}$  is the group of homothecies preserving  $X_{a,\lambda}$ , that is, satisfying  $X_{a,\lambda}(T_\eta(z_1, z_2)) = DT_\eta X_{a,\lambda}(z_1, z_2)$ . We will call  $H_{\nu-1}$  the *group of automorphisms of the vector field  $X_{a,\lambda}$* .

**5.1. Proof, main result (revisited).** . The notation and language in the following paragraphs emulate those of §3.

Let  $E$  be the complement in the exceptional divisor of  $\widetilde{\mathbf{CP}(2)}$  of  $\{\infty, 0, 1, a_4, \dots, a_{\nu+1}\}$ . Every generic leaf,  $\mathcal{L} \in \widetilde{\mathcal{F}_{a,\lambda}}$  defines a covering space  $\pi|_{\mathcal{L}} : \mathcal{L} \rightarrow E$ , whose monodromy is given by the holonomy of  $\widetilde{\mathcal{F}_{a,\lambda}}$  relative to the fibration defined by  $\pi$ . This holonomy is

generated by the maps  $z \rightsquigarrow e^{\frac{2\pi i \lambda_j}{\nu-1}} z$ , for  $j = 1, \dots, \nu + 1$ . Let  $\Sigma_{a,\lambda}$  denote the  $\nu + 1$ -punctured sphere obtained by identifying two copies of the polygon  $P$  in their common sides. We note  $f : E \rightarrow \Sigma_{a,\lambda}$  the analytic continuation of the SCHWARZ-CHRISTOFFEL transformation

$$(11) \quad t \rightsquigarrow \int_0^t \xi^{\lambda_2-1} (\xi - 1)^{\lambda_3-1} \prod_{j=4}^{\nu+1} (\xi - a_j)^{\lambda_j-1} d\xi, \quad \text{Im}(t) > 0$$

to the exceptional (punctured) divisor  $E$ . Let  $\mathcal{L} \in \widetilde{\mathcal{F}}_{a,\lambda}$  be a generic leaf of the foliation on  $\widetilde{\mathbf{CP}(2)}$  induced by  $\mathcal{F}_{a,\lambda}$ . A straightforward calculation shows that, in the local coordinates  $z_2 = tz_1$ , the projection of the vector field  $\widetilde{X}_{a,\lambda}|_{\mathcal{L}}$  to  $E$  is given by the branches of  $kt^{1-\lambda_2}(t-1)^{1-\lambda_3} \prod_{j=4}^{\nu+1} (t-a_j)^{1-\lambda_j} \partial/\partial t$ , where  $k \in \mathbf{C}^*$  is a constant depending on the leaf  $\mathcal{L}$ . Therefore, (11) locally rectifies the real foliation,  $\widetilde{\mathcal{F}}_{a,\lambda,0}$ , induced in  $\widetilde{\mathbf{CP}(2)}$  by  $\mathcal{F}_{a,\lambda,0}$ . The biholomorphism  $f$  lifts to a biholomorphism  $\tilde{f} : \mathcal{L} \rightarrow S(P)$  if the coverings  $\pi|_{\mathcal{L}} : \mathcal{L} \rightarrow E$  and  $\Pi_P : S(P) \rightarrow \Sigma_{a,\lambda}$  are in the same analytic class. Such is not always the case.

Indeed, let  $\gamma_1, \dots, \gamma_{\nu+1}$  be simple loops generating  $\pi_1(E, f^{-1}(*))$ . We suppose that the action of the loop  $\gamma_j$  in the fiber  $\Pi^{-1}(*)$  is given by the holonomy generator  $z \rightsquigarrow e^{\frac{2\pi i \lambda_j}{\nu-1}} z$ . On the other hand, up to a permutation of indexes,  $d(h \circ \phi)(\gamma_j) = e^{2\pi i \lambda_j} Id_{\mathbf{R}^2}$ . Therefore, if  $\nu > 2$ , the image of the stabilizer of the monodromy action of  $\pi|_{\mathcal{L}} : \mathcal{L} \rightarrow E$ , under the isomorphism  $\phi$ , can be a proper subgroup of the stabilizer of the monodromy action of  $\Pi_P : S(P) \rightarrow \Sigma_{a,\lambda}$ . This implies that, in some ‘‘non typical’’ cases, the generic leaf of the foliation  $\mathcal{F}_{a,\lambda}$  and the translation surface  $S(P)$  are not even homeomorphic.

*Example.* The translation surface generated by any rectangular figure has genus one, whereas the closure in  $\mathbf{CP}(2)$  of the generic leave of the holomorphic homogeneous foliation  $\mathcal{F}_{a,\lambda}$ , having as first integral  $F_{a,\lambda}(z_1, z_2) = z_1^{1/2} z_2^{1/2} (z_2 - z_1)^{1/2} (z_2 + z_1)^{1/2}$ , has genus three.

