Heisenberg Uncertainty Principle for the q-Bessel Fourier transform

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Abstract

In this paper we uses an I.I. Hirschman-W. Beckner entropy argument to give an uncertainty inequality for the q-Bessel Fourier transform:

$$\mathcal{F}_{q,v}f(x) = c_{q,v}\int_0^\infty f(t)j_v(xt,q^2)t^{2v+1}d_qt,$$

where $j_v(x,q)$ is the normalized Hahn-Exton q-Bessel function.

1 Introduction

I.I.Hirschman-W. Beckner entropy argument is one further variant of Heisenbergs uncertainty principle.

Let $\widehat{f}$ be the Fourier transform of f defined by

$$\widehat{f}(x) = \int_{-\infty}^{\infty} f(y)e^{2i\pi xy}f(y)dy, \quad x \in \mathbb{R}.$$

If $f \in L^2(\mathbb{R})$ with L^2-norme $\|f\|_2 = 1$, then by Plancherel's theorem $\|\widehat{f}\|_2 = 1$, so that $|f(x)|^2$ and $|\widehat{f}(x)|^2$ are probability frequency functions. The variance of a probability frequency g is defined by

$$V[g] = \int_{\mathbb{R}} x^2 g(x)dx.$$

The Heisenberg uncertainty principle can be stated as follows

$$V[|f|^2]V[|\widehat{f}|^2] \geq \frac{1}{16\pi^2}. \tag{1}$$

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If g is a probability frequency function, then the entropy of g is defined by

$$E(g) = \int_{\mathbb{R}} g(x)\log(x)dx.$$

With f as above, Hirschman [10] proved that

$$E(|f|^2) + E(|\widehat{f}|^2) \leq 0. \tag{2}$$

By an inequality of Shannon and Weaver it follows that (2) implies (1). Using the Babenko-Beckner inequality

$$\|\widehat{f}\|_{p'} \leq A(p)\|f\|_p, \quad 1 < p < 2, \quad A(p) = \left[p^{1/p}(p')^{-1/p'}\right]^{1/2},$$

in Hirschman's proof of (2) another uncertainty inequality is deduced. For more detail the reader can consult [8,10,11].

In this paper we use I.I. Hirschman entropy argument de give an uncertainty inequality for the q-Bessel Fourier transform (also called q-Hankel transform).

Note that other versions of the Heisenberg uncertainty principle for the q-Fourier transform have recently appeared in the literature [1,2,6]. There are some differences of the results cited above and our result:

• In [1] the uncertainty inequality is established for the q-cosine and q-sine transform but here is established for the q-Bessel transform.

• In [2] the uncertainty inequality is for the q^2-Fourier transform but here is for the q-Hankel transform.

• In [6] the uncertainty inequality is established for functions in q-Schwartz space. In this paper the uncertainty inequality is established for functions in $\mathcal{L}_{q,2,v}$ space.

The inequality discuss here is a quantitative uncertainty principles which give an information about how a function and its q-Bessel Fourier transform relate. A qualitative uncertainty principles give an information about how a function (and its Fourier transform) behave under certain circumstances. A classical qualitative uncertainty principle called Hardy's theorem. In [4,7] a q-version of the Hardy's theorem for the q-Bessel Fourier transform was established.

In the end, our objective is to develop a coherent harmonic analysis attached to the q-Bessel operator

$$\Delta_{q,v}f(x) = \frac{1}{x^2}\left[f(q^{-1}x) - (1+q^{2v})f(x) + q^{2v}f(qx)\right].$$

Thus, this paper is an opportunity to implement the arguments of the q-Bessel Fourier analysis proved before, as the Plancherel formula, the positivity of the q-translation operator, the q-convolution product, the q-Gauss kernel...

2 The q-Bessel Fourier transform

In the following we will always assume $0 < q < 1$ and $v > -1$. We denote by

$$\mathbb{R}_q = \{\pm q^n, \quad n \in \mathbb{Z}\}, \qquad \mathbb{R}_q^+ = \{q^n, \quad n \in \mathbb{Z}\}.$$

For more informations on the q-series theory the reader can see the references [9,12,14] and the references [3,5,13] about the q-bessel Fourier analysis. Also for details of the proofs of the following results in this section can be fond in [3].

