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Heisenberg Uncertainty Principle for the q -Bessel Fourier transform

Lazhar Dhaouadi *

Abstract

In this paper we give a q -analogue of The Heisenberg Uncertainty Principle for the q -Bessel Fourier transform:

$$\mathcal{F}_{q,v}f(x) = c_{q,v} \int_0^\infty f(t)j_v(xt, q^2)t^{2v+1}d_qt,$$

where $j_v(x, q)$ is the normalized Hahn-Exton q -Bessel function.

1 The q -Foueir Bessel transform

Throughout this paper we consider $0 < q < 1$ and we adopt the standard conventional notations of [2]. We put

$$\mathbb{R}_q^+ = \{q^n, \quad n \in \mathbb{Z}\},$$

and for complex a

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i), \quad n = 1 \dots \infty.$$

Jackson's q -integral (see [3]) in the interval $[0, \infty[$ is defined by

$$\int_0^\infty f(x)d_qx = (1 - q) \sum_{n=-\infty}^\infty q^n f(q^n).$$

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We introduce the following functional spaces $\mathcal{L}_{q,p,v}$ of even functions f defined on \mathbb{R}_q^+ such that

$$\|f\|_{q,p,v} = \left[\int_0^\infty |f(x)|^p x^{2v+1} d_q x \right] < \infty.$$

The normalized Hahn-Exton q -Bessel function of order $v > -1$ (see [5]) is defined by

$$j_v(z, q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{n(n-1)}{2}}}{(q, q)_n (q^{v+1}, q)_n} z^n,$$

and the q -Bessel Fourier transform $\mathcal{F}_{q,v}$ introduced in [1,4] as follow

$$\mathcal{F}_{q,v} f(x) = c_{q,v} \int_0^\infty f(t) j_v(xt, q^2) t^{2v+1} d_q t,$$

where

$$c_{q,v} = \frac{1}{1-q} \frac{(q^{2v+2}, q^2)_\infty}{(q^2, q^2)_\infty}.$$

Define the q -Bessel translation operator as follows:

$$T_{q,x}^v f(y) = c_{q,v} \int_0^\infty \mathcal{F}_{q,v}(f)(t) j_v(xt, q^2) j_v(yt, q^2) t^{2v+1} d_q t, \quad \forall x, y \in \mathbb{R}_q^+, \forall f \in \mathcal{L}_{q,v,1}.$$

The q -convolution product of both functions $f, g \in \mathcal{L}_{q,v,1}$ is defined by

$$f *_q g(x) = c_{q,v} \int_0^\infty T_{q,x}^v f(y) g(y) y^{2v+1} d_q y.$$

The followings results was proved in [1].

Proposition 1 For any function $f \in \mathcal{L}_{q,v,1}$ we have

$$\int_0^\infty T_{q,x}^v f(y) y^{2v+1} d_q y = \int_0^\infty f(y) y^{2v+1} d_q y.$$

Proposition 2 Given two functions $f, g \in \mathcal{L}_{q,v,1}$ then

$$f *_q g \in \mathcal{L}_{q,v,1},$$

and

$$\mathcal{F}_{q,v}(f *_q g) = \mathcal{F}_{q,v}(f) \mathcal{F}_{q,v}(g).$$

Theorem 1

1. The q -Bessel Fourier transform $\mathcal{F}_{q,v}$ possess an extension

$$U : \mathcal{L}_{q,2,v} \rightarrow \mathcal{L}_{q,2,v}.$$

2. For $f \in \mathcal{L}_{q,2,v}$, we have

$$\|U(f)\|_{q,2,v} = \|f\|_{q,2,v}.$$

3. The application U is bijective and

$$U^{-1} = U.$$

Proposition 3 Given $1 < p \leq 2$ and $\frac{1}{p} + \frac{1}{\bar{p}} = 1$. If $f \in \mathcal{L}_{q,p,v}$, then

$$\mathcal{F}_{q,v}(f) \in \mathcal{L}_{q,\bar{p},v}$$

and

$$\|\mathcal{F}_{q,v}(f)\|_{q,\bar{p},v} \leq B_{q,v}^{\left(\frac{2}{p}-1\right)} \|f\|_{q,p,v},$$

where

$$B_{q,v} = \frac{1}{1-q} \frac{(-q^2; q^2)_\infty (-q^{2v+2}; q^2)_\infty}{(q^2; q^2)_\infty}.$$

