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# SELF-SIMILAR RANDOM FIELDS AND RESCALED RANDOM BALLS MODELS

HERMINE BIERMÉ, ANNE ESTRADE, AND INGEMAR KAJ

ABSTRACT. We study generalized random fields which arise as rescaling limits of spatial configurations of uniformly scattered random balls as the mean radius of the balls tends to 0 or infinity. Assuming that the radius distribution has a power law behavior, we prove that the centered and renormalized random balls field admits a limit with strong spatial dependence. In particular, our approach provides a unified framework to obtain all self-similar, stationary and isotropic Gaussian fields. In addition to investigating stationarity and self-similarity properties, we give  $L^2$ -representations of the asymptotic generalized random fields viewed as continuous random linear functionals.

## INTRODUCTION

In this work we construct essentially all Gaussian, stationary and isotropic, self-similar random fields on  $\mathbb{R}^d$  in a unified manner as scaling limits of a Poisson germ-grain type model. This is a random balls model that arises by aggregation of spherical grains attached to uniformly scattered germs given by a Poisson point process in  $d$ -dimensional space. The grains have random radius, independent and identically distributed, with a distribution which is assumed to have a power law behavior either in zero or at infinity. The resulting configuration of mass, obtained by counting the number of balls that cover any given point in space, suitably centered and normalized exhibits limit distributions under scaling. For the case of the random balls radius distribution being heavy-tailed at infinity, the corresponding scaling operation amounts to zooming out over larger areas of space while re-normalizing the mass. In the opposite case, when the radius of balls is given by an intensity with prescribed power-law behavior close to zero, the scaling which is applied entails zooming in successively smaller regions of space. Infinitesimally small microballs will emerge and eventually shape the resulting limit fields.

The rescaled limit configurations are conveniently described in a random fields setting which allows us to construct in this manner stationary Gaussian self-similar random fields of index  $H$ , for arbitrary non-integer  $H > -d/2$ . We obtain also non-Gaussian random fields with interesting properties, in particular a model of the type "fractional Poisson motion". Our results unify and extend in some directions the previous works on similar topics in Kaj *et al.* [14] and Biermé and Estrade [4]. For example, in terms of the self-similarity index  $H$ , [14] covers the interval  $-d/2 < H < 0$  and [4] the interval  $0 < H < 1/2$ . On the other hand, the prelimit model in the present work restricts to spherical grains only whereas [4] and [14] consider non-isotropic variations of germ-grain

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models. Preliminary and less general versions of some of the results presented here have appeared in Biermé et al. [5].

Dobrushin [8] characterized the stationary self-similar Gaussian generalized random fields in their spectral form. It is the subclass of isotropic such random fields that we are able to obtain in this work, since the random balls models we work with are rotationally symmetric. In order to obtain the whole range of self-similarity behavior it is necessary to work not only with stationary random fields but with the wider class of generalized random fields with stationary increments or stationary  $n$ th increments. In this sense our approach also links to the line of work initiated by Matheron [16].

This paper is organized as follows. After having introduced the framework model and the setting of the investigation we discuss in section 2 some principles for scaling limit analysis and state two main results, which cover a Gaussian limit regime and an intermediate limit regime. Section 3 is devoted to stationarity and self-similarity properties of generalized random fields, and representation of continuous random linear functionals. We conclude with some further topics on aggregate similarity and discuss a few explicit examples.

## 1. SETTING

We present first a unified framework which includes and extends both the situations of [14] and [4]. Let  $B(x, r)$  denote the ball in  $\mathbb{R}^d$  with center at  $x$  and radius  $r$  and consider a family of grains  $X_j + B(0, R_j)$  in  $\mathbb{R}^d$  generated by a Poisson point process  $(X_j, R_j)_j$  in  $\mathbb{R}^d \times \mathbb{R}^+$ . Equivalently, we let  $N(dx, dr)$  be a Poisson random measure on  $\mathbb{R}^d \times \mathbb{R}^+$  and associate with each random point  $(x, r) \in \mathbb{R}^d \times \mathbb{R}^+$  the random ball  $B(x, r)$ . We assume that the intensity measure of  $N$  is given by  $dx F(dr)$  where  $F$  is a  $\sigma$ -finite non-negative measure on  $\mathbb{R}^+$ .

Throughout it is assumed that the ball radius intensity  $F(dr)$  is such that

$$(1) \quad \int_{\mathbb{R}^+} r^d F(dr) < +\infty.$$

Note that if  $F$  is a probability measure, this assumption implies that the expected balls volume is finite.

For measurable sets  $A \subset \mathbb{R}^d \times \mathbb{R}^+$  we let the map  $A \mapsto N(A)$  count the number of balls subject to  $A$  with values viewed as real-valued random variables on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . We recall the basic facts that  $N(A)$  is Poisson distributed with mean  $\int_A dx F(dr)$  (if the integral diverges then  $N(A)$  is countably infinite with probability one) and if  $A_1, \dots, A_n$  are disjoint then  $N(A_1), \dots, N(A_n)$  are independent. We also recall that for measurable functions  $k : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}$ , the stochastic integral  $\int k(x, r) N(dx, dr)$  of  $k$  with respect to  $N$  exists  $\mathbb{P}$ -a.s. if and only if

$$(2) \quad \int_{\mathbb{R}^d \times \mathbb{R}^+} \min(|k(x, r)|, 1) dx F(dr) < \infty.$$

**1.1. Power-law assumption.** For  $\beta \neq d$  we introduce the following asymptotic power law assumption for the behavior of  $F$  near 0 or at infinity:

$$\mathbf{H}(\beta) : F(dr) = f(r)dr \text{ with } f(r) \sim C_\beta r^{-\beta-1}, \text{ as } r \rightarrow 0^{d-\beta},$$

where  $C_\beta > 0$  and by convention  $0^\alpha = 0$  if  $\alpha > 0$  and  $0^\alpha = +\infty$  if  $\alpha < 0$ .

Let us remark that according to (1), under assumption  $\mathbf{H}(\beta)$  it is natural to consider the asymptotic behavior of  $F$  near 0 for  $\beta < d$  and at infinity for  $\beta > d$ .

**1.2. Random field.** We consider random fields defined on a space of measures, in the same spirit as the random functionals of [14] or the generalized random fields of [3]. Let  $\mathcal{M}$  denote the space of signed measures  $\mu$  on  $\mathbb{R}^d$  with finite total variation  $|\mu|(\mathbb{R}^d)$ , with  $|\mu|$  the total variation measure of  $\mu$ . For any  $\mu \in \mathcal{M}$ ,  $\mu(B(x, r))$  is a measurable function on  $\mathbb{R}^d \times \mathbb{R}^+$  for which

$$(3) \quad \int_{\mathbb{R}^d \times \mathbb{R}^+} |\mu(B(x, r))| dx F(dr) \leq |B(0, 1)| |\mu|(\mathbb{R}^d) \int_{\mathbb{R}^+} r^d F(dr) < +\infty,$$

in view of (1). In particular, (2) applies for this choice of  $k$ . We may hence introduce a generalized random field  $X$  defined on  $\mathcal{M}$  by

$$(4) \quad X(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}^+} \mu(B(x, r)) N(dx, dr), \quad \forall \mu \in \mathcal{M}.$$

The condition (3) is even sufficient and necessary for  $X(\mu)$  to have finite expected value, and in this case

$$\mathbb{E}X(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}^+} \mu(B(x, r)) dx F(dr) = c_d \mu(\mathbb{R}^d) \int_{\mathbb{R}^+} r^d F(dr),$$

where  $c_d = |B(0, 1)|$  is the volume of the unit ball in  $\mathbb{R}^d$ . Let us also note that the random field  $X$  is linear on each vectorial subspace of  $\mathcal{M}$  in the sense that for all  $\mu_1, \dots, \mu_n \in \mathcal{M}$  and  $a_1, \dots, a_n \in \mathbb{R}$ , almost surely,

$$X(a_1\mu_1 + \dots + a_n\mu_n) = a_1X(\mu_1) + \dots + a_nX(\mu_n).$$

Our first proposition adds to this a simple topological structure.

**Proposition 1.1.** *The random field  $X$  is a continuous random linear functional on  $\mathcal{M}$ .*

*Proof.* It is natural to consider the weak topology on  $\mathcal{M}$  where the weak convergence  $\mu_n \rightharpoonup \mu$  is equivalent to

$$\int_{\mathbb{R}^d} f(z) \mu_n(dz) \rightarrow \int_{\mathbb{R}^d} f(z) \mu(dz),$$

for all  $f \in \mathcal{C}_b(\mathbb{R}^d)$  the space of continuous and bounded functions on  $\mathbb{R}^d$ . A sequence of signed measures  $(\mu_n)_{n \in \mathbb{N}}$  converges weakly to  $\mu \in \mathcal{M}$  if and only if the sequence of its total variations  $(|\mu_n|)_{n \in \mathbb{N}}$  converges weakly to  $|\mu|$  in  $\mathcal{M}$ . This is a consequence of the Radon Nikodym decomposition theorem (see [18] for instance). Then, in particular, if  $\mu_n \rightharpoonup 0$  we have  $|\mu_n|(\mathbb{R}^d) \rightarrow 0$ .

Now, for any  $\mu \in \mathcal{M}$  the random variable  $X(\mu)$  is in  $L^2(\Omega, \mathcal{A}, \mathbb{P})$  and so  $X$  can be considered as a linear functional  $X : \mathcal{M} \rightarrow L^2(\Omega, \mathcal{A}, \mathbb{P})$ . Indeed, for any  $\mu \in \mathcal{M}$

$$\begin{aligned} \text{Var}(X(\mu)) &= \int_{\mathbb{R}^d \times \mathbb{R}^+} \mu(B(x, r))^2 dx F(dr) \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \mu(dy) \mu(dy') \int_{\mathbb{R}^+} |B(y, r) \cap B(y', r)| F(dr) \\ &\leq c_d \left( \int_0^{+\infty} r^d F(dr) \right) |\mu|(\mathbb{R}^d)^2 < \infty. \end{aligned}$$

Similarly, as noticed above  $|\mathbb{E}(X(\mu))| \leq c_d \left( \int_0^{+\infty} r^d F(dr) \right) |\mu|(\mathbb{R}^d)$ . Then  $X$  is continuous since  $\mu_n \rightarrow 0$  in  $\mathcal{M}$  implies that  $\mathbb{E}(X(\mu_n)^2) \rightarrow 0$ .  $\square$

The random linear functional  $X - \mathbb{E}(X)$  is also continuous from  $\mathcal{M}$  to  $L^2(\Omega, \mathcal{A}, \mathbb{P})$ . Moreover, by identifying the space  $\mathcal{M}$  to  $\mathcal{C}_0(\mathbb{R}^d)'$ , the topological dual space of continuous functions that tend to 0 at infinity (see [18] p.161 for instance), which is a normed space, we can consider the subordinated norm of  $X - \mathbb{E}(X)$ . For  $\mu = \delta_0$ , the Dirac mass at the origine of  $\mathbb{R}^d$ , we get  $\text{Var}(X(\delta_0)) = c_d \left( \int_0^{+\infty} r^d F(dr) \right)$  and may conclude that

$$(5) \quad \|X - \mathbb{E}(X)\| = \sqrt{\left( c_d \int_0^{+\infty} r^d F(dr) \right)}.$$

## 2. SCALING LIMIT

**2.1. Scaled random fields.** Let us introduce now the notion of “scaling”, by which we indicate an action: a change of scale acts on the size of the grains. The following procedure is performed in [14] where grains of volume  $v$  are changed by shrinking into grains of volume  $\rho v$  with a small parameter  $\rho$  (“small scaling” behavior). The same is performed in [4] in the context of a homogenization, but the scaling acts in the opposite way: the radii  $r$  of grains are changed into  $r/\varepsilon$  (which is a “large scaling” behavior). Note also that both scalings are performed for  $\alpha$ -stable measures in [7] in the special case of  $d = 1$ .