**Definition 4.** A polygon having angles  $\{\lambda_j \pi\}_{j=1}^{\nu+1}$  is called reasonable if and only if every integer of the form  $\sum_{j=1}^{\nu+1} n_j \lambda_j$ ,  $n_j \in \mathbf{Z}$ ,  $\forall j$ , is congruent to  $\nu - 1$  modulo  $\mathbf{Z}$ .

Clearly, every triangle is reasonable, whereas rectangles are not. For reasonable polygons the stabilizers of the monodromies of the coverings having projections  $\pi|_{\mathcal{L}}$  and  $\Pi_P$  are isomorphic via  $\phi$ . Hence,

reasonable polygons form the natural context in which theorem 2 remains valid.

**Theorem 8.** *Let  $P$  be a reasonable polygon. Then for every generic leaf  $\mathcal{L} \in \mathcal{F}_{a,\lambda}$  there exists a direction  $\theta \in \mathbf{R}/2\pi\mathbf{Z}$  such that the foliation  $\mathcal{F}_{a,\lambda,0|_{\mathcal{L}}}$  is holomorphically conjugated to the geodesic foliation defined by the direction  $\theta$  on the translation surface  $S(P)$ . Furthermore, for every  $\theta' \in \mathbf{R}/2\pi\mathbf{Z}$  and  $\rho \in \mathbf{R}^*$ ,  $\mathcal{F}_{a,\lambda,0|_{\rho e^{i\theta'}\mathcal{L}}}$  is holomorphically conjugated to the geodesic foliation defined by the direction  $\theta + (\nu - 1)\theta'$  on  $S(P)$ .*

*Remarks.*

**i:** The factor  $(\nu - 1)\theta'$  appears because the projection of the vector field  $\widetilde{X_{a,\lambda}|_{\rho e^{i\theta'}\mathcal{L}}}$ ,  $\Pi(\mathcal{L}) = F_{a,\lambda}^{-1}(k)$ , to the exceptional divisor  $E$  is given by the branches of  $(\rho e^{i\theta'})^{\nu-1} k t^{1-\lambda_2} (t-1)^{1-\lambda_3} \prod_{j=4}^{\nu+1} (t-a_j)^{1-\lambda_j} \partial/\partial t$ .

**ii:** Let  $P$  be a polygon and  $\mathcal{F}_{a,\lambda}$  the corresponding polygonal foliation. Let  $H(\mathcal{F}_{a,\lambda})$  be the homothety subgroup formed by those  $T_k$  such that  $z \rightsquigarrow kz$  is an element of the holonomy of the foliation  $\widetilde{\mathcal{F}_{a,\lambda}}$ , relative to the fibration defined by  $\pi$ . Then, the polygon  $P$  is reasonable if and only if the intersection of  $H(\mathcal{F}_{a,\lambda})$  with  $H_{\nu-1}$ , the group of automorphisms of the vector field  $X_{a,\lambda}$ , is trivial.

*Non reasonable case.* Let  $H_\lambda$  be the intersection of the homothety subgroups  $H_{\nu-1}$  and  $H(\mathcal{F}_{a,\lambda})$ . Consider the action

$$(12) \quad \mathbf{C}^2 \times H_\lambda \ni ((z_1, z_2), T_k) \rightsquigarrow T_k(z_1, z_2)$$

and let  $\mathbf{C}_\lambda^2$  be the quotient of  $\mathbf{C}^2$  by this action. This quotient is a (complex) ruled surface with an isolated singularity. We denote the natural projection as  $\Pi_\lambda : \mathbf{C}^2 \longrightarrow \mathbf{C}_\lambda^2$ . The foliation  $\mathcal{F}_{a,\lambda}$  defines a foliation on  $\mathbf{C}_\lambda^2$ , which we denote  $\widetilde{\mathcal{F}_{a,\lambda}}$ . Due to the fact that  $H_\lambda$  contains only automorphisms of the holomorphic vector field  $X_{a,\lambda}$ , the foliation  $\mathcal{F}_{a,\lambda,0}$  defines a real foliation on  $\mathbf{C}_\lambda^2$ , which we denote  $\widehat{\mathcal{F}_{a,\lambda,0}}$ .

**Corollary 7.** *Let  $P$  be a non reasonable polygon. Then for every generic leaf  $\mathcal{L} \in \widetilde{\mathcal{F}_{a,\lambda}}$  there exists a direction  $\theta \in \mathbf{R}/2\pi\mathbf{Z}$  such that the foliation  $\widehat{\mathcal{F}_{a,\lambda,0}|_{\mathcal{L}}}$  is holomorphically conjugated to the geodesic foliation defined by the direction  $\theta$  on the translation surface  $S(P)$ . Furthermore, for every  $\theta' \in \mathbf{R}/2\pi\mathbf{Z}$  and*

$\rho \in \mathbf{R}^*$ ,  $\widehat{\mathcal{F}_{a,\lambda,0} |_{\rho e^{i\theta'} \mathcal{L}}}$  is holomorphically conjugated to the geodesic foliation defined by the direction  $\theta + (\nu - 1)\theta'$  on  $S(P)$ .

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