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On the Graf's addition theorem for Hahn Exton q -Bessel function

Lazhar Dhaouadi *

Abstract

It is the purpose of this note to give a new interpretation of the Graf's addition formula for Hahn Exton q -Bessel function using the properties of the q -Bessel Fourier transform. In the rest of this note we establish a connection between this result and the positivity of the q -Bessel translation operator.

1 Preliminaries

Throughout this paper we consider $0 < q < 1$ and we adopt the standard conventional notations of [2]. We put

$$\mathbb{R}_q^+ = \{q^n, \quad n \in \mathbb{Z}\},$$

and for complex a

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i), \quad n = 1 \dots \infty.$$

Jackson's q -integral (see [3]) in the interval $[0, \infty[$ is defined by

$$\int_0^\infty f(x) d_q x = (1 - q) \sum_{n=-\infty}^{\infty} q^n f(q^n).$$

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Let $\mathcal{L}_{q,p,v}$ be the space of even functions f defined on \mathbb{R}_q^+ such that

$$\|f\|_{q,p,v} = \left[\int_0^\infty |f(x)|^p x^{2v+1} d_q x \right]^{1/p} < \infty.$$

The q -Bessel function of third kind and of order v is defined by (see [6])

$$J_v(x, q) = \frac{(q^{v+1}; q)_\infty}{(q; q)_\infty} x^v {}_1\phi_1(0, q^{v+1}, q, qx^2), \quad \Re(v) > -1,$$

and its normalized form is given by

$$\begin{aligned} j_v(x, q) &= \frac{(q; q)_\infty}{(q^{v+1}; q)_\infty} x^{-v} J_v(x, q) = {}_1\phi_1(0, q^{v+1}, q, qx^2) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{n(n+1)}{2}}}{(q; q)_n (q^{v+1}; q)_n} x^{2n}. \end{aligned}$$

It's an entire analytic function in z . The q -Bessel Fourier transform $\mathcal{F}_{q,v}$ introduced in [1,4] as follow

$$\mathcal{F}_{q,v} f(x) = c_{q,v} \int_0^\infty f(t) j_v(xt, q^2) t^{2v+1} d_q t,$$

where

$$c_{q,v} = \frac{1}{1-q} \frac{(q^{2v+2}, q^2)_\infty}{(q^2, q^2)_\infty}.$$

The q -Bessel translation operator is defined as follows:

$$T_{q,x}^v f(y) = c_{q,v} \int_0^\infty \mathcal{F}_{q,v}(f)(t) j_v(xt, q^2) j_v(yt, q^2) t^{2v+1} d_q t, \quad \forall x, y \in \mathbb{R}_q^+, \forall f \in \mathcal{L}_{q,1,v}.$$

Recall that $T_{q,x}^v$ is said positive if $T_{q,x}^v f \geq 0$ for $f \geq 0$. In the following we put

$$Q_v = \{q \in]0, 1[, \quad T_{q,x}^v \text{ is positive for all } x \in \mathbb{R}_q^+\}.$$

Tack $q \in Q_v$ then the q -convolution product of both functions $f, g \in \mathcal{L}_{q,1,v}$ is defined by

$$f *_q g(x) = c_q \int_0^\infty T_{q,x}^v f(y) g(y) y^{2v+1} d_q y.$$

The followings results was proved in [1].

Proposition 1

$$|j_v(q^n, q^2)| \leq \frac{(-q^2; q^2)_\infty (-q^{2v+2}; q^2)_\infty}{(q^{2v+2}; q^2)_\infty} \begin{cases} 1 & \text{if } n \geq 0 \\ q^{n^2+(2v+1)n} & \text{if } n < 0 \end{cases}.$$