Before stating our main results we introduce the field obtained by applying the rescaling of measures  $\mu \mapsto \mu^\rho$ , where  $\mu^\rho(B) = \mu(\rho B)$  for  $\rho > 0$  and measurable subsets  $B$  of  $\mathbb{R}^d$ . Let us denote by  $F_\rho(dr)$  the image measure of  $F(dr)$  by the change of scale  $r \mapsto \rho r$  and remark that

$$X(\mu^\rho) = \int_{\mathbb{R}^d \times \mathbb{R}^+} \mu(B(x, r)) N(d\rho^{-1}x, d\rho^{-1}r), \quad \forall \mu \in \mathcal{M},$$

where the intensity measure of  $N(d\rho^{-1}x, d\rho^{-1}r)$  is  $\rho^{-d} dx F_\rho(dr)$ . It is natural from this viewpoint to have  $\mu$  representing an observation window and interpret limits  $\rho \rightarrow 0$  as *zoom-out* and limits  $\rho \rightarrow \infty$  as *zoom-in* of the random configurations of balls in space. We will see that during zoom-out the appearance of occasional balls of very large radius will provide enough overlap between balls to create strong positive dependence (the case  $\beta > d$ ), while zooming in among microballs will generate asymptotically a variety of spatially dependent random fields (the case  $\beta < d$ ).

Let us multiply the intensity measure by  $\lambda > 0$  and consider the associated random field on  $\mathcal{M}$  given by

$$\int_{\mathbb{R}^d \times \mathbb{R}^+} \mu(B(x, r)) N_{\lambda, \rho}(dx, dr),$$

where  $N_{\lambda, \rho}(dx, dr)$  is the Poisson random measure with intensity measure  $\lambda dx F_\rho(dr)$  and  $\mu \in \mathcal{M}$ . Results are expected concerning the asymptotic behavior of this scaled random balls model under hypothesis  $\mathbf{H}(\beta)$  when  $\rho \rightarrow 0$  or  $\rho \rightarrow +\infty$ . We choose  $\rho$  as

the basic model parameter, consider  $\lambda = \lambda(\rho)$  as a function of  $\rho$ , and define on  $\mathcal{M}$  the random field

$$(6) \quad X_\rho(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}^+} \mu(B(x, r)) N_{\lambda(\rho), \rho}(dx, dr) .$$

Then, we are looking for a normalization term  $n(\rho)$  such that the centered field converges in distribution,

$$(7) \quad \frac{X_\rho(\cdot) - \mathbb{E}(X_\rho(\cdot))}{n(\rho)} \xrightarrow{fdd} W(\cdot)$$

and we are interested in the nature of the limit field  $W$ . The convergence (7) holds whenever

$$\mathbb{E} \left( \exp \left( i \frac{X_\rho(\mu) - \mathbb{E}(X_\rho(\mu))}{n(\rho)} \right) \right) \rightarrow \mathbb{E} (\exp (iW(\mu))) ,$$

for all  $\mu$  in a convenient subspace of  $\mathcal{M}$ . A scaling analysis of power law tails reveals that under  $\mathbf{H}(\beta)$  we expect

$$\text{Var}(X_\rho(\mu)) \sim \lambda(\rho) \rho^\beta \text{Var}(X(\mu)), \quad \rho^{d-\beta} \rightarrow 0,$$

which suggests the asymptotic relation  $n(\rho)^2 \sim \lambda(\rho) \rho^\beta$ . However, in view of (5), the norm of  $(X_\rho - \mathbb{E}(X_\rho))/n(\rho)$  as a linear functional from  $\mathcal{M}$  to  $L^2(\Omega, \mathcal{A}, \mathbb{P})$  is given by

$$(8) \quad \left\| \frac{X_\rho - \mathbb{E}(X_\rho)}{n(\rho)} \right\| = \sqrt{\left( c_d \int_0^{+\infty} r^d F(dr) \right)} \sqrt{\frac{\lambda(\rho) \rho^d}{n(\rho)^2}} .$$

In particular, (8) is not bounded for  $n(\rho)^2 = \lambda(\rho) \rho^\beta$  as  $\rho \rightarrow 0^{\beta-d}$  and the Banach Steinhaus Theorem states that there exists a dense subset of  $\mathcal{M}$  on which the rescaled process  $(X_\rho(\mu) - \mathbb{E}(X_\rho(\mu)))/\sqrt{\lambda(\rho) \rho^\beta}$  can not converge. Therefore we study in the sequel the convergence (7) on convenient subspaces of  $\mathcal{M}$ .

**2.2. Main results.** For  $\beta \neq d$  let us define the space of measures

$$\mathcal{M}^\beta = \left\{ \mu \in \mathcal{M} : \exists \alpha \text{ s. t. } \alpha < \beta < d \text{ or } d < \beta < \alpha \text{ and } \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\alpha} |\mu|(dz) |\mu|(dz') < +\infty \right\},$$

where  $|\mu|$  is the total variation measure of  $\mu$ . Let us remark that the integral assumption is a finite Riesz energy assumption for  $\beta > d$  and that  $\mathcal{M}^\beta = \{0\}$  when  $\beta \geq 2d$ .

Let us emphasize that, in both cases ( $\beta > d$  or  $\beta < d$ ), if  $\mu \in \mathcal{M}$  satisfies

$\int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\alpha} |\mu|(dz) |\mu|(dz') < +\infty$  for some  $\alpha$  ( $\alpha < \beta < d$  or  $d < \beta < \alpha$ ) then the same holds for any  $\gamma$  between  $\beta$  and  $\alpha$ . In particular, for any  $\mu \in \mathcal{M}^\beta$ ,

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\beta} |\mu|(dz) |\mu|(dz') < +\infty .$$

We also need to impose some vanishing moments assumptions on the measures through the following subspaces. For any  $n \in \mathbb{N} \setminus \{0\}$ , we denote by  $\mathcal{M}_n$  the subspace of finite signed measures  $\mu \in \mathcal{M}$ , which satisfy

$$(9) \quad \int_{\mathbb{R}^d} z^j \mu(dz) = \int_{\mathbb{R}^d} z_1^{j_1} \dots z_d^{j_d} \mu(dz) = 0$$

for all  $j = (j_1, \dots, j_d) \in \mathbb{N}^d$  with  $0 \leq j_1 + \dots + j_d < n$  (see [16] where similar spaces of measures are introduced). Let us point out that a simple but tedious computation shows that

$$\mathcal{M}_n = \left\{ \mu \in \mathcal{M} : \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{2k} \mu(dz) \mu(dz') = 0 \text{ for all integers } 0 \leq k < n \right\}.$$

By convention, we set  $\mathcal{M}_0 = \mathcal{M}$ . For any real number  $\alpha$  we denote by  $[\alpha] \in \mathbb{Z}$  the unique integer such that  $[\alpha] - 1 < \alpha \leq [\alpha]$  and we consider the subspaces

$$(10) \quad \widetilde{\mathcal{M}}_\beta = \begin{cases} \mathcal{M}^\beta & \text{for } \beta > d \\ \mathcal{M}^\beta \cap \mathcal{M}_{n_\beta} & \text{for } \beta < d \text{ with } n_\beta = \lceil \frac{d-\beta}{2} \rceil \end{cases}.$$

**Theorem 2.1.** *Let  $\beta < 2d$  such that  $d - \beta \notin 2\mathbb{N}$ . Let  $F$  be a non-negative measure on  $\mathbb{R}^+$  satisfying  $\mathbf{H}(\beta)$ . For all positive functions  $\lambda$  such that  $\lambda(\rho)\rho^\beta \xrightarrow{\rho \rightarrow 0^{\beta-d}} +\infty$ , the limit*

$$\frac{X_\rho(\mu) - \mathbb{E}(X_\rho(\mu))}{\sqrt{\lambda(\rho)\rho^\beta}} \xrightarrow[\rho \rightarrow 0^{\beta-d}]{fdd} W_\beta(\mu)$$

holds for all  $\mu \in \widetilde{\mathcal{M}}_\beta$ , in the sense of finite dimensional distributions of the random functionals. Here  $W_\beta$  is the centered Gaussian random linear functional on  $\widetilde{\mathcal{M}}_\beta$  with

$$(11) \quad \text{Cov}(W_\beta(\mu), W_\beta(\nu)) = \mathbb{E}(W_\beta(\mu)W_\beta(\nu)) = c_\beta \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\beta} \mu(dz) \nu(dz'),$$

for a constant  $c_\beta$  depending on  $\beta$ .

*Proof.* We first introduce two lemmas. The first one describes the covariance function and is based on some technical estimates for the intersection volume of two balls. The second one, inspired by Lemma 1 of [14], stands for Lebesgue's Theorem with assumptions that are well adapted to the present setting.

**Lemma 2.2.** *Let  $\beta < 2d$  with  $d - \beta \notin 2\mathbb{N}$ . Let  $C_\beta > 0$ . There exists a real constant  $c_\beta$  such that for all  $\mu \in \widetilde{\mathcal{M}}_\beta$ ,*

$$0 < C_\beta \int_{\mathbb{R}^d \times \mathbb{R}^+} \mu(B(x, r))^2 r^{-\beta-1} dr dx = c_\beta \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\beta} \mu(dz) \mu(dz') < +\infty.$$

**Remark 2.3.** *The above identity ensures that equation (11) defines a covariance function, called generalized covariance function in [16].*

*Proof.* Let us introduce the function  $\gamma$  defined on  $\mathbb{R}$  by

$$(12) \quad \gamma(u) = |B(0, 1) \cap B(ue, 1)|$$

for any unit vector  $e \in \mathbb{R}^d$ , where  $|\cdot|$  stands for the Lebesgue measure. The function  $\gamma$  is clearly even, supported by  $[-2, 2]$ , bounded by  $\gamma(0) = c_d$ , the volume of the unit ball in  $\mathbb{R}^d$ , and smooth on  $(-2, 2)$ .

Consider  $\beta < d$  such that  $n_\beta = \lceil \frac{d-\beta}{2} \rceil \geq 1$ . Define  $\gamma_\beta$  as the remainder of the Taylor expansion of  $\gamma$  around 0 at order  $2n_\beta - 1$ , i.e.  $\forall u \in \mathbb{R}$ ,

$$\gamma_\beta(u) = \gamma(u) - \sum_{k=0}^{2n_\beta-1} \frac{\gamma^{(k)}(0)}{k!} u^k = \gamma(u) - \sum_{0 \leq k < n_\beta} \frac{\gamma^{(2k)}(0)}{(2k)!} u^{2k}.$$

Note that the second equality is due to the fact that  $\gamma$  is even. We notice that

$$\forall u \in (-2, 2), |\gamma_\beta(u)| \leq C|u|^{2n_\beta} \quad \text{and} \quad \forall |u| \geq 2, |\gamma_\beta(u)| \leq C|u|^{2n_\beta-2},$$

for some constant  $C > 0$ . Therefore, for any  $\beta < d$  with  $d - \beta \notin 2\mathbb{N}$ , one can find  $C > 0$  such that the function  $\gamma_\beta$  satisfies

$$(13) \quad \forall u \in \mathbb{R}, |\gamma_\beta(u)| \leq C|u|^{d-\alpha}.$$

for any  $\alpha \in [d - 2n_\beta, \beta]$ .