Definition 1 *The q-Bessel operator is defined as follows*

$$\Delta_{q,v} f(x) = \frac{1}{x^2}\left[f(q^{-1}x) - (1+q^{2v})f(x) + q^{2v} f(qx)\right].$$

Definition 2 *The normalized q-Bessel function of Hahn-Exton is defined by*

$$j_v(x, q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^{2v+2}, q^2)_n (q^2, q^2)_n} x^{2n}.$$

Proposition 1 *The function*

$$x \mapsto j_v(\lambda x, q^2)$$

is the eigenfunction of the operator $\Delta_{q,v}$ associated with the eigenvalue $-\lambda^2$.

Definition 3 *The q-Jackson integral of a function f defined on $\mathbb{R}_q$ is*

$$\int_0^{\infty} f(t) d_q t = (1-q) \sum_{n \in \mathbb{Z}} q^n f(q^n).$$

Definition 4 *We denote by $\mathcal{L}_{q,p,v}$ the space of even functions f defined on $\mathbb{R}_q$ such that*

$$\|f\|_{q,p,v} = \left[\int_0^{\infty} |f(x)|^p x^{2v+1} d_q x\right]^{1/p} < \infty.$$

Definition 5 *We denote by $\mathcal{C}_{q,0}$ the space of even functions defined on $\mathbb{R}_q$ tending to 0 as $x \to \pm\infty$ and continuous at 0 equipped with the topology of uniform convergence. The space $\mathcal{C}_{q,0}$ is complete with respect to the norm*

$$\|f\|_{q,\infty} = \sup_{x\in\mathbb{R}_q} |f(x)|.$$

Definition 6 *The q-Bessel Fourier transform $\mathcal{F}_{q,v}$ (also called q-Hankel transform) is defined by*

$$\mathcal{F}_{q,v}f(x) = c_{q,v}\int_0^\infty f(t)j_v(xt,q^2)t^{2v+1}d_qt, \qquad \forall x \in \mathbb{R}_q.$$

where

$$c_{q,v} = \frac{1}{1-q}\frac{(q^{2v+2};q^2)_\infty}{(q^2;q^2)_\infty}.$$

Proposition 2 *Let $f \in \mathcal{L}_{q,1,v}$ then $\mathcal{F}_{q,v}f$ existe and $\mathcal{F}_{q,v}f \in \mathcal{C}_{q,0}$.*

Definition 7 *The q-translation operator is given as follows*

$$T^v_{q,x}f(y) = c_{q,v}\int_0^\infty \mathcal{F}_{q,v}f(t)j_v(yt,q^2)j_v(xt,q^2)t^{2v+1}d_qt \quad \forall f \in \mathcal{L}_{q,1,v}.$$

Definition 8 *The operator $T^v_{q,x}$ is said positive if $T^v_{q,x}f \geq 0$ when $f \geq 0$ for all $x \in \mathbb{R}_q$. We denote by Q_v the domain of positivity of $T^v_{q,x}$*

$$Q_v = \left\{q \in]0,1[, \qquad T^v_{q,x} \text{ is positive}\right\}.$$

In the following we assume that $q \in Q_v$.

Proposition 3 *If $f \in \mathcal{L}_{q,1,v}$ then*

$$\int_0^\infty T^v_{q,x}f(y)y^{2v+1}d_qy = \int_0^\infty f(y)y^{2v+1}d_qy.$$

Definition 9 *The q-convolution product is defined as follows*

$$f *_q g(x) = c_{q,v}\int_0^\infty T^v_{q,x}f(y)g(y)y^{2v+1}d_qy.$$

Proposition 4 *Let $f, g \in \mathcal{L}_{q,1,v}$ then $f *_q g \in \mathcal{L}_{q,1,v}$ and we have*

$$\mathcal{F}_{q,v}(f *_q g) = \mathcal{F}_{q,v}(g) \times \mathcal{F}_{q,v}(f).$$