Theorem 1 *The operator $\mathcal{F}_{q,v}$ satisfying*

1. *For all functions $f \in \mathcal{L}_{q,1,v}$,*

$$\mathcal{F}_{q,v}^2 f(x) = f(x), \quad \forall x \in \mathbb{R}_q^+.$$

2. *If $f \in \mathcal{L}_{q,1,v}$, and $\mathcal{F}_{q,v} f \in \mathcal{L}_{q,1,v}$ then*

$$\|\mathcal{F}_{q,v} f\|_{q,v,2} = \|f\|_{q,v,2}.$$

3. *For all functions $f, g \in \mathcal{L}_{q,1,v}$,*

$$\mathcal{F}_{q,v}(f *_q g)(x) = \mathcal{F}_{q,v} f(x) \times \mathcal{F}_{q,v} g(x), \quad \forall x \in \mathbb{R}_q^+.$$

2 The Graf's addition formula

The Graf's addition formula for Hahn-Exton q -Bessel function is (see [5])

$$\begin{aligned} & J_v(Rq^{1/2(y+z+v)}, q) J_{x-v}(q^{1/2z}, q) \\ &= \sum_{k \in \mathbb{Z}} J_k(Rq^{1/2(x+y+k)}, q) J_{v+k}(Rq^{1/2(y+k+v)}, q) J_x(q^{1/2(z-k)}, q). \end{aligned}$$

This formula is valid for $z \in \mathbb{Z}$ and $R, x, y, v \in \mathbb{C}$ satisfying

$$|R|^2 q^{1+\Re(x)+\Re(y)} < 1, \quad \Re(x) > -1, \quad R \neq 0.$$

This formula has originally been derived for $v, x, y \in \mathbb{Z}$, $R > 0$ by Koelink using the interpretation of the Hahn-Exton q -Bessel function as matrix elements of irreducible unitary representation of the quantum group of plane motions. Now replace q by q^2 and R by q^r we obtain

$$\begin{aligned} & J_v(q^{y+z+v+r}, q^2) J_{x-v}(q^z, q^2) \\ &= \sum_{k \in \mathbb{Z}} J_k(q^{x+y+k+r}, q^2) J_{v+k}(q^{y+k+v+r}, q^2) J_x(q^{z-k}, q^2), \end{aligned}$$

and let

$$m = y + z + v + r,$$

then

$$\begin{aligned} x + y + k + r &= m + k + x - z - v, \\ y + k + v + r &= m + k - z, \end{aligned}$$

which give

$$\begin{aligned} &J_v(q^m, q^2)J_{x-v}(q^z, q^2) \\ &= \sum_{k \in \mathbb{Z}} J_k(q^{m+k+x-z-v}, q^2)J_{v+k}(q^{m+k-z}, q^2)J_x(q^{z-k}, q^2). \end{aligned}$$

This formula is valid for $z \in \mathbb{Z}$ and $r, x, y, v \in \mathbb{C}$ satisfying

$$1 + 2\Re(r) + \Re(x) + \Re(y) = 1 + \Re(r) + \Re(m) - \Re(z) - \Re(v) > 0, \quad \Re(x) > -1.$$

In the above sum we replace $z - k$ by k we get

$$\begin{aligned} &J_v(q^m, q^2)J_{x-v}(q^z, q^2) \\ &= \sum_{k \in \mathbb{Z}} J_{z-k}(q^{m+x-v-k}, q^2)J_{v+z-k}(q^{m-k}, q^2)J_x(q^k, q^2), \end{aligned}$$

This sum exist

$$\forall z \in \mathbb{Z}, \quad \forall m, v, x \in \mathbb{C}, \quad \Re(x) > -1.$$