In the case  $\beta > d$ , we will denote by  $\gamma_\beta$  the single function  $\gamma$  such that (13) holds also for any  $\alpha \geq 0$ .

1st Step. For  $\mu \in \widetilde{\mathcal{M}}_\beta$ , let us prove that  $\int_{\mathbb{R}^d \times \mathbb{R}^+} \mu(B(x, r))^2 r^{-\beta-1} dr dx < +\infty$ . We introduce the function  $\varphi$  defined by

$$(14) \quad \varphi(r) = \int_{\mathbb{R}^d} \mu(B(x, r))^2 dx, \quad r > 0.$$

Using successively Fubini's Theorem, homogeneity and (12) we get

$$\varphi(r) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |B(z, r) \cap B(z', r)| \mu(dz) \mu(dz') = r^d \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma(|z - z'|/r) \mu(dz) \mu(dz').$$

Therefore  $\varphi(r) \leq c_d |\mu|(\mathbb{R}^d)^2 r^d$ . Moreover, since  $\mu \in \widetilde{\mathcal{M}}_\beta$ ,

$$(15) \quad \varphi(r) = r^d \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma_\beta(|z - z'|/r) \mu(dz) \mu(dz'),$$

and we can choose  $\alpha > \beta$  (if  $\beta > d$ ) or  $\alpha \in [d - 2n_\beta, \beta]$  (if  $\beta < d$ ) such that  $\int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\alpha} |\mu|(dz) |\mu|(dz') < +\infty$  and (13) holds. Then

$$\varphi(r) \leq C r^\alpha \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\alpha} |\mu|(dz) |\mu|(dz').$$

Finally, one can find  $C > 0$  such that

$$(16) \quad \varphi(r) \leq C \min(r^d, r^\alpha),$$

and

$$\int_0^{+\infty} \varphi(r) r^{-\beta-1} dr = \int_{\mathbb{R}^d \times \mathbb{R}^+} \mu(B(x, r))^2 r^{-\beta-1} dr dx < +\infty.$$

2nd Step. We prove the equality stated in the Lemma, which can be formulated as

$$C_\beta \int_0^{+\infty} \varphi(r) r^{-\beta-1} dr = c_\beta \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\beta} \mu(dz) \mu(dz'),$$

using the previous notations. For that purpose we want to replace  $\varphi$  by (15) in the left hand side integral. Using the estimates (13) on  $|\gamma_\beta|$ , one can show that the integral defined by

$$I_\beta(u) := \int_{\mathbb{R}^+} \gamma_\beta(u/r) r^{d-\beta-1} dr,$$

is well defined for all  $u \in \mathbb{R}$ . Since  $\gamma$  is even, so is  $\gamma_\beta$ . Hence  $I_\beta$  is an even function of  $u$ . Furthermore,  $I_\beta$  is homogeneous of order  $d - \beta$  so that

$$\forall u \in \mathbb{R}, I_\beta(u) = I_\beta(1) |u|^{d-\beta}.$$



This proves that

$$\int_0^{+\infty} \varphi(r)r^{-\beta-1}dr = I_\beta(1) \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\beta} \mu(dz)\mu(dz'),$$

and concludes for the proof with  $c_\beta = C_\beta I_\beta(1)$ .  $\square$

Now let us state a second lemma which is the main tool to establish our scaling limit results.

**Lemma 2.4.** *Let  $F$  be a non-negative measure on  $\mathbb{R}^+$  satisfying  $\mathbf{H}(\beta)$  for  $\beta \neq d$ .*

(i) *Assume that  $g$  is a continuous function on  $\mathbb{R}^+$  such that for some  $0 < p < \beta < q$ , there exists  $C > 0$  such that*

$$|g(r)| \leq C \min(r^q, r^p).$$

Then

$$\int_{\mathbb{R}^+} g(r)F_\rho(dr) \sim C_\beta \rho^\beta \int_0^\infty g(r)r^{-\beta-1}dr \text{ as } \rho \rightarrow 0^{\beta-d}.$$

(ii) *Let  $g_\rho$  be a family of continuous functions on  $\mathbb{R}^+$ . Assume that*

$$\lim_{\rho \rightarrow 0^{\beta-d}} \rho^\beta g_\rho(r) = 0, \quad \text{and} \quad \rho^\beta |g_\rho(r)| \leq C \min(r^p, r^q),$$

for some  $0 < p < \beta < q$  and  $C > 0$ . Then

$$\lim_{\rho \rightarrow 0^{\beta-d}} \int_{\mathbb{R}^+} g_\rho(r)F_\rho(dr) = 0.$$

*Proof.* (i) Let us assume for instance that  $\beta < d$  (the proof of the case  $\beta > d$  is similar and can be found in [14]). Let  $\varepsilon > 0$ . Since  $F$  satisfies  $\mathbf{H}(\beta)$  there exists  $\delta > 0$  such that

$$(17) \quad r < \delta \Rightarrow \left| f(r) - C_\beta r^{-\beta-1} \right| \leq \varepsilon r^{-\beta-1}.$$

Let us remark that the assumptions on  $g$  ensure that

$$\int_0^{+\infty} g(r)r^{-\beta-1}dr < +\infty.$$

On the one hand, since  $\int_0^{\delta\rho} g(r)F_\rho(dr) = \int_0^{\delta\rho} g(r)f\left(\frac{r}{\rho}\right)\frac{dr}{\rho}$ , we get by (17)

$$\left| \int_0^{\delta\rho} g(r)F_\rho(dr) - C_\beta \rho^\beta \int_0^{\delta\rho} g(r)r^{-\beta-1}dr \right| \leq \varepsilon \rho^\beta \int_0^\infty g(r)r^{-\beta-1}dr.$$

On the other hand, for  $\delta\rho > 1$ , since  $|g(r)| \leq Cr^p$ ,

$$\begin{aligned} \left| \int_{\delta\rho}^\infty g(r)F_\rho(dr) - C_\beta \rho^\beta \int_{\delta\rho}^\infty g(r)r^{-\beta-1}dr \right| &\leq C_1(\delta)\rho^p + \frac{C_\beta}{\beta-p}\delta^{p-\beta}\rho^p \\ &\leq C(\delta)\rho^{p+1}, \end{aligned}$$

where  $C_1(\delta) = \int_\delta^{+\infty} r^p F(dr) \leq \delta^{p-d} \int_0^{+\infty} r^d F(dr) < \infty$ . Since  $p < \beta$ , we obtain (i).

(ii) We follow the same lines as for (i) and can assume similarly that  $\beta < d$ . Since  $F$  satisfies  $\mathbf{H}(\beta)$  there exists  $\delta > 0$  such that

$$(18) \quad r < \delta \Rightarrow |f(r)| \leq (C_0 + 1)r^{-\beta-1}.$$

The assumptions on  $g$  ensure that for all  $\rho > 0$ ,

$$\int_0^{+\infty} \rho^\beta |g_\rho(r)| r^{-\beta-1} dr < +\infty.$$

Since  $\int_0^{\delta\rho} g_\rho(r) F_\rho(dr) = \int_0^{\delta\rho} g_\rho(r) f\left(\frac{r}{\rho}\right) \frac{dr}{\rho}$ , we get by (18)

$$\left| \int_0^{\delta\rho} g_\rho(r) F_\rho(dr) \right| \leq (C_0 + 1) \int_0^\infty \rho^\beta |g_\rho(r)| r^{-\beta-1} dr.$$

Thus, by Lebesgue's Theorem

$$(19) \quad \lim_{\rho \rightarrow +\infty} \int_0^{\delta\rho} g_\rho(r) F_\rho(dr) = 0.$$

Moreover, for  $\delta\rho > 1$ , since  $C_1(\delta) = \int_\delta^{+\infty} r^p F(dr) < +\infty$  and  $|g_\rho(r)| \leq C\rho^{-\beta} r^p$ ,

$$(20) \quad \begin{aligned} \left| \int_{\delta\rho}^\infty g_\rho(r) F_\rho(dr) \right| &\leq C\rho^{-\beta} \int_{\delta\rho}^\infty r^p F_\rho(dr) \\ &\leq CC_1(\delta)\rho^{-(\beta-p)}. \end{aligned}$$

We conclude the proof using (19) and (20), since  $p < \beta$ .  $\square$

We start now with the proof of Theorem 2.1. Let us denote

$$n(\rho) := \sqrt{\lambda(\rho)\rho^\beta}$$

and define the function  $\varphi_\rho$  on  $\mathbb{R}^+$  by

$$\varphi_\rho(r) = \int_{\mathbb{R}^d} \Psi\left(\frac{\mu(B(x,r))}{n(\rho)}\right) dx,$$

where

$$(21) \quad \Psi(v) = e^{iv} - 1 - iv.$$

The characteristic function of the normalized field  $(X_\rho(\cdot) - \mathbb{E}(X_\rho(\cdot)))/n(\rho)$  is then given by

$$\mathbb{E}\left(\exp\left(i\frac{X_\rho(\mu) - \mathbb{E}(X_\rho(\mu))}{n(\rho)}\right)\right) = \exp\left(\int_{\mathbb{R}^+} \lambda(\rho)\varphi_\rho(r) F_\rho(dr)\right).$$

By assumption,  $n(\rho)$  tends to  $+\infty$  as  $\rho \rightarrow 0^{\beta-d}$  so that  $\Psi\left(\frac{\mu(B(x,r))}{n(\rho)}\right)$  behaves like  $-\frac{1}{2}\left(\frac{\mu(B(x,r))}{n(\rho)}\right)^2$ . Therefore, we write

$$(22) \quad \int_{\mathbb{R}^+} \lambda(\rho)\varphi_\rho(r) F_\rho(dr) = -\frac{1}{2} \int_{\mathbb{R}^+} \varphi(r)\lambda(\rho)n(\rho)^{-2} F_\rho(dr) + \int_{\mathbb{R}^+} \Delta_\rho(r) F_\rho(dr)$$

where the function  $\varphi$  is introduced in (14) and

$$(23) \quad \begin{aligned} \Delta_\rho(r) &= \lambda(\rho)\varphi_\rho(r) + \frac{1}{2}\lambda(\rho)n(\rho)^{-2}\varphi(r) \\ &= \lambda(\rho) \int_{\mathbb{R}^d} \left( \Psi\left(\frac{\mu(B(x,r))}{n(\rho)}\right) + \frac{1}{2}\left(\frac{\mu(B(x,r))}{n(\rho)}\right)^2 \right) dx. \end{aligned}$$

Since  $\mu \in \widetilde{\mathcal{M}}_\beta$ , the function  $\varphi$  satisfies (16) and thus the assumptions of Lemma 2.4(i). Thus, the first term of the right hand side of (22) converges to  $C_\beta \int_{\mathbb{R}^+} \varphi(r) r^{-\beta-1} dr$ , and using Lemma 2.2 we obtain

$$\lim_{\rho \rightarrow 0^{\beta-d}} \int_{\mathbb{R}^+} \varphi(r) \lambda(\rho) n(\rho)^{-2} F_\rho(dr) = c_\beta \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\beta} \mu(dz) \mu(dz').$$