Proposition 5 *Let $f \in \mathcal{L}_{q,1,v}$ and $g \in \mathcal{L}_{q,2,v}$ then $f *_q g \in \mathcal{L}_{q,2,v}$ and we have*

$$\mathcal{F}_{q,v}(f *_q g) = \mathcal{F}_{q,v}(f) \times \mathcal{F}_{q,v}(g).$$

Theorem 1 *The q-Bessel Fourier transform $\mathcal{F}_{q,v}$ satisfies*

1. *$\mathcal{F}_{q,v}$ sends $\mathcal{L}_{q,2,v}$ to $\mathcal{L}_{q,2,v}$.*
2. *For $f \in \mathcal{L}_{q,2,v}$, we have*

$$\|\mathcal{F}_{q,v}(f)\|_{q,2,v} = \|f\|_{q,2,v}.$$

3. *The operator $\mathcal{F}_{q,v} : \mathcal{L}_{q,2,v} \to \mathcal{L}_{q,2,v}$ is bijective and*

$$\mathcal{F}_{q,v}^{-1} = \mathcal{F}_{q,v}.$$

Proposition 6 *Given $1 < p \leq 2$ and $\frac{1}{p} + \frac{1}{\overline{p}} = 1$. If $f \in \mathcal{L}_{q,p,v}$ then*

$$\mathcal{F}_{q,v}(f) \in \mathcal{L}_{\overline{p},2,v}$$

and

$$\|\mathcal{F}_{q,v}(f)\|_{q,\overline{p},v} \leq B_{q,v}^{(\frac{2}{p}-1)} \|f\|_{q,p,v},$$

where

$$B_{q,v} = \frac{1}{1-q} \frac{(-q^2;q^2)_\infty (-q^{2v+2};q^2)_\infty}{(q^2;q^2)_\infty}.$$

Definition 10 *The q-exponential function is defined by*

$$e(z,q) = \sum_{n=0}^{\infty} \frac{z^n}{(q,q)_n} = \frac{1}{(z;q)_\infty}, \quad |z| < 1.$$

Proposition 7 *The q-Gauss kernel*

$$G^v(x,t^2,q^2) = \frac{(-q^{2v+2}t^2, -q^{-2v}/t^2;q^2)_\infty}{(-t^2,-q^2/t^2;q^2)_\infty} e\Big(-\frac{q^{-2v}}{t^2}x^2, q^2\Big), \quad \forall x,t \in \mathbb{R}_q^+$$

satisfies

$$\mathcal{F}_{q,v}\left\{e(-t^2y^2,q^2)\right\}(x) = G^v(x,t^2,q^2),$$

and for all function $f \in \mathcal{L}_{q,2,v}$

$$\lim_{n\to\infty} \|G^v(x,q^{2n},q^2) *_q f - f\|_{q,2,v} = 0.$$

3 Uncertainty Principle

The following Lemma are crucial for the proof of our main result. First we enunciate the Jensens inequality

Lemma 1 *Let γ be a probability measure on $\mathbb{R}_q^+$. Let g be a convex function on a subset I of $\mathbb{R}$. If $\psi : \mathbb{R}_q^+ \to I$ satisfies*

$$\int_0^\infty \psi(u) d\gamma(u) \in I,$$

then we have

$$g\left(\int_0^\infty \psi(x) d\gamma(x)\right) \leq \int_0^\infty g \circ \psi(x) d\gamma(x).$$

Proof. Let

$$t = \int_0^\infty \psi(u) d\gamma(u).$$

There exist $c \in \mathbb{R}$ such that for all $y \in I$ it holds

$$g(y) \geq g(t) + c(y - t).$$

Now let $y = \psi(x)$ we obtain

$$g\left(\psi(x)\right) \geq g(t) + c(\psi(x) - t).$$

Integrating both sides and using the special value of t gives

$$\int_0^\infty g\left(\psi(x)\right) d\gamma(x) \geq \int_0^\infty [g(t) + c(\psi(x) - t)] d\gamma(x) = g(t).$$