In fact there exist an infinity complex number $r \in \mathbb{C}$ for which

$$1 + \Re(r) + \Re(m) - \Re(z) - \Re(v) > 0.$$

Now using the definition of the normalized q -Bessel function

$$J_x(q^k, q^2) = \frac{(q^{2x+2}, q^2)_\infty}{(q^2, q^2)_\infty} q^{vk} j_x(q^k, q^2) = (1-q)c_{q,x} q^{xk} j_x(q^k, q^2),$$

we obtain

$$\begin{aligned} &q^{mv+z(x-v)}(1-q)^2 c_{q,v} c_{q,x-v} j_v(q^m, q^2) j_{x-v}(q^z, q^2) \\ &= (1-q)c_{q,x} \sum_{k \in \mathbb{Z}} q^{xk} J_{z-k}(q^{x-v+m-k}, q^2) J_{v+z-k}(q^{m-k}, q^2) j_x(q^k, q^2). \end{aligned}$$

Let

$$\lambda = q^n, \quad n \in \mathbb{Z},$$

and replace

$$k \rightarrow k + n, m \rightarrow m + n, z \rightarrow z + n.$$

This implies

$$\begin{aligned} & q^{mv+z(x-v)}(1-q)^2 c_{q,v} c_{q,x-v} j_v(q^m \lambda, q^2) j_{x-v}(q^z \lambda, q^2) \\ &= (1-q) c_{q,x} \sum_{k \in \mathbb{Z}} q^{xk} J_{z-k}(q^{x-v+m-k}, q^2) J_{v+z-k}(q^{m-k}, q^2) j_x(q^k \lambda, q^2) \\ &= (1-q) c_{q,x} \sum_{k \in \mathbb{Z}} q^{2k(x+1)} q^{-k(x+2)} J_{z-k}(q^{x-v+m-k}, q^2) J_{v+z-k}(q^{m-k}, q^2) j_x(q^k \lambda, q^2). \end{aligned}$$

Now, put

$$E_{v,x}(q^m, q^z, q^k) = \frac{1}{(1-q)^2 c_{q,v} c_{q,x-v}} q^{-k(x+2)-mv-z(x-v)} J_{z-k}(q^{x-v+m-k}, q^2) J_{v+z-k}(q^{m-k}, q^2).$$

Using the Proposition 1, we see that the following function

$$\lambda \mapsto j_v(q^m \lambda, q^2) j_{x-v}(q^z \lambda, q^2), \quad \forall m, z \in \mathbb{Z},$$

belongs to the space $\mathcal{L}_{q,1,v}$, and by the inversion formula in Theorem 1, the following result hold true

Proposition 2 *For $z, m \in \mathbb{Z}$ and $x, v \in \mathbb{C}$ satisfying $\Re(x) > -1$ we have*

$$j_v(q^m \lambda, q^2) j_{x-v}(q^z \lambda, q^2) = c_{q,x} \int_0^\infty E_{v,x}(q^m, q^z, t) j_x(\lambda t, q^2) t^{2x+1} d_q t,$$

and

$$E_{v,x}(q^m, q^z, q^k) = c_{q,x} \int_0^\infty j_v(q^m \lambda, q^2) j_{x-v}(q^z \lambda, q^2) j_x(q^k \lambda, q^2) \lambda^{2x+1} d_q \lambda.$$

3 Application

In q -Fourier analysis the positivity of the q -Bessel translation operator is crucial. In general case, the fact that $f \in \mathcal{L}_{q,1,v}$ does not imply $T_{q,x}^v f \in \mathcal{L}_{q,1,v}$. So that it is natural to look for when this property holds for $T_{q,x}^v$.

In [1] the following result was proved

Proposition 3 *Let $f \in \mathcal{L}_{q,1,v}$ then*

$$T_{q,x}^v f(y) = \int_0^\infty f(z) D_v(x, y, z) z^{2v+1} d_q z,$$

where

$$D_v(x, y, z) = c_{q,v}^2 \int_0^\infty j_v(xt, q^2) j_v(yt, q^2) j_v(zt, q^2) t^{2v+1} d_q t.$$