For the second term, let us verify that  $\Delta_\rho$  given by (23) satisfies the assumptions of Lemma 2.4(ii). First let us remark that the function  $\Delta_\rho$  is continuous on  $\mathbb{R}^+$  since  $\mu \in \mathcal{M}$ . Because  $\left| \Psi(v) - \left(-\frac{v^2}{2}\right) \right| \leq \frac{|v|^3}{6}$  and

$$\int_{\mathbb{R}^d} \mu(B(x, r))^3 dx \leq \|\mu\|_1^2 \int_{\mathbb{R}^d} |\mu(B(x, r))| dx \leq C_d \|\mu\|_1^3 r^d,$$

we also check that

$$|\lambda(\rho)^{-1} n(\rho)^2 \Delta_\rho(r)| \leq \frac{C_d \|\mu\|_1^3}{6} n(\rho)^{-1} r^d.$$

Finally, since  $|\Psi(v)| \leq \frac{|v|^2}{2}$ , there exists  $\alpha$  with  $(\alpha - \beta)(\beta - d) > 0$  such that by (16)

$$|\lambda(\rho)^{-1} n(\rho)^2 \Delta_\rho(r)| \leq cr^\alpha.$$

Therefore,  $\int_{\mathbb{R}^+} \Delta_\rho(r) F_\rho(dr)$  tends to 0 according to Lemma 2.4(ii), so we get

$$\lim_{\rho \rightarrow 0^{\beta-d}} \mathbb{E} \left( \exp \left( i \frac{X_\rho(\mu) - \mathbb{E}(X_\rho(\mu))}{n(\rho)} \right) \right) = \exp \left( -\frac{1}{2} c_\beta \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\beta} \mu(dz) \mu(dz') \right).$$

Hence  $(X_\rho(\mu) - \mathbb{E}(X_\rho(\mu)))/n(\rho)$  converges in distribution to the centered Gaussian random variable  $W(\mu)$  whose variance is equal to

$$\mathbb{E}(W(\mu)^2) = c_\beta \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\beta} \mu(dz) \mu(dz').$$

By linearity, the covariance of  $W$  satisfies (11).  $\square$

With similar arguments we can state an intermediate scaling result.

**Theorem 2.5.** *Let  $\beta < 2d$  such that  $d - \beta \notin 2\mathbb{N}$ . Let  $F$  be a non-negative measure on  $\mathbb{R}^+$  satisfying  $\mathbf{H}(\beta)$ . For all positive functions  $\lambda$  such that  $\lambda(\rho) \rho^\beta \xrightarrow[\rho \rightarrow 0^{\beta-d}]{} a^{d-\beta}$ , for some  $a > 0$ , we have in the sense of finite-dimensional distributions of random functionals the scaling limit*

$$X_\rho(\mu) - \mathbb{E}(X_\rho(\mu)) \xrightarrow{fdd} J_\beta(\mu_a),$$

for all  $\mu \in \widetilde{\mathcal{M}}_\beta$ . Here  $J_\beta$  is the centered random linear functional on  $\widetilde{\mathcal{M}}_\beta$  defined as

$$J_\beta(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}^+} \mu(B(x, r)) \widetilde{N}_\beta(dx, dr),$$

where  $\widetilde{N}_\beta$  is a compensated Poisson random measure with intensity  $C_\beta dx r^{-\beta-1} dr$ , and  $\mu_a$  is defined by  $\mu_a(A) = \mu(a^{-1}A)$ .

*Proof.* Let us recall (see [15] for instance) that for measurable functions  $k : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}$ , the stochastic integral  $\int k(x, r) \tilde{N}(dx, dr)$  of  $k$  with respect to a compensated Poisson measure  $\tilde{N}$  of intensity  $n$  exists  $\mathbb{P}$ -a.s. if and only if

$$(24) \quad \int_{\mathbb{R}^d \times \mathbb{R}^+} \min(|k(x, r)|, k(x, r)^2) n(dx, dr) < \infty.$$

Let  $\varphi$  be the function introduced in (14) and let us recall that by Lemma 2.2 for all  $\mu \in \tilde{\mathcal{M}}_\beta$ ,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^+} \mu(B(x, r))^2 r^{-\beta-1} dr dx = \int_{\mathbb{R}^+} \varphi(r) r^{d-\beta-1} dr < +\infty.$$

Hence, in view of (24) with  $n(dx, dr) = C_\beta dx r^{-\beta-1} dr$  and  $k(x, r) = \mu(B(x, r))$ , the random field  $J_\beta$  is well defined on  $\tilde{\mathcal{M}}_\beta$ , with characteristic function

$$(25) \quad \mathbb{E}(\exp(iJ_\beta(\mu))) = \exp\left(\int_{\mathbb{R}^+ \times \mathbb{R}^d} \Psi(\mu(B(x, r))) C_\beta dx r^{-\beta-1} dr\right),$$

where  $\Psi$  is given by (21). Let us recall that the characteristic function for the centered Poisson random balls model is

$$\mathbb{E}(\exp(i(X_\rho(\mu) - \mathbb{E}(X_\rho(\mu)))) = \exp\left(\int_{\mathbb{R}^+ \times \mathbb{R}^d} \lambda(\rho) \Psi(\mu(B(x, r))) dx F_\rho(dr)\right),$$

and let us define for  $r > 0$ ,

$$\tilde{\varphi}(r) = \int_{\mathbb{R}^d} \Psi(\mu(B(x, r))) dx.$$

For  $\mu \in \tilde{\mathcal{M}}_\beta$ , using  $|\Psi(v)| \leq |v|^2/2$  and (16), there exists  $C > 0$  such that

$$|\tilde{\varphi}(r)| \leq C \min(r^d, r^\alpha),$$

for some  $\alpha$  with  $(\alpha - \beta)(\beta - d) > 0$ . Therefore, from Lemma 2.4 (i),

$$\int_{\mathbb{R}^+} \lambda(\rho) \tilde{\varphi}(r) F_\rho(dr) \underset{\rho \rightarrow 0^{\beta-d}}{\sim} C_\beta a^{d-\beta} \int_0^\infty \tilde{\varphi}(r) r^{-\beta-1} dr.$$

Then

$$\lim_{\rho \rightarrow 0^{\beta-d}} \mathbb{E}(\exp(i(X_\rho(\mu) - \mathbb{E}(X_\rho(\mu)))) = \exp\left(C_\beta a^{d-\beta} \int_{\mathbb{R}^+} \tilde{\varphi}(r) r^{-\beta-1} dr\right).$$

Finally, it is sufficient to remark that

$$a^{d-\beta} \int_{\mathbb{R}^+} \tilde{\varphi}(r) r^{-\beta-1} dr = a^d \int_{\mathbb{R}^+} \tilde{\varphi}(a^{-1}r) r^{-\beta-1} dr,$$

with

$$a^d \tilde{\varphi}(a^{-1}r) = a^d \int_{\mathbb{R}^d} \Psi(\mu(B(x, a^{-1}r))) dx = \int_{\mathbb{R}^d} \Psi(\mu_a(B(x, r))) dx,$$

to obtain

$$\lim_{\rho \rightarrow 0^{\beta-d}} \mathbb{E}(\exp(i(X_\rho(\mu) - \mathbb{E}(X_\rho(\mu)))) = \mathbb{E}(\exp(iJ_\beta(\mu_a))).$$

□

From Lemma 2.2 and (11), we get the following remark.

**Remark 2.6.** *The covariance function of  $J_\beta$  is given for all  $\mu, \nu \in \widetilde{\mathcal{M}}_\beta$  by*

$$\begin{aligned} \text{Cov}(J_\beta(\mu), J_\beta(\nu)) &= \int_{\mathbb{R}^d \times \mathbb{R}^+} \mu(B(x, r)) \nu(B(x, r)) C_\beta \, dx \, r^{-\beta-1} dr \\ &= c_\beta \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\beta} \mu(dz) \nu(dz'), \end{aligned}$$

and so  $J_\beta$  and  $W_\beta$  have the same covariance function on  $\widetilde{\mathcal{M}}_\beta$ .

We should also obtain what is called the small-grain scaling in [14]. In this particular case, the limit is an independently scattered  $\alpha$ -stable random measure on  $\mathbb{R}^d$  with Lebesgue measure and unit skewness.

### 3. PROPERTIES OF SELF-SIMILAR RANDOM GENERALIZED FIELDS

This section is devoted to the main properties of the fields we obtain as scaling limits. Due to the stationarity of the initial field on the one hand and to the performed scaling on the other hand, the limits inherit a stationarity property as well as a self-similarity property that we discuss here.

**3.1. Translation invariant self-similar generalized random fields.** Following the same ideas as in [8] or [16] we define a notion of stationarity which characterizes the translation invariance of a random linear functional over a subset of signed measures. We say as usual that a subspace  $\mathcal{S} \subset \mathcal{M}$  is closed for translations if, for any  $\mu \in \mathcal{S}$  and any  $s \in \mathbb{R}^d$ , we have  $\tau_s \mu \in \mathcal{S}$ , where  $\tau_s \mu$  is defined by  $\tau_s \mu(A) = \mu(A - s)$ , for any Borel set  $A$ . In particular the subspaces  $\mathcal{M}_n$ , defined by (9), are closed for translations for any  $n \in \mathbb{N}$ .

**Definition 3.1.** *Let  $n \in \mathbb{N}$ . Let  $X$  be a random field defined on a subspace  $\mathcal{S} \subset \mathcal{M}_n$  closed for translations. The field  $X$  is translation invariant if*

$$(26) \quad \forall \mu \in \mathcal{S}, \forall s \in \mathbb{R}^d, X(\tau_s \mu) \stackrel{fdd}{=} X(\mu).$$

More precisely, one says that  $X$  is stationary when  $n = 0$  and has stationary  $n$ th increments when  $n > 0$ .

Let us remark that if  $X$  has stationary  $n$ th increments on a subspace  $\mathcal{S} \subset \mathcal{M}_n$ , then its restriction on  $\mathcal{S} \cap \mathcal{M}_{n+1} \subset \mathcal{M}_{n+1}$  has stationary  $(n + 1)$ th increments.

By the translation invariance of the Lebesgue measure, for any  $\rho > 0$  the random field  $X_\rho$  defined by (6) is stationary on  $\mathcal{M}$ . The fields  $W_\beta$  and  $J_\beta$  obtained as limit fields on  $\widetilde{\mathcal{M}}_\beta$  in Theorem 2.1 and Theorem 2.5 are not defined on the full space  $\mathcal{M}$ . But  $\widetilde{\mathcal{M}}_\beta$  is closed for translations. Therefore, when considering the limiting random fields on  $\widetilde{\mathcal{M}}_\beta$ , one has the following property.

**Proposition 3.2.** *Let  $\beta \neq d$  with  $d - \beta \notin 2\mathbb{N}$ . Then  $W_\beta$  as well as  $J_\beta$  are translation invariant on  $\widetilde{\mathcal{M}}_\beta$ .*

In other words, from (10),  $W_\beta$  and  $J_\beta$  defined on  $\widetilde{\mathcal{M}}_\beta$  are both stationary if  $\beta > d$  and they have stationary  $n_\beta$ th increments if  $\beta < d$ , where  $n_\beta = \lceil \frac{d-\beta}{2} \rceil > 0$ .

We turn now to the self-similarity property. We note  $\mu_a$  the dilated measure defined by  $\mu_a(A) = \mu(a^{-1}A)$  for any Borel set  $A$ . We say that a subspace  $\mathcal{S} \subset \mathcal{M}$  is closed for dilations if, for any  $\mu \in \mathcal{S}$  and any  $a > 0$ , we have  $\mu_a \in \mathcal{S}$ .

We introduce the following definition, which coincides with the usual definition of self-similar punctual random fields.