This finish the proof. ■

Lemma 2 *Let f be an even function defined on $\mathbb{R}_q$. Assume $\psi : \mathbb{R} \to \mathbb{R}_+$ is a convexe function and $\psi \circ f \in \mathcal{L}_{q,1,v}$. If ϱ_n is a sequence of non-negative function in $\mathcal{L}_{q,1,v}$ such that*

$$\mathcal{F}_{q,v}(\varrho_n)(0) = c_{q,v} \int_0^\infty \varrho_n(x) x^{2v+1} d_q x = 1$$

and $\varrho_n *_q f \to f$ then $\psi \circ \Big(\varrho_n *_q f\Big)$ is in $\mathcal{L}_{q,1,v}$ and

$$\lim_{n\to\infty} \int_0^\infty \psi \circ \Big(\varrho_n *_q f\Big)(x) x^{2v+1} d_q x = \int_0^\infty \psi \circ f(x) x^{2v+1} d_q x.$$

Proof. For a given x and by Proposition 3 we have

$$c_{q,v}\int_0^\infty T^v_{q,x}\varrho_n(y)y^{2v+1}d_qy = 1$$

From the positivity of $T^v_{q,x}$ we see that

$$c_{q,v}T^v_{q,x}\varrho_n(y)y^{2v+1}d_qy$$

is a probability measure on $\mathbb{R}^+_q$. The following holds by Jensens Inequality

$$\begin{aligned}\psi\circ\Big(\varrho_n *_q f\Big)(x) &= \psi\left[c_{q,v}\int_0^\infty f(y)T^v_{q,x}\varrho_n(y)y^{2v+1}d_qy\right]\\ &\le c_{q,v}\int_0^\infty \psi\circ f(y)T^v_{q,x}\varrho_n(y)y^{2v+1}d_qy\\ &= \varrho_n *_q \psi\circ f(x).\end{aligned}$$

By the use of the Fatou's Lemma and Proposition 4 we obtain

$$\begin{aligned}&\int_0^\infty \psi\circ f(x)x^{2v+1}d_qx\\ &= \int_0^\infty \liminf_{n\to\infty}\psi\circ\Big(\varrho_n *_q f\Big)(x)x^{2v+1}d_qx\\ &\le \liminf_{n\to\infty}\int_0^\infty \psi\circ\Big(\varrho_n *_q f\Big)(x)x^{2v+1}d_qx\\ &\le \limsup_{n\to\infty}\int_0^\infty \psi\circ\Big(\varrho_n *_q f\Big)(x)x^{2v+1}d_qx\\ &\le \lim_{n\to\infty}\int_0^\infty \varrho_n *_q \psi\circ f(x)x^{2v+1}d_qx\\ &= \frac{1}{c_{q,v}}\lim_{n\to\infty}\mathcal{F}_{q,v}(\varrho_n)(0)\times\mathcal{F}_{q,v}\Big(\psi\circ f\Big)(0)\\ &= \int_0^\infty \psi\circ f(x)x^{2v+1}d_qx.\end{aligned}$$

This finish the proof. ■

Definition 11 *For a positive function ϕ define the entropy of ϕ to be*

$$E(\phi) = \int_0^\infty \phi(x)\log\phi(x)x^{2v+1}d_qx.$$

$E(\phi)$ can have values in $[-\infty,\infty]$.

Remark 1 *For a given $c \in \mathbb{R}_q^+$ let*

$$d\gamma(x) = k_c^{-1} \exp\left(-|cx|^a\right) x^{2v+1} d_q x$$

where

$$\sigma_a = \int_0^\infty \exp\left(-|x|^a\right) x^{2v+1} d_q x, \quad k_c = \frac{\sigma_a}{c^{2v+2}}.$$

Then $d\gamma(x)$ is a probability measure on $\mathbb{R}_q^+$.