A direct consequence of the above result is the fact that $T_{q,x}^v$ is positive if and only if $D_v(x, y, z) \geq 0$ for all $x, y, z \in \mathbb{R}_q^+$. Another consequence is the fact that if the operator $T_{q,x}$ is positive and $f \in \mathcal{L}_{q,1,v}$ then $T_{q,x} f \in \mathcal{L}_{q,1,v}$:

$$\begin{aligned} \int_0^\infty |T_{q,x}^v f(y)| y^{2v+1} d_q y &\leq \int_0^\infty T_{q,x}^v |f|(y) y^{2v+1} d_q y \\ &= \int_0^\infty |f(z)| \left[\int_0^\infty D_v(x, y, z) y^{2v+1} d_q y \right] z^{2v+1} d_q z \end{aligned}$$

Let

$$\phi : t \mapsto j_v(xt, q^2) j_v(zt, q^2),$$

then

$$D_v(x, y, z) = c_{q,v} \mathcal{F}_{q,v} \phi(y).$$

This implies with the inversion formula in Theorem 1

$$\int_0^\infty D_v(x, y, z) y^{2v+1} d_q y = \mathcal{F}_{q,v}^2 \phi(0) = \phi(0) = 1.$$

Finally we prove that

$$f \in \mathcal{L}_{q,1,v} \Rightarrow T_{q,x}^v f \in \mathcal{L}_{q,1,v}.$$

In [1] it also proved that

$$\begin{aligned} D_{-1/2}(q^m, q^r, q^k) &= \frac{q^{2(r-m)(k-m)-m}}{(1-q)(q; q)_\infty} (q^{2(r-m)+1}; q)_\infty \phi_1(0, q^{2(r-m)+1}, q; q^{2(k-m)+1}) \\ &= \frac{1}{1-q} q^{-m} J_{2(r-m)}(q^{k-m}, q), \end{aligned}$$

which implies (see [1])

$$Q_{-1/2} = [0, q_0],$$

where q_0 is the first zero of the following function:

$$q \mapsto {}_1\phi_1(0, q, q, q).$$

Now using the Proposition 2 for $x = v = 0$ we obtain

$$\begin{aligned} E_{0,0}(q^m, q^z, q^k) &= c_{q,0} \int_0^\infty j_0(q^m \lambda, q^2) j_0(q^z \lambda, q^2) j_0(q^k \lambda, q^2) \lambda d_q \lambda \\ &= \frac{1}{c_{q,0}} D_0(q^m, q^z, q^k), \end{aligned}$$

and then

$$\begin{aligned} D_0(q^m, q^z, q^k) &= c_{q,0} E_{0,0}(q^m, q^z, q^k) \\ &= \frac{1}{(1-q)} q^{-2k} [J_{z-k}(q^{m-k}, q^2)]^2, \end{aligned}$$

In the end we obtain

$$Q_0 =]0, 1[.$$

Proposition 4 For $z, m \in \mathbb{Z}$ and $v \in \mathbb{C}$ satisfying $\Re(v) > -1$ we have

$$E_{v,v}(q^m, q^z, q^k) = (1-q) \sum_{i=0}^{\infty} \frac{(q^{-2v}, q^2)_i}{(q^2, q^2)_i} q^{i(2+2v)} D_v(q^m, q^{i+z}, q^k).$$

Proof. From the following formula

$$J_{x-v}(\lambda, q^2) = \lambda^{-v} \sum_{i=0}^{\infty} \frac{(q^{-2v}, q^2)_i}{(q^2, q^2)_i} q^{i(2+x)} J_x(\lambda q^i, q^2),$$

we obtain

$$(1-q) c_{q,x-v} j_{x-v}(\lambda, q^2) = (1-q) c_{q,x} \sum_{i=0}^{\infty} \frac{(q^{-2v}, q^2)_i}{(q^2, q^2)_i} q^{i(2+2x)} j_x(\lambda q^i, q^2).$$