**Definition 3.3.** *Let  $H \in \mathbb{R}$ . A random field  $X$ , defined on a subspace  $\mathcal{S}$  of  $\mathcal{M}$  which is closed for dilations, is said to be self-similar with index  $H$  if*

$$\forall \mu \in \mathcal{S}, \forall a > 0, X(\mu_a) \stackrel{fdd}{=} a^H X(\mu).$$

Once noticed that  $\widetilde{\mathcal{M}}_\beta$  is closed for dilations, and by observing the consequence of dilation on the covariance of  $W_\beta$ , the following property is straightforward.

**Proposition 3.4.** *The field  $W_\beta$ , defined on  $\widetilde{\mathcal{M}}_\beta$ , is self-similar with index  $H = \frac{d-\beta}{2}$  that runs over  $(-\frac{d}{2}, +\infty) \setminus \mathbb{N}$ .*

**Remark 3.5.** *In contrast to the Gaussian field  $W_\beta$ , the intermediate scaling limit field  $J_\beta$  is not self-similar.*

Hence the Gaussian field  $W_\beta$  is both translation invariant and self-similar. Dobrushin [8] gives a complete description of Gaussian translation invariant self-similar generalized random fields on  $\mathbb{R}^d$ . For this purpose he considers continuous random linear functionals of  $\mathcal{S}(\mathbb{R}^d)'$ , where  $\mathcal{S}(\mathbb{R}^d)'$  is the topological dual of the Schwartz space  $\mathcal{S}(\mathbb{R}^d)$  of all infinitely differentiable rapidly decreasing functions on  $\mathbb{R}^d$  (see [9] for instance). As usual  $\mathcal{S}(\mathbb{R}^d)$  is equipped with the topology that corresponds to the following notion of convergence:  $\varphi_n \rightarrow \varphi$  if and only if for all  $N \in \mathbb{N}$  and  $j \in \mathbb{N}^d$

$$\sup_{z \in \mathbb{R}^d} (1 + |z|)^N |D^j(\varphi_n - \varphi)(z)| \rightarrow 0,$$

where  $D^j\varphi(z) = \frac{\partial^{j_1} \dots \partial^{j_d}}{\partial z_1^{j_1} \dots \partial z_d^{j_d}} \varphi(z)$  denotes the partial derivative of order  $j$ . Then, a linear functional  $X : \mathcal{S}(\mathbb{R}^d) \rightarrow L^2(\Omega, \mathcal{A}, \mathbb{P})$  is continuous if and only if  $\varphi_n \rightarrow 0$  in  $\mathcal{S}(\mathbb{R}^d)$  implies that

$$\mathbb{E}(X(\varphi_n)^2) \rightarrow 0.$$

Let us remark that, to each function  $\varphi \in \mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$  one can uniquely associate a signed measure  $\tilde{\varphi} \in \mathcal{M}$  defined by  $\tilde{\varphi}(dz) = \varphi(z)dz$ . For the sake of notational simplicity we identify any function  $\varphi \in L^1(\mathbb{R}^d)$  with its image  $\tilde{\varphi}$  in  $\mathcal{M}$  so that  $L^1(\mathbb{R}^d) \subset \mathcal{M}$ . Therefore any random linear functional on  $\mathcal{M}$ , when restricted to  $\mathcal{S}(\mathbb{R}^d)$ , can be viewed as a linear functional on  $\mathcal{S}(\mathbb{R}^d)$ .

**Proposition 3.6.** *Let  $\rho > 0$ . The random field  $X_\rho$  induces a continuous random linear functional on  $\mathcal{S}(\mathbb{R}^d)$ .*

*Proof.* From (8), the random field  $X_\rho$  is a continuous random linear functional on  $\mathcal{M}$ . Then, to prove the continuity of  $X_\rho$  on  $\mathcal{S}(\mathbb{R}^d)$  it is sufficient, using Lebesgue's theorem, to notice that the previous identification implies that if  $\mu_n = \tilde{\varphi}_n \rightarrow 0$  in  $\mathcal{S}(\mathbb{R}^d)$  then  $\mu_n \rightarrow 0$  in  $\mathcal{M}$ .  $\square$

We are now interested in the continuity properties of  $W_\beta$  and  $J_\beta$ . Let us remark that  $\mathcal{S}(\mathbb{R}^d) \cap \widetilde{\mathcal{M}}_\beta = \mathcal{S}(\mathbb{R}^d)$  when  $d < \beta < 2d$ , while  $\mathcal{S}(\mathbb{R}^d) \cap \widetilde{\mathcal{M}}_\beta = \mathcal{S}(\mathbb{R}^d) \cap \mathcal{M}_{n_\beta}$  for  $\beta < d$ .

**Proposition 3.7.** *Let  $\beta < 2d$  with  $d - \beta \notin 2\mathbb{N}$ . The random fields  $W_\beta$  and  $J_\beta$  induce continuous random linear functionals on  $\mathcal{S}_n(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d) \cap \mathcal{M}_n$  for any integer  $n > \frac{d-\beta}{2}$ .*

*Proof.* Note that by (11) and Remark 2.6, for any  $\mu \in \widetilde{\mathcal{M}}_\beta$ ,

$$(27) \quad \mathbb{E}(W_\beta(\mu)^2) = \mathbb{E}(J_\beta(\mu)^2) \leq |c_\beta| \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\beta} |\mu|(dz) |\mu|(dz').$$

A straightforward use of Lebesgue's theorem concludes for the proof.  $\square$

Then, restricted to  $\mathcal{S}_n(\mathbb{R}^d)$  the Gaussian field  $W_\beta$  is a translation invariant self-similar generalized fields. We address to [6] for other examples of self-similar generalized field. In [8] Dobrushin focuses on the spectral representation of such Gaussian fields. Since the law of a centered Gaussian field is characterized by its covariance function, let us introduce a second order self-similarity property. For  $H \in \mathbb{R}$  we say that  $X$ , a random linear functional on  $\mathcal{S}_n(\mathbb{R}^d)$  is a second order self-similar field of order  $H$  if, for all  $a > 0$ ,  $\varphi, \psi \in \mathcal{S}_n(\mathbb{R}^d)$ ,

$$(28) \quad \text{Cov}(X(\varphi_a), X(\psi_a)) = a^{2H} \text{Cov}(X(\varphi), X(\psi)), \text{ where } \varphi_a(x) = a^{-d} \varphi(a^{-1}x).$$

We denote by  $\widehat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-iz \cdot \xi} \mu(dz)$  the Fourier transform of any  $\mu \in \mathcal{M}$ . Then Theorem 3.2 of [8] can be reformulated as follows.

**Theorem 3.8.** *Let  $n \geq 0$  and  $X$  be a continuous random linear functional on  $\mathcal{S}_n(\mathbb{R}^d)$ . Then  $X$  is a second order self-similar field of order  $H \in \mathbb{R}$  if and only if for all  $\varphi, \psi \in \mathcal{S}_n(\mathbb{R}^d)$ ,*

$$(29) \quad \text{Cov}(X(\varphi), X(\psi)) = \int_{S^{d-1}} \int_0^{+\infty} \widehat{\varphi}(r\theta) \overline{\widehat{\psi}(r\theta)} r^{-2H-1} dr d\sigma(\theta) + \sum_{|j|=|k|=n} A_{j,k} \alpha_j(\varphi) \overline{\alpha_k(\psi)},$$

where  $\sigma$  is a finite positive measure on the unit sphere  $S^{d-1}$ ,  $\alpha_j(\varphi) = \int_{\mathbb{R}^d} \varphi(x) x^j dx = i^{|j|} D^j \widehat{\varphi}(0)$ , for  $j \in \mathbb{N}^d$  with  $|j| = n$ , and  $A = (A_{j,k})_{|j|=|k|=n}$  is a symmetric positive definite real matrix. Moreover, if  $H < n$  then  $A = 0$ ; if  $H = n$  then  $\sigma = 0$ ; and if  $H > n$  then  $A = 0$  and  $\sigma = 0$ .

Note that  $W_\beta$  and  $J_\beta$  share the same covariance function by Remark 2.6 so that they are both second order self-similar fields of order  $\frac{d-\beta}{2}$ . Therefore, by Proposition 3.7 and Theorem 3.8 they have to be considered on  $\mathcal{S}_n(\mathbb{R}^d)$  for  $n > \frac{d-\beta}{2}$ . Moreover, due to the isotropy of balls and the rotation invariance of Lebesgue's measure it is straightforward to conclude that  $W_\beta$  and  $J_\beta$  are isotropic random fields. Therefore the measure  $\sigma$  obtained from Theorem 3.8 must be invariant under rotation and, up to a constant, it is the Lebesgue's measure on the sphere. We obtain the following result, which is of Plancherel's type and gives the covariance function of  $W_\beta$  and  $J_\beta$  in spectral form.

**Proposition 3.9.** Fix  $\beta < 2d$  with  $\beta - d \notin 2\mathbb{N}$  and let  $n_\beta = \lceil \frac{d-\beta}{2} \rceil$ . There exists  $k_\beta > 0$  such that, for any  $\varphi, \psi \in \mathcal{S}_{n_\beta}(\mathbb{R}^d)$ ,

$$\begin{aligned} \text{Cov}(W_\beta(\varphi), W_\beta(\psi)) &= \text{Cov}(J_\beta(\varphi), J_\beta(\psi)) \\ &= c_\beta \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\beta} \varphi(z) \psi(z') \, dz dz' = k_\beta \int_{\mathbb{R}^d} \widehat{\varphi}(\xi) \overline{\widehat{\psi}(\xi)} |\xi|^{\beta-2d} \, d\xi. \end{aligned}$$

We make the further comment that generalized random fields defined on  $\mathcal{S}_n(\mathbb{R}^d)$  for some  $n > 0$  correspond roughly speaking to some derivatives of different orders of random fields defined on  $\mathcal{S}(\mathbb{R}^d)$ . More precisely, since the Schwartz class is closed under differentiation, if  $X$  is a continuous random linear functional on  $\mathcal{S}(\mathbb{R}^d)$  one can define for any  $j \in \mathbb{N}^d$  the partial derivative of  $X$  of order  $j$  as the continuous random linear functional defined by

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^d), \quad D^j X(\varphi) = (-1)^{|j|} X(D^j \varphi).$$

Moreover, Lemma 1.2.1 p.23 of [3] states the following equality.

**Proposition 3.10.** For any  $n \in \mathbb{N}$ ,  $\mathcal{S}_n(\mathbb{R}^d) = \text{Span} \{ D^j \varphi, \varphi \in \mathcal{S}(\mathbb{R}^d), |j| = n \}$ .

Therefore, the knowledge of a generalized random field  $X$  on  $\mathcal{S}_n(\mathbb{R}^d)$  is equivalent to the knowledge of all its partial derivatives  $D^j X$  of order  $j$  with  $|j| = n$ . Furthermore,  $X$  has stationary  $n$ th increments if and only if its partial derivatives  $D^j X$  of order  $j$  with  $|j| = n$  are stationary. A natural question is then to find some continuous random linear functional  $B$  on  $\mathcal{S}(\mathbb{R}^d)$  such that for any  $j \in \mathbb{N}^d$  with  $|j| = n_\beta$ ,

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^d), \quad D^j B(\varphi) = (-1)^{|j|} W_\beta(D^j \varphi).$$

We will use the representation of  $W_\beta$  as defined by Matheron [16] to provide an answer to this question (see also the links between “generalized random fields” and “punctual random fields” in [3]).