Lemma 3 *Let $a > 0$. For a positive function $\phi \in \mathcal{L}_{q,1,v}$ such that*

$$\|\phi\|_{q,1,v} = 1$$

and

$$M_a(\phi) = \left(\int_0^\infty |x|^a \, \phi(x) x^{2v+1} d_q x \right)^{\frac{1}{a}}$$

is finite, we have

$$-E(\phi) \leq \log k_c + c^a M_a^a(\phi). \tag{3}$$

Proof. Indeed, defining

$$\psi(x) = k_c \exp\left(|cx|^a\right) \phi(x),$$

From Remark 1 we see that

$$\int_0^\infty \psi(x) d\gamma(x) = 1.$$

According to the fact that $g : t \mapsto t \log t$ is convex on $\mathbb{R}_+^*$, so Jensen's inequality gives

$$g\left[\int_0^\infty \psi(x) d\gamma(x) \right] \leq \int_0^\infty g \circ \psi(x) d\gamma(x).$$

Hence,

$$0 = \left[\int_0^\infty \psi(x) d\gamma(x) \right] \log \left[\int_0^\infty \psi(x) d\gamma(x) \right] \leq \int_0^\infty \psi(x) \log \psi(x) d\gamma(x).$$

This implies

$$\begin{aligned} 0 &\leq \int_0^\infty \phi(x) \log \left[k_c \exp\left(|cx|^a\right) \phi(x) \right] x^{2v+1} d_q x \\ &= \int_0^\infty \phi(x) \left[\log k_c + |cx|^a + \log \phi(x) \right] x^{2v+1} d_q x. \end{aligned}$$

$$0 \leq \log k_c + c^a \int_0^\infty |x|^a \, \phi(x) x^{2v+1} d_q x + \int_0^\infty \phi(x) \log \phi(x) x^{2v+1} d_q x.$$

In the end

$$0 \leq \log k_c + c^a M_a^a(\phi) + E(\phi).$$

This finish the proof. ■

Lemma 4 *Let $f \in \mathcal{L}_{q,1,v} \cap \mathcal{L}_{q,2,v}$ then we have*

$$E\left(|f|^2\right) + E\left(|\mathcal{F}_{q,v} f|^2\right) \leq 2\|f\|_{q,v,2}^2 \log\left(B_{q,v}\|f\|_{q,v,2}^2\right). \tag{4}$$

Proof. Hölder inequality implies that f will be in $\mathcal{L}_{q,p,v}$ for $1 < p \leq 2$. With

$$\frac{1}{p} + \frac{1}{\overline{p}} = 1,$$

Hausdorff-Young's inequality (Proposition 6) tells us that $\mathcal{F}_{q,v} f$ is in $\mathcal{L}_{q,\overline{p},v}$. So we can define the functions

$$A(p) = \int_0^\infty |f(x)|^p \, d_q x \quad \text{and } B(\overline{p}) = \int_0^\infty |\mathcal{F}_{q,v} f(x)|^{\overline{p}} \, x^{2v+1} d_q x.$$

Now define

$$\begin{aligned} C(p) &= \log \|\mathcal{F}_{q,v} f\|_{q,\overline{p},v} - \log\left(B_{q,v}^{\frac{2}{p}-1} \|f\|_{q,p,v}\right) \\ &= \frac{1}{\overline{p}} \log B(\overline{p}) - \frac{1}{p} \log A(p) - \left(\frac{2}{p} - 1\right) \log B_{q,v}. \end{aligned}$$

By Hausdorff-Young's inequality

$$C(p) \leq 0, \text{ for } 1 < p < 2,$$

and by Plancherel equality (Theorem 1 part 2)

$$C(2) = 0.$$

Then

$$C'(2^-) \geq 0.$$

On the other hand for $1 < p < 2$ we have

$$C'(p) = \frac{\overline{p}'}{\overline{p}} \frac{B'(\overline{p})}{B(\overline{p})} - \frac{\overline{p}'}{\overline{p}^2} \log B(\overline{p}) - \frac{1}{p} \frac{A'(p)}{A(p)} + \frac{1}{p^2} \log A(p) + \frac{2}{p^2} \log B_{q,v}.$$