Tack $x = v$ and replace λ by $q^z \lambda$ we obtain

$$j_0(q^z \lambda, q^2) = (1-q) c_{q,v} \sum_{i=0}^{\infty} \frac{(q^{-2v}, q^2)_i}{(q^2, q^2)_i} q^{i(2+2v)} j_v(\lambda q^{i+z}, q^2),$$

which gives from Proposition 2

$$\begin{aligned}
E_{v,v}(q^m, q^z, q^k) &= c_{q,v} \int_0^\infty j_v(q^m \lambda, q^2) j_0(q^z \lambda, q^2) j_v(q^k \lambda, q^2) \lambda^{2v+1} d_q \lambda \\
&= (1-q) \sum_{i=0}^\infty \frac{(q^{-2v}, q^2)_i}{(q^2, q^2)_i} q^{i(2+2v)} \\
&\quad \times \left[c_{q,v}^2 \int_0^\infty j_v(q^m \lambda, q^2) j_v(q^{i+z} \lambda, q^2) j_v(q^k \lambda, q^2) \lambda^{2v+1} d_q \lambda \right] \\
&= (1-q) \sum_{i=0}^\infty \frac{(q^{-2v}, q^2)_i}{(q^2, q^2)_i} q^{i(2+2v)} D_v(q^m, q^{i+z}, q^k).
\end{aligned}$$

This finish the proof. ■

Remark 1 If $v \in \mathbb{N}$ then for all $i \geq v+1$ we have

$$(q^{-2v}, q^2)_i = 0,$$

which implies

$$E_{v,v}(q^m, q^z, q^k) = (1-q) \sum_{i=0}^v \frac{(q^{-2v}, q^2)_i}{(q^2, q^2)_i} q^{i(2+2v)} D_v(q^m, q^{i+z}, q^k).$$

Remark 2 For $m, n, k \in \mathbb{Z}$ and $\Re(v) > -1$ we have

$$\begin{aligned}
E_{v,v}(q^m, q^z, q^k) &= \frac{1}{(1-q)c_{q,v}} q^{-k(v+2)-mv} J_{z-k}(q^{m-k}, q^2) J_{v+z-k}(q^{m-k}, q^2) \\
&= (1-q) \sum_{i=0}^\infty \frac{(q^{-2v}, q^2)_i}{(q^2, q^2)_i} q^{i(2+2v)} D_v(q^m, q^{i+z}, q^k).
\end{aligned}$$

In particular

$$E_{v,v}(1, 1, 1) = \frac{1}{(1-q)c_{q,v}} J_0(1, q^2) J_v(1, q^2).$$

Therefore if $\Re(v) > -1$ and $v \neq 0$ then there exist $q \in]0, 1[$ such that

$$J_0(1, q^2) J_v(1, q^2) < 0,$$

this implies

$$Q_v \not\subseteq]0, 1[.$$

On the other hand if $q \rightarrow 0^+$ then for all $m, z, k \in \mathbb{Z}$

$$0 < E_{v,v}(q^m, q^z, q^k) \sim D_v(q^m, q^z, q^k),$$

therefore

$$Q_v \neq \emptyset.$$

References

- [1] L. Dhaouadi, A. Fitouhi and J. El Kamel, Inequalities in q-Fourier Analysis, Journal of Inequalities in Pure and Applied Mathematics, Volume 7, Issue 5, Article 171, 2006.
- [2] G. Gasper and M. Rahman, Basic hypergeometric series, Encyclopedia of mathematics and its applications 35, Cambridge university press, 1990.
- [3] F. H. Jackson, On a q-Definite Integrals, Quarterly Journal of Pure and Application Mathematics 41, 1910, 193-203.
- [4] T. H. Koornwinder and R. F. Swarttouw, On q-Analogues of the Hankel and Fourier Transform, Trans. A. M. S. 1992, 333, 445-461.
- [5] H. T. Koelink and R. F. Swarttouw , A q-Analogue of Graf's Addition Formula for the Hahn-Exton q-Bessel Function, Journal of Approximation theory 81. 260-273(1995).
- [6] R. F. Swarttouw, The Hahn-Exton q-Bessel functions PhD Thesis The Technical University of Delft ((1992).