**3.2. Representation of continuous random linear functionals.** Let  $X$  be a continuous random linear functional on a subset  $\mathcal{S}$  of  $\mathcal{M}$ . We say that a continuous function  $\widetilde{X} : \mathbb{R}^d \rightarrow L^2(\Omega, \mathcal{A}, \mathbb{P})$  is a representation of  $X$  if, for any  $\mu \in \mathcal{S}$

$$X(\mu) \stackrel{L^2(\Omega, \mathcal{A}, \mathbb{P})}{=} \int_{\mathbb{R}^d} \widetilde{X}(t) \mu(dt).$$

When such a representation holds an easy way to obtain  $\widetilde{X}(t)$  for any  $t \in \mathbb{R}^d$  is to consider  $X(\delta_t)$  with  $\delta_t$  the Dirac mass at point  $t$ , as soon as  $\delta_t \in \mathcal{S}$ . Unfortunately  $\delta_t \notin \mathcal{M}^\beta$  but a natural way to overcome this difficulty is to consider an approximation of  $\delta_t$ .

Following the same ideas as [16], let us choose  $\theta \in \mathcal{S}(\mathbb{R}^d)$  a positive even function such that its Fourier transform  $\widehat{\theta}$  satisfies  $\widehat{\theta}(0) = \int_{\mathbb{R}^d} \theta(z) dz = 1$  and for any  $j \in \mathbb{N}^d$ ,

$$D^j \widehat{\theta}(0) = (-i)^{|j|} \int_{\mathbb{R}^d} z^j \theta(z) dz = 0, \quad \text{with } j \neq 0 \text{ and } |j| \leq n_\beta,$$



if  $n_\beta = \lceil \frac{d-\beta}{2} \rceil \geq 1$ . Let us observe that  $\int_{\mathbb{R}^d} z^k D^l \theta(z) dz = i^{|k|} D^k \widehat{D^l \theta}(0)$  for any  $k, l \in \mathbb{N}^d$ , with  $\widehat{D^l \theta}(\xi) = (i\xi)^l \widehat{\theta}(\xi)$ . Thus, for any  $l \in \mathbb{N}^d$  with  $|l| < n_\beta$ ,

$$\int_{\mathbb{R}^d} z^k D^l \theta(z) (dz) = \delta_l^k \text{ for } |k| < n_\beta, \text{ with } \delta_l^k = 0 \text{ if } k \neq l \text{ and } \delta_l^k = 1 \text{ if } k = l.$$

Let  $n \in \mathbb{N}$  with  $n \neq 0$  and set  $\theta_n(z) = n^d \theta(nz)$ . Then for all  $t \in \mathbb{R}^d$ , the measures defined by

$$\Theta_t^n = \tau_t \theta_n - \sum_{|l| < n_\beta} \frac{(-1)^{|l|}}{l!} t^l D^l \theta_n$$

belongs to  $\widetilde{\mathcal{M}}_\beta$ , where by convention the sum is empty when  $n_\beta \leq 0$ .

Let us recall that for any function  $f \in \mathcal{C}_b(\mathbb{R}^d)$  we have  $f * \theta_n(t) \rightarrow f(t)$  with  $f * \theta_n(t) = \int_{\mathbb{R}^d} f(z) \theta_n(t-z) dz = \int_{\mathbb{R}^d} f(z) \tau_t \theta_n(z) dz$ , since  $\theta$  is even. Therefore  $\tau_t \theta_n \rightarrow \delta_t$  and if  $\widetilde{X}$  is a representation of  $X$  we should obtain  $\widetilde{X}(t)$  as  $\lim_{n \rightarrow +\infty} X(\tau_t \theta_n)$  when this limit makes

sense. Since  $\Theta_t^n \in \widetilde{\mathcal{M}}_\beta$  for all  $t \in \mathbb{R}^d$ , we can consider the sequence of random functions defined by  $(W_\beta(\Theta^n))_{n \geq 1}$  and  $(J_\beta(\Theta^n))_{n \geq 1}$ , where  $W_\beta(\Theta^n) : t \mapsto W_\beta(\Theta_t^n)$  for all  $n \geq 1$  and similarly for  $J_\beta(\Theta^n)$ .

**Theorem 3.11.** *Let  $\beta < d$  with  $d - \beta \notin 2\mathbb{N}$ . The finite dimensional distributions of  $(W_\beta(\Theta^n))_{n \geq 1}$  converge in  $L^2(\Omega, \mathcal{A}, \mathbb{P})$  to a representation  $\widetilde{W}_\beta$  of  $W_\beta$  on  $\mathcal{S}_{n_\beta}(\mathbb{R}^d)$  with a covariance function given by*

$$\begin{aligned} \Gamma_\beta(t, s) &= k_\beta \int_{\mathbb{R}^d} \left( e^{-it \cdot \xi} - \sum_{k=0}^{n_\beta-1} \frac{(it \cdot \xi)^k}{k!} \right) \overline{\left( e^{-is \cdot \xi} - \sum_{k=0}^{n_\beta-1} \frac{(is \cdot \xi)^k}{k!} \right)} |\xi|^{\beta-2d} d\xi \\ (30) \quad &= c_\beta \left( |t-s|^{d-\beta} - \sum_{|l| < n_\beta} \frac{(-1)^{|l|}}{l!} \left( s^l D^l |t|^{d-\beta} + t^l D^l |s|^{d-\beta} \right) \right) \end{aligned}$$

Similarly, the finite dimensional distributions of  $(J_\beta(\Theta^n))_{n \geq 1}$  converge in  $L^2(\Omega, \mathcal{A}, \mathbb{P})$  to a representation  $\widetilde{J}_\beta$  of  $J_\beta$  on  $\mathcal{S}_{n_\beta}(\mathbb{R}^d)$  with the same covariance function as  $\widetilde{W}_\beta$ .

*Proof.* Let  $n \in \mathbb{N} \setminus \{0\}$  and  $t \in \mathbb{R}^d$ . By choice of  $\theta$  we have  $\Theta_t^n \in \mathcal{S}_{n_\beta}(\mathbb{R}^d)$ . Moreover, for  $\xi \in \mathbb{R}^d$ ,

$$\widehat{\Theta}_t^n(\xi) = \widehat{\theta}_n(\xi) \left( e^{-it \cdot \xi} - \sum_{|l| < n_\beta} \frac{1}{l!} t^l (-i\xi)^l \right) = \widehat{\theta} \left( \frac{\xi}{n} \right) \left( e^{-it \cdot \xi} - \sum_{k=0}^{n_\beta-1} \frac{(it \cdot \xi)^k}{k!} \right),$$

since  $\sum_{|l|=k} \frac{1}{l!} t^l (-i\xi)^l = \frac{(it \cdot \xi)^k}{k!}$  for any  $k \in \mathbb{N}$ , by a generalization of the binomial theorem.

Let  $n, m \in \mathbb{N} \setminus \{0\}$  and  $t, s \in \mathbb{R}^d$ . From Proposition 3.9, there exists  $k_\beta > 0$  such that the covariance

$$\Gamma_{n,m}(t, s) = \text{Cov}(W_\beta(\Theta_t^n), W_\beta(\Theta_s^m)) = \text{Cov}(J_\beta(\Theta_t^n), J_\beta(\Theta_s^m))$$

can be written

$$\Gamma_{n,m}(t,s) = k_\beta \int_{\mathbb{R}^d} \widehat{\Theta}_t^n(\xi) \overline{\widehat{\Theta}_s^m(\xi)} |\xi|^{\beta-2d} d\xi.$$

Then Lebesgue's theorem implies that the limit in  $\Gamma_{n,m}(t,s) \xrightarrow{n,m \rightarrow +\infty} \Gamma_\beta(t,s)$ , is given by

$$\Gamma_\beta(t,s) = C_\beta \int_{\mathbb{R}^d} \left( e^{-it \cdot \xi} - \sum_{k=0}^{n_\beta-1} \frac{(it \cdot \xi)^k}{k!} \right) \overline{\left( e^{-is \cdot \xi} - \sum_{k=0}^{n_\beta-1} \frac{(is \cdot \xi)^k}{k!} \right)} |\xi|^{\beta-2d} d\xi.$$

Therefore, the finite dimensional distributions of  $(W_\beta(\Theta^n))_{n \geq 1}$ , converge in  $L^2(\Omega, \mathcal{A}, \mathbb{P})$  to a centered random field  $\widetilde{W}_\beta$ . The finite dimensional distributions of  $(J_\beta(\Theta^n))_{n \geq 1}$  converge similarly to a limit  $\widetilde{J}_\beta$ . Both limit fields have covariance function given by  $\Gamma_\beta$ .

Let us prove that  $\widetilde{W}_\beta$  is a representation of  $W_\beta$  on  $\mathcal{S}_{n_\beta}(\mathbb{R}^d)$ . The covariance function  $\Gamma_\beta$  of  $\widetilde{W}_\beta$  is continuous with respect to each variable and so  $\widetilde{W}_\beta : \mathbb{R}^d \rightarrow L^2(\Omega, \mathcal{A}, \mathbb{P})$  is continuous. Then, the random linear functional  $X : \mu \in \mathcal{S}(\mathbb{R}^d) \mapsto \int_{\mathbb{R}^d} \widetilde{W}_\beta(t) \mu(dt)$  is well defined since

$$\text{Var}(X(\mu)) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \text{Cov}(\widetilde{W}_\beta(t), \widetilde{W}_\beta(s)) \mu(dt) \mu(ds) < +\infty,$$

using the fact that  $\text{Var}(\widetilde{W}_\beta)(t) \leq C|t|^{d-\beta}$ . Finally, for any  $\mu \in \mathcal{S}_{n_\beta}(\mathbb{R}^d)$  we have  $\text{Var}(X(\mu)) = \text{Var}(W_\beta(\mu))$  by Proposition 3.9, since  $\int_{\mathbb{R}^d} t^l \mu(dt) = 0$  for  $|l| < n_\beta$ , which proves that  $\widetilde{W}_\beta$  is a representation of  $W_\beta$  on  $\mathcal{S}_{n_\beta}(\mathbb{R}^d)$ . The same arguments hold to prove that  $\widetilde{J}_\beta$  is a representation of  $J_\beta$  on  $\mathcal{S}_{n_\beta}(\mathbb{R}^d)$ .