The derivative of $\overline{p}$ with respect to p is

$$\overline{p}' = -\frac{1}{(p-1)^2}.$$

For a given $x > 0$ we have

$$\lim_{p\to 2} \frac{x^p - x^2}{p-2} = x^2 \log x.$$

Then

$$A'(2^-) = \lim_{p\to 2^-} \frac{A(p) - A(2)}{p-2} = \frac{1}{2} E\left(|f|^2\right),$$

$$B'(2^+) = \lim_{\overline{p}\to 2^+} \frac{B(\overline{p}) - B(2)}{\overline{p}-2} = \frac{1}{2} E\left(|\mathcal{F}_{q,v} f|^2\right).$$

Since

$$p \mapsto \frac{x^p - x^2}{p-2}$$

is an increasing function, the exchange of the signs limit and integral is valid sense. On the other hand

$$\lim_{p\to 2^-} A(p) = \|f\|_{q,v,2}^2, \quad \lim_{\overline{p}\to 2^+} B(p) = \|\mathcal{F}_{q,v} f\|_{q,v,2}^2 = \|f\|_{q,v,2}^2.$$

So

$$C'(2^-) = \lim_{p\to 2^-} \frac{C(p) - C(2)}{p-2} = -\frac{1}{2\|f\|_{q,v,2}^2} \left[A'(2^-) + B'(2^+)\right] + \frac{1}{2} \log\left(B_{q,v} \|f\|_{q,v,2}^2\right).$$

Therefore

$$A'(2^-) + B'(2^+) - \|f\|_{q,v,2}^2 \log\left(B_{q,v} \|f\|_{q,v,2}^2\right) \leq 0,$$

and then

$$E\left(|f|^2\right) + E\left(|\mathcal{F}_{q,v} f|^2\right) \leq 2\|f\|_{q,v,2}^2 \log\left(B_{q,v} \|f\|_{q,v,2}^2\right).$$

This finish the proof. ■

Lemma 5 *Let $f \in \mathcal{L}_{q,2,v}$ then we have*

$$E\left(|f|^2\right) + E\left(|\mathcal{F}_{q,v} f|^2\right) \leq 2\|f\|_{q,v,2}^2 \log\left(B_{q,v} \|f\|_{q,v,2}^2\right). \tag{5}$$

Proof. Assume that $E(|f|^2)$ and $E(|\mathcal{F}_{q,v}f|^2)$ are defined and then approximate f by functions in $\mathcal{L}_{q,1,v} \cap \mathcal{L}_{q,2,v}$. Let

$$h_n(x) = e(-q^{2n}x^2, q^2).$$

The function h_n is in $\mathcal{L}_{q,2,v}$ then $h_n f \in \mathcal{L}_{q,1,v}$. On the other hand $h_n \in \mathcal{C}_{q,0}$ then $h_n f \in \mathcal{L}_{q,2,v}$. We obtain

$$h_n f \in \mathcal{L}_{q,1,v} \cap \mathcal{L}_{q,2,v}.$$

The following holds by (2)

$$E\left(|h_n f|^2\right) + E\left(|\mathcal{F}_{q,v}(h_n f)|^2\right) \leq 2\|h_n f\|_{q,2,v}^2 \log\left(B_{q,v}\|h_n f\|_{q,2,v}^2\right). \quad (6)$$

One can see by the Lebesgue Dominated Convergence Theorem that

$$\lim_{n\to\infty} \|h_n f\|_{q,2,v} = \|f\|_{q,2,v} \quad (7)$$

and

$$\lim_{n\to\infty} E\left(|h_n f|^2\right) = E\left(|f|^2\right). \quad (8)$$

By the use of Proposition 5 and the inversion formula (Theorem 1 part 3) we see that

$$\mathcal{F}_{q,v}(h_n f) = \mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f.$$

We will prove that

$$\lim_{n\to\infty} E\left(|\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f|^2\right) = E\left(|\mathcal{F}_{q,v}f|^2\right).$$

The functions

$$\phi_1(x) = x^2 \log^+|x| \text{ and } \phi_2(x) = x^2\left(-\log^-|x| + \frac{3}{2}\right),$$

are convex on $\mathbb{R}$, where

$$\log^+ x = \max\{0, \log x\} \text{ and } \log^- x = \min\{0, \log x\}.$$

Note that

$$2\phi_1(x) - 2\phi_2(x) + 3x^2 = x^2 \log|x|^2.$$

Since

- From the inversion formula we see that

$$c_{q,v}\int_0^\infty \mathcal{F}_{q,v}h_n(t)t^{2v+1}d_q t = h_n(0) = 1.$$

- The function $\mathcal{F}_{q,v}h_n \geq 0$.
- The functions ϕ_i are convex on $\mathbb{R}$.
- $E(\mathcal{F}_{q,v}f)$ is finite then $\phi_i(\mathcal{F}_{q,v}f)$ is in $\mathcal{L}_{q,1,v}$.
- From Proposition 7 we have

$$\lim_{n\to\infty} \mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f(x) = \mathcal{F}_{q,v}f(x)$$

we deduce that $\mathcal{F}_{q,v}h_n$ and ϕ_i satisfy the conditions of Lemma 2. Then we obtain

$$\lim_{n\to\infty} \int_0^\infty \phi_i \circ (\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f)(x) x^{2v+1} d_q x = \int_0^\infty \phi_i \circ (\mathcal{F}_{q,v}f)(x) x^{2v+1} d_q x, \quad i = 1, 2.$$

It also hold

$$\begin{aligned} E\left(|\mathcal{F}_{q,v}f|^2\right) &= 2\int_0^\infty \phi_1\left(\mathcal{F}_{q,v}f\right) x^{2v+1} d_q \\ &\quad - 2\int_0^\infty \phi_2\left(\mathcal{F}_{q,v}f\right) x^{2v+1} d_q x + 3\left\|\mathcal{F}_{q,v}f\right\|_{q,2,v}^2, \end{aligned}$$

and

$$\begin{aligned} E\left(|\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f|^2\right) &= 2\int_0^\infty \phi_1\left(\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f\right) x^{2v+1} d_q x \\ &\quad - 2\int_0^\infty \phi_2\left(\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f\right) x^{2v+1} d_q x \\ &\quad + 3\left\|\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f\right\|_{q,2,v}^2. \end{aligned}$$

Then

$$\lim_{n\to\infty} E\left(|\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f|^2\right) = E\left(|\mathcal{F}_{q,v}f|^2\right). \tag{9}$$

With (6) and the limits (7), (8) and (9) we complete the proof of (5).

Note that these limits also hold in the case where $E(|f|^2)$ and $E(|\mathcal{F}_{q,v}f|^2)$ are ∞ or $-\infty$. ■

Now we are in position to state and prove the uncertainty inequality for the q-Bessel Fourier transform.

Theorem 2 *Given $a, b > 0$. Then for all $c, d \in \mathbb{R}_q^+$ satisfying*

$$0 < B_{q,v}^2 \frac{\sigma_a \sigma_b}{(cd)^{2v+2}} < 1,$$

the following hold for any function $f \in \mathcal{L}_{q,2,v}$

$$c^a \left\|x^{a/2} f\right\|_{q,2,v}^2 + d^b \left\|x^{b/2} \mathcal{F}_{q,v}f\right\|_{q,2,v}^2 \geq -\log\left(B_{q,v}^2 \frac{\sigma_a \sigma_b}{(cd)^{2v+2}}\right) \|f\|_{q,2,v}^2.$$

Proof. Assume that $\|f\|_{q,2,v} = 1$. By (3) we can write

$$-E(|f|^2) \leq \log k_c + c^a \left\| x^{a/2} f \right\|_{q,2,v}^2$$

$$-E\left(|\mathcal{F}_{q,v} f|^2\right) \leq \log k_d + d^b \left\| x^{b/2} \mathcal{F}_{q,v} f \right\|_{q,2,v}^2 .$$