To conclude it remains to prove (30). By Proposition 3.9, for all  $n, m \in \mathbb{N} \setminus \{0\}$ ,

$$\Gamma_{n,m}(t,s) = c_\beta \int_{\mathbb{R}^d \times \mathbb{R}^d} |z - z'|^{d-\beta} \Theta_t^n(z) \Theta_s^m(z') dz dz'.$$

For any  $z' \in \mathbb{R}^d$ , the function  $f_{z'}(z) = |z - z'|^{d-\beta}$  admits continuous derivatives of order  $l$  on  $\mathbb{R}^d$  for any  $|l| < d - \beta$ . Therefore, for any  $z' \in \mathbb{R}^d$ ,

$$\begin{aligned} \int_{\mathbb{R}^d} |z - z'|^{d-\beta} \Theta_t^n(z) dz &= f_{z'} * \theta_n(t) - \sum_{|l| < n_\beta} \frac{t^l}{l!} D^l f_{z'} * \theta_n(0) \\ &\xrightarrow{n \rightarrow +\infty} |t - z'|^{d-\beta} - \sum_{|l| < n_\beta} \frac{(-1)^{|l|}}{l!} t^l D^l |z'|^{d-\beta}. \end{aligned}$$

By Lebesgue's theorem, as  $n \rightarrow +\infty$  we get

$$\lim_{n \rightarrow +\infty} \Gamma_{n,m}(t,s) = c_\beta \int_{\mathbb{R}^d} \left( |t - z'|^{d-\beta} - \sum_{|l| < n_\beta} \frac{(-1)^{|l|}}{l!} t^l D^l |z'|^{d-\beta} \right) \Theta_s^m(z') dz'.$$

As previously we obtain

$$\int_{\mathbb{R}^d} |t - z'|^{d-\beta} \Theta_s^m(z') dz' \xrightarrow{m \rightarrow +\infty} |t - s|^{d-\beta} - \sum_{|l| < n_\beta} (-1)^{|l|} \frac{s^l}{l!} D^l |t|^{d-\beta},$$

while

$$\int_{\mathbb{R}^d} D^l |z'|^{d-\beta} \Theta_s^m(z') dz' \xrightarrow{m \rightarrow +\infty} D^l |s|^{d-\beta}.$$

Therefore  $\Gamma_\beta(t, s) = \lim_{n, m \rightarrow +\infty} \Gamma_{n, m}(t, s)$  is also equal to (30).  $\square$

Since  $W_\beta$  is Gaussian,  $\widetilde{W}_\beta$  is also Gaussian as a limit in  $L^2(\Omega, \mathcal{A}, \mathbb{P})$  of a Gaussian functional. The spectral representation of  $\widetilde{W}_\beta$  is given by

$$(31) \quad \widetilde{W}_\beta(t) \stackrel{fdd}{=} \sqrt{k_\beta} \int_{\mathbb{R}^d} \left( e^{-it \cdot \xi} - \sum_{k=0}^{n_\beta-1} \frac{(-it \cdot \xi)^k}{k!} \right) |\xi|^{\beta/2-d} W(d\xi),$$

where  $W$  is the complex Brownian measure. This field is called elliptic Gaussian self-similar random field in [2].

For  $0 < \frac{d-\beta}{2} < 1$ , the random field  $\widetilde{W}_\beta$  corresponds to the well known fractional Brownian field with Hurst parameter equal to  $\frac{d-\beta}{2}$  and (31) is known as the harmonizable representation of the fractional Brownian field (see [12] for a review).

Let us also remark that when  $d = 1$ , the covariance function  $\Gamma_\beta$  given by (30) is equal to

$$c_\beta \left( |t-s|^{d-\beta} - \sum_{l=0}^{n_\beta-1} (-1)^l \binom{d-\beta}{l} \left( \left( \frac{s}{t} \right)^l |t|^{d-\beta} + \left( \frac{t}{s} \right)^l |s|^{d-\beta} \right) \right),$$

where  $\binom{d-\beta}{l} = (d-\beta) \cdot \dots \cdot (d-\beta-l+1)/l!$ . Therefore  $\sqrt{\frac{C_{d-\beta}}{2(-1)^{n_\beta} c_\beta}} \widetilde{W}_\beta$  is a  $n_\beta$ th-order fractional Brownian motion as defined in [17], with  $C_{d-\beta}^{-1} = \Gamma(1+d-\beta) |\sin((d-\beta)\pi/2)|$ .

One can define the  $n_\beta$ th increments of  $\widetilde{W}_\beta$  with lag  $h \in \mathbb{R}^d$ , which corresponds to the discrete differentiation of order  $n_\beta$ , by

$$\Delta_h^{n_\beta} \widetilde{W}_\beta(t) = \sum_{p=0}^{n_\beta} \binom{n_\beta}{p} (-1)^{n_\beta-p} \widetilde{W}_\beta(t+ph).$$

Then

$$\Delta_h^{n_\beta} \widetilde{W}_\beta(t) = \lim_{n \rightarrow +\infty} W_\beta \left( \sum_{p=0}^{n_\beta} \binom{n_\beta}{p} (-1)^{n_\beta-p} \tau_{t+ph} \theta_n \right)$$

and the stationarity of  $W_\beta$  implies that  $\widetilde{W}_\beta$  has stationary  $n_\beta$ th increments in the wide sense: for all  $t, s, h, h' \in \mathbb{R}^d$ , the covariances  $\text{Cov} \left( \Delta_h^{n_\beta} \widetilde{W}_\beta(s), \Delta_{h'}^{n_\beta} \widetilde{W}_\beta(s+t) \right)$  do not depend on  $s$  (see [19] or [11] for instance).

**Proposition 3.12.** *Let  $\beta < d$  with  $d - \beta \notin 2\mathbb{N}$  and  $n_\beta = \lceil \frac{d-\beta}{2} \rceil$ . Then the Gaussian random field  $\widetilde{W}_\beta$  has stationary  $n_\beta$ th increments. Moreover, this field admits continuous partial derivatives of order  $l \in \mathbb{N}^d$  in mean square for any  $|l| < n_\beta$ , such that  $D^l \widetilde{W}_\beta$  has stationary  $(n_\beta - |l|)$  increments, is self-similar of order  $\frac{d-\beta}{2} - |l|$ , and satisfies  $D^l \widetilde{W}_\beta(0) = 0$  almost surely.*

*Proof.* Recall that  $\Gamma_\beta$  denotes the covariance function of  $\widetilde{W}_\beta$ . Since  $n_\beta = \lceil \frac{d-\beta}{2} \rceil > 0$ , it is straightforward to see that  $\Gamma_\beta$  admits symmetric partial derivatives of order  $l \in \mathbb{N}^d$  for any  $|l| < n_\beta$  with  $\frac{\partial^{2l}\Gamma_\beta}{\partial s^l \partial t^l}(s, t)$  given by

$$C_\beta \int_{\mathbb{R}^d} \left( e^{-it \cdot \xi} - \sum_{k=0}^{n_\beta - |l| - 1} \frac{(it \cdot \xi)^k}{k!} \right) \overline{\left( e^{-is \cdot \xi} - \sum_{k=0}^{n_\beta - |l| - 1} \frac{(is \cdot \xi)^k}{k!} \right)} \xi^{2l} |\xi|^{\beta - 2d} d\xi.$$

By Theorem 2.2.2 of [1], this means that  $\widetilde{W}_\beta$  admits a continuous partial derivative of order  $l$  in mean square,  $D^l \widetilde{W}_\beta$ , which is a Gaussian random field with covariance given by  $\text{Cov}(D^l \widetilde{W}_\beta(t), D^l \widetilde{W}_\beta(s)) = \frac{\partial^{2l}\Gamma_\beta}{\partial s^l \partial t^l}(s, t)$ . A straightforward change of variable yields for all  $a > 0$

$$\text{Cov}(D^l \widetilde{W}_\beta(at), D^l \widetilde{W}_\beta(as)) = a^{d-\beta-2l} \text{Cov}(D^l \widetilde{W}_\beta(t), D^l \widetilde{W}_\beta(s)).$$

Since  $D^l \widetilde{W}_\beta$  is Gaussian this implies that  $D^l \widetilde{W}_\beta$  is self-similar of order  $\frac{d-\beta}{2} - |l|$  ie

$$\left\{ D^l \widetilde{W}_\beta(at), t \in \mathbb{R}^d \right\} \stackrel{fdd}{=} a^{\frac{d-\beta}{2} - |l|} \left\{ D^l \widetilde{W}_\beta(t), t \in \mathbb{R}^d \right\} \text{ for all } a > 0.$$

Moreover, for all  $t, s, h, h' \in \mathbb{R}^d$ ,

$$\begin{aligned} & \text{Cov} \left( \Delta_h^{n_\beta - |l|} D^l \widetilde{W}_\beta(s), \Delta_{h'}^{n_\beta - |l|} D^l \widetilde{W}_\beta(s+t) \right) \\ &= C_\beta \int_{\mathbb{R}^d} e^{-it \cdot \xi} (e^{-ih \cdot \xi} - 1)^{n_\beta - |l|} (e^{ih' \cdot \xi} - 1)^{n_\beta - |l|} \xi^{2l} |\xi|^{\beta - 2d} d\xi, \end{aligned}$$

and  $D^l \widetilde{W}_\beta$  has stationary  $(n_\beta - |l|)$ th increments. We conclude since  $\text{Var} \left( D^l \widetilde{W}_\beta(0) \right) = 0$  implies that  $D^l \widetilde{W}_\beta(0) = 0$  almost surely.  $\square$

**Remark 3.13.** *One can prove that  $\widetilde{W}_\beta$  is the only Gaussian random field with stationary  $n_\beta$ th increments, which is self-similar of order  $\frac{d-\beta}{2}$  and isotropic.*

**Remark 3.14.** *The representation  $\widetilde{J}_\beta$  of  $J_\beta$  obtained in Theorem 3.11 is not Gaussian but shares the same covariance function as  $\widetilde{W}_\beta$ . Therefore it satisfies the same second order properties: stationary  $n_\beta$ th increments, self-similarity of order  $\frac{d-\beta}{2}$  and isotropy.*

**Remark 3.15.** *In the case  $\beta > d$  one can not find any representation of neither  $W_\beta$  nor  $J_\beta$  on  $\mathcal{S}(\mathbb{R}^d)$ . This is due to the fact that a random field can not be second order self-similar of order  $H < 0$  and continuous in mean square around 0.*

## 4. FURTHER TOPICS

**4.1. Aggregate similarity.** We have seen that the Gaussian limit field  $W_\beta$  is self-similar whereas the intermediate regime scaling limit  $J_\beta$  is not. A similarity property which applies in great generality to long-range dependent processes is discussed in [13]. The following is a version for spatial random fields.

**Definition 4.1.** A random field  $X$  with  $\mathbb{E}X = 0$ , defined on a subspace  $\mathcal{S}$  of  $\mathcal{M}$  which is closed for dilations, is said to be aggregate-similar if there exists a sequence of positive real numbers  $(a_m)_{m \geq 1}$ , such that

$$\forall \mu \in \mathcal{S}, \forall m \geq 1, X(\mu_{a_m}) \stackrel{fdd}{=} \sum_{i=1}^m X^i(\mu),$$

where  $(X^i)_{i \geq 1}$  are i.i.d copies of  $X$ .

Thus, a random field is aggregate-similar if the path  $\mu_{a_m} \mapsto X(\mu_{a_m})$  as we trace along the sequence of dilations given by  $a_m$  passes all aggregates  $\sum_{i=1}^m X^i$  of  $X$ , in the distributional sense. We may write, equivalently,

$$\forall \mu \in \mathcal{S}, \forall m \geq 1, X(\mu) \stackrel{fdd}{=} \sum_{i=1}^m X^i(\mu_{a_m^{-1}}),$$

which immediately shows that an aggregate-similar random field is also infinitely divisible.

Any self-similar, zero mean Gaussian random field is aggregate-similar. Indeed, if  $X_H$  is Gaussian with  $\mathbb{E}X_H = 0$  and self-similar with index  $H$  then letting  $a_m = m^{1/2H}$  we have

$$X_H(\mu_{a_m}) \stackrel{fdd}{=} m^{1/2} X_H(\mu) \stackrel{fdd}{=} \sum_{i=1}^m X_H^i(\mu), \quad m \geq 1.$$

In particular,  $W_\beta$  is aggregate similar on  $\widetilde{\mathcal{M}}_\beta$  with respect to the sequence  $a_m = m^{1/(d-\beta)}$ . For  $\beta < d$  we have  $a_m^{-1} \rightarrow 0$  and hence  $\mu_{a_m}$  represents a zoom-in of  $W_\beta$ , as  $m \rightarrow \infty$ . This is in contrast to the case  $d < \beta < 2d$ . In this case  $a_m^{-1} \rightarrow \infty$ , and consequently the succession of aggregates  $\sum_{i=1}^m W_\beta^i(\mu)$  of  $W_\beta(\mu)$  appears as the sequence of measures  $\mu_{a_m}$  performs a zoom-out, in the limit  $m \rightarrow \infty$ .