Which implies with (5)

$$\begin{aligned} -2\log B_{q,v} &\leq -E\left(|f|^2\right) - E\left(|\mathcal{F}_{q,v} f|^2\right) \\ &\leq \log\left(k_c k_d\right) + c^a \left\| x^{a/2} f \right\|_{q,2,v}^2 + d^b \left\| x^{b/2} \mathcal{F}_{q,v} f \right\|_{q,2,v}^2 . \end{aligned}$$

By replacing f by $\dfrac{f}{\|f\|_{q,2,v}}$ we get

$$c^a \left\| x^{a/2} f \right\|_{q,2,v}^2 + d^b \left\| x^{b/2} \mathcal{F}_{q,v} f \right\|_{q,2,v}^2 \geq -\log\left(B_{q,v}^2 k_c k_d\right) \|f\|_{q,2,v}^2 .$$

This finish the proof. ■

Corollary 1 *There exist $k > 0$ such that for any function $f \in \mathcal{L}_{q,2,v}$ we have*

$$\|xf\|_{q,2,v} \|x\mathcal{F}_{q,v} f\|_{q,2,v} \geq k \|f\|_{q,2,v}^2 .$$

Proof. Let $a = b = 2$ and $c = d$ then by Theorem 3

$$\|xf\|_{q,2,v}^2 + \|x\mathcal{F}_{q,v} f\|_{q,2,v}^2 \geq -\frac{1}{c^2} \log\left(B_{q,v}^2 \frac{\sigma_2^2}{c^{4(v+1)}}\right) \|f\|_{q,2,v}^2 ,$$

where

$$0 < \left(B_{q,v}^2 \frac{\sigma_2^2}{c^{4(v+1)}}\right) < 1.$$

Now put

$$f_t(x) = f(tx), \quad t \in \mathbb{R}_q^+,$$

then

$$\mathcal{F}_{q,v} f_t(x) = \frac{1}{t^{2v+2}} \mathcal{F}_{q,v} f(x/t), \quad \|x\mathcal{F}_{q,v} f_t\|_{q,2,v}^2 = \frac{1}{t^{2v}} \|\mathcal{F}_{q,v} f\|_{q,2,v}^2,$$

and

$$\|f_t\|_{q,2,v}^2 = \frac{1}{t^{2v+2}} \|f\|_{q,2,v}^2, \quad \|xf_t\|_{q,2,v}^2 = \frac{1}{t^{2v+4}} \|xf\|_{q,v,2}^2,$$

which gives

$$t^4\|x\mathcal{F}_{q,v}f\|_{q,2,v}^2+t^2\frac{1}{c^2}\log\left(B_{q,v}^2\frac{\sigma_2^2}{c^{4(v+1)}}\right)\|f\|_{q,v,2}^2+\|xf\|_{q,2,v}^2\geq 0,$$

and then

$$\|xf\|_{q,2,v}\|x\mathcal{F}_{q,v}f\|_{q,2,v}\geq\psi(c)\|f\|_{q,2,v}^2.$$

where

$$\psi(c)=\frac{v+1}{[\sigma_2 B_{q,v}]^{\frac{1}{v+1}}}|z_c\log(z_c)|,\quad z_c=\frac{[\sigma_2 B_{q,v}]^{\frac{1}{v+1}}}{c^2},\quad 0<z_c<1.$$

One can see that

$$\sup_{0<z_c<1}\psi(c)=\psi(q^\alpha),\quad \alpha=\frac{\log[\sigma_2 B_{q,v}]}{2(1+v)\log q}+\frac{1}{2\log q}.$$

Let

$$n_1=\lfloor\alpha\rfloor,\quad n_2=\lceil\alpha\rceil,$$

where$\lfloor.\rfloor$ and $\lceil.\rceil$ are respectively the floor and ceiling functions. Now the constant k is given as follows

$$k=\psi(q^{n_1}),\quad \text{if}\quad \lceil\alpha\rceil\geq\alpha-\frac{1}{2\log q}$$

and

$$k=\max\{\psi(q^{n_1}),\psi(q^{n_2})\},\quad \text{if}\quad \lceil\alpha\rceil<\alpha-\frac{1}{2\log q}.$$

This finish the proof. ■

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