Turning next to the non-Gaussian field  $J_\beta$ , by (25)

$$\log \mathbb{E}(\exp(iJ_\beta(\mu_a))) = a^{\beta-d} \log \mathbb{E}(\exp(iJ_\beta(\mu))).$$

Thus,  $J_\beta$  is aggregate-similar with respect to  $a_m$ , given by  $a_m^{d-\beta} = m$ . This property provides an interpretation of the dilation parameter  $a$  in Theorem 2.5. If we assume in the theorem that  $\lambda(\rho)\rho^\beta \rightarrow a_m^{d-\beta}$  as  $\rho^{\beta-d} \rightarrow 0$ , for arbitrary  $m \geq 1$ , then

$$X_\rho(\mu) - \mathbb{E}(X_\rho(\mu)) \xrightarrow{fdd} J_\beta(\mu_{a_m}) \stackrel{fdd}{=} \sum_{i=1}^m J_\beta^i(\mu).$$

The guiding asymptotic quantity  $\lambda\rho^\beta$  may be interpreted as the expected number of very large ( $\beta > d$ ) or very small ( $\beta < d$ ) balls which cover a point asymptotically. Thus, the more of such extreme grains are allowed asymptotically, the larger number of i.i.d. copies of the basic field  $J_\beta$  appears in the limit.

We may continue this line of reasoning by providing a limit result for  $J_\beta(\mu_{a_m})$  as  $m \rightarrow \infty$ .

**Proposition 4.2.** *As  $a^{d-\beta} \rightarrow \infty$ , for all  $\mu$  in  $\widetilde{\mathcal{M}}_\beta$*

$$\frac{1}{a^{(d-\beta)/2}} J_\beta(\mu_a) \xrightarrow{fdd} W_\beta(\mu)$$

*Proof.* Consider the subsequence  $a_m = m^{1/(d-\beta)}$ . It follows immediately from aggregate-similarity and the central limit theorem that

$$\frac{1}{a_m^{(d-\beta)/2}} J_\beta(\mu_{a_m}) \xrightarrow{fdd} \frac{1}{\sqrt{m}} \sum_{i=1}^m J_\beta^i(\mu) \xrightarrow{fdd} W_\beta(\mu), \quad m \rightarrow \infty,$$

since  $J_\beta(\mu)$  and  $W_\beta(\mu)$  have the same variance. A standard argument completes the proof of convergence in distribution along an arbitrary sequence.  $\square$

**4.2. Fractional Brownian field and fractional Poisson field.** We consider the special case  $d-2 < \beta < d$ . For this range of parameters we have  $n_\beta = 1$  and

$$\widetilde{\mathcal{M}}_\beta = \mathcal{M}^\beta \cap \mathcal{M}_1, \quad \mathcal{M}_1 = \left\{ \mu \in \mathcal{M} : \int_{\mathbb{R}^d} \mu(dz) = 0 \right\}.$$

It follows that all point-wise increment measures  $\delta_x - \delta_0$ ,  $x \in \mathbb{R}^d$ , belong to  $\in \widetilde{\mathcal{M}}_\beta$ , and are hence admissible for evaluating the limit fields  $W_\beta$  and  $J_\beta$ . We introduce  $H = \frac{d-\beta}{2} \in (0, 1)$  and

$$B_H(x) = W_\beta(\delta_x - \delta_0), \quad P_H(x) = J_\beta(\delta_x - \delta_0), \quad x \in \mathbb{R}^d.$$

Let us mention that using representations  $\widetilde{W}_\beta$  and  $\widetilde{J}_\beta$  obtained in Theorem 3.11 one can prove that  $B_H(x) \xrightarrow{fdd} \widetilde{W}_\beta(x) - \widetilde{W}_\beta(0) = \widetilde{W}_\beta(x)$  since  $\widetilde{W}_\beta(0) = 0$  almost surely and similarly  $P_H(x) \xrightarrow{fdd} \widetilde{J}_\beta(x)$

Then  $B_H(x)$  has a zero mean Gaussian distribution for any  $x \in \mathbb{R}^d$ , and it is straightforward to check from (11) that, for any  $x, x' \in \mathbb{R}^d$ ,

$$\text{Cov}(B_H(x), B_H(x')) = (-c_\beta) (|x|^{2H} + |x'|^{2H} - |x - x'|^{2H}).$$

Thus,  $B_H$  is fractional Brownian field on  $\mathbb{R}^d$  with Hurst parameter  $H$ .

To analyze the properties of  $P_H$  we observe using (25),

$$(32) \quad \log \mathbb{E}(\exp(i P_H(x))) = \int_{\mathbb{R}^+ \times \mathbb{R}^d} \Psi(\delta_x(B(y, r)) - \delta_0(B(y, r))) C_\beta \, dy \, r^{-\beta-1} dr.$$

Here,

$$\delta_x(B(y, r)) - \delta_0(B(y, r)) = \begin{cases} 1, & |x - y| < r < |y| \\ -1, & |y| < r < |x - y| \\ 0 & \text{otherwise} \end{cases},$$

and hence we may recast (32) into

$$\begin{aligned} \log \mathbb{E}(\exp(i \theta P_H(x))) &= \Psi(\theta) \int_{\mathbb{R}^+ \times \mathbb{R}^d} \mathbf{1}_{\{|x-y| < r < |y|\}} C_\beta \, dy \, r^{-\beta-1} dr \\ &\quad + \Psi(-\theta) \int_{\mathbb{R}^+ \times \mathbb{R}^d} \mathbf{1}_{\{|y| < r < |x-y|\}} C_\beta \, dy \, r^{-\beta-1} dr \\ &= (-c_\beta) |x|^{d-\beta} (\Psi(\theta) + \Psi(-\theta)). \end{aligned}$$

This is the logarithmic characteristic functional of the difference of two independent random variables both having a Poisson distribution with intensity  $(-c_\beta)|x|^{d-\beta}$ . Hence,  $P_H(x)$ ,  $x \in \mathbb{R}^d$ , defines a mean zero integer-valued symmetrized Poisson distributed random field, such that for any  $x, x' \in \mathbb{R}^d$ ,

$$\text{Cov}(P_H(x), P_H(x')) = (-c_\beta) (|x|^{2H} + |x'|^{2H} - |x - x'|^{2H}) .$$

By analogy with fractional Brownian field this makes it natural to view  $P_H$  as a fractional Poisson field.

The fractional Poisson field  $P_H$  shares with  $B_H$  a well known interesting invariance property under restriction to lower-dimensional hyperplanes. For example, any cut along a line through a planar fractional field in  $\mathbb{R}^2$  generates a one-dimensional fractional process of the same kind. To see this, let  $H_k$  be a  $k$ -dimensional hyperplane in  $\mathbb{R}^d$ . We consider  $\mathbb{R}^d = H_k \oplus H_k^\perp$  and write  $\bar{x}_k$  for the restriction to  $H_k$  of  $x = \bar{x}_k + (x - \bar{x}_k) \in \mathbb{R}^d$ . To emphasize the dimensional dependence we write here  $B_{H,d}(x)$  and  $P_{H,d}(x)$  respectively, if the fractional fields are defined on  $\mathbb{R}^d$ .

**Proposition 4.3.** *Given  $\beta \in (d-2, d)$ , let  $\beta' = \beta - d + k \in (k-2, k)$ . Then the measure  $\delta_{\bar{x}_k} - \delta_0$  belongs to  $\widetilde{\mathcal{M}}_{\beta'}$ , and we have*

$$B_{H,d}(\bar{x}_k) \stackrel{fdd}{=} B_{H',k}(\bar{x}_k)$$

and

$$P_{H,d}(\bar{x}_k) \stackrel{fdd}{=} P_{H',k}(\bar{x}_k)$$

for  $H' = k - \beta' = d - \beta = H$ .

*Proof.* It is enough to consider hyperplanes of the form  $x = (x_1, \dots, x_k, 0, \dots, 0)$ . Then, clearly,  $|\bar{x}_k|^{d-\beta} = |\bar{x}_k|^{k-\beta'}$ , which carries over to showing that the covariances of the pair of relevant random fields coincide. □

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#### REFERENCES

- [1] Adler, R. J. *The Geometry of Random Field*, John Wiley & Sons, (1981).
- [2] Benassi, A. and Jaffard, S. and Roux, D., *Elliptic Gaussian random processes*, Rev. Matem. Iberoamericana, **13**, 1, 19–89, (1997).
- [3] Biermé, H.: *Champs aléatoires : autosimilarité, anisotropie et étude directionnelle*, PhD report, <http://www.math-info.univ-paris5.fr/~bierme/recherche/Thesehb.pdf>, (2005).
- [4] Biermé, H. and Estrade A.: *Poisson random balls: self-similarity and X-ray images*, Adv. Appl. Prob., **38**, 1–20, (2006).
- [5] Biermé, H., Estrade A. and Kaj I.: *About scaling behavior of random balls models*,  $S^4G$  6th Int. Conference, published by Union of Czech mathematicians and physicists, 63–68, (2006).
- [6] Chi, Z.: *Construction of stationary self-similar generalized fields by random wavelet expansion*, Probab. Theory Relat. Fields, **121**, 269–300, (2001).
- [7] Cohen, S. and Taqqu M.: *Small and large scale behavior of the Poissonized Telecom Process*, *Methodology and Computing in Applied Probability*, **6**, 363–379, (2004).
- [8] Dobrushin, R. L.: *Gaussian and their subordinated self-similar random generalized fields*, Ann. Probab., **7**(1), 1–28, (1979).
- [9] Guelfand, I. M. and Chilov, G. E. *Les Distributions I*, Dunod, (1962).

- [10] Guelfand, I. M. and N. Y. Vilenkin, *Les Distributions IV: Applications de l'Analyse Harmonique*, Dunod, (1967).
- [11] Guérin, C. A., *Wavelet analysis and covariance structure of non-stationary processes*, J. Fourier Anal. Appl., **6**, 403–425, (2000).
- [12] Herbin, E. *From N-parameter fractional Brownian motions to N-parameter multifractional Brownian motion*, Rocky Mountain J. of Math, **36**, n4, 1249–1284, (2006).
- [13] Kaj, I.: *Limiting fractal random processes in heavy-tailed systems*, In: *Fractals in Engineering, New Trends in Theory and Applications*, 199–218, Springer-Verlag London, (2005).
- [14] Kaj, I., Leskelä, L., Norros, I. and Schmidt, V. *Scaling limits for random fields with long-range dependence*, Ann. Probab. **35**, 528–550, (2007).
- [15] Kallenberg, O. *Foundations of Modern Probability. Second edition*, Springer-Verlag. (2002).
- [16] Matheron, G. *The intrinsic random functions and their applications*, Adv. Appl. Prob., **5**, 439–468, (1973).
- [17] Perrin, E., Harba, R., Berzin-Joseph, C., Iribarren, I. and Bonami, A., *nth-order fractional Brownian motion and fractional Gaussian noises*, IEEE Trans. Sign. Proc., **45**, 1049–1059, (2001).
- [18] Rudin, W. *Real and complex analysis*, Mc-Graw-Hill, (1966).
- [19] Yaglom, A. M. *Correlation Theory of Stationary and Related Random Functions (I)*, Springer-Verlag, (1997).

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