The q -exponential function is defined by

$$e(z, q) = \sum_{n=0}^{\infty} \frac{z^n}{(q, q)_n} = \frac{1}{(z; q)_\infty}, \quad |z| < 1.$$

Proposition 4 The q -Gauss kernel

$$G^v(x, t, q^2) = \frac{(-q^{2v+2}t, -q^{-2v}/t; q^2)_\infty}{(-t, -q^2/t; q^2)_\infty} e\left(-\frac{q^{-2v}}{t}x^2, q^2\right),$$

satisfying

$$\mathcal{F}_{q,v} \{e(-ty^2, q^2)\} (x) = G^v(x, t, q^2),$$

and for all function $f \in \mathcal{L}_{q,2,v}$

$$\lim_{n \rightarrow \infty} \|c_{q,v} G^v(x, q^{2n}, q^2) *_q f - f\|_{q,2,v} = 0.$$

2 Uncertainty Principale

The following theorem is Jensens inequality and is crucial for the proof of our main result.

Theorem 2 *Let μ be a probability measure on \mathbb{R}_q^+ , If f is a positive function in $\mathcal{L}_{q,v,1}$ and g is convex on \mathbb{R}^+ then*

$$g\left(\int_0^\infty f d\mu(x)\right) \leq \int_0^\infty g \circ f(x) d\mu(x).$$

Proof. There exists $c \in \mathbb{R}$ such that for all x, y in a convex set \mathbb{R}^+ it holds

$$g(x) \geq g(y) + c(x - y).$$

Let $t = \int_0^\infty f(x) d\mu(x)$. Then $t \in \mathbb{R}^+$. Now let $x \in \mathbb{R}_q^+$ then

$$g(f(x)) \geq g(t) + c(f(x) - t).$$

Integrating both sides and using the special value of t gives

$$\int_0^\infty g(f(x)) d\mu(x) \geq \int_0^\infty [g(t) + c(f(x) - t)] d\mu(x) = g(t).$$

This finish the proof. ■

Now we are in a position to state and prove the q -analogue of the Uncertainty Principale.

Theorem 3 *Given $a, b > 0$ and let*

$$\sigma_a = \int_0^\infty \exp(-|x|^a) x^{2v+1} d_q x,$$

then for all $c, d \in \mathbb{R}_q^+$ satisfying

$$0 < B_{q,v} \frac{\sigma_a \sigma_b}{(cd)^{2v+2}} < 1,$$

the following hold for any function $f \in \mathcal{L}_{q,2,v}$

$$c^a \|x^{a/2} f\|_{q,v,2}^2 + d^b \|x^{b/2} \mathcal{F}_{q,v} f\|_{q,v,2}^2 \geq -\log\left(B_{q,v} \frac{\sigma_a \sigma_b}{(cd)^{2v+2}}\right) \|f\|_{q,v,2}^2.$$

Proof. For a positive function $\phi \in \mathcal{L}_{q,v,1}$ with $\|\phi\|_{q,v,1} = 1$ define the entropy of ϕ to be

$$E(\phi) = \int_0^\infty \phi(x) \log \phi(x) x^{2v+1} d_q x,$$

and for some $a > 0$ defining

$$M_a(\phi) = \left(\int_0^\infty |x|^a \phi(x) x^{2v+1} d_q x \right)^{\frac{1}{a}}.$$

In the following assume that $E(\phi)$ and $M_a(\phi)$ exist. For a given $c \in \mathbb{R}_q^+$ let

$$d\gamma(x) = k_c^{-1} \exp(-|cx|^a) x^{2v+1} d_q x \text{ where } k_c = \frac{\sigma_a}{c^{2v+2}}.$$

Then $d\gamma(x)$ is a probability measure. Defining

$$\psi(x) = k_c \exp(|cx|^a) \phi(x),$$

we get

$$\int_0^\infty \psi(x) d\gamma(x) = 1.$$

According to the fact that $t \mapsto t \log t$ is convex on \mathbb{R}^+ , so Jensen's inequality gives

$$0 = \int_0^\infty \psi(x) d\gamma(x) \log \left[\int_0^\infty \psi(x) d\gamma(x) \right] \leq \int_0^\infty \psi(x) \log \psi(x) . d\gamma(x).$$

$$\begin{aligned} 0 &\leq \int_0^\infty \phi(x) \log [k_c \exp(|cx|^a) \phi(x)] . x^{2v+1} d_q x \\ &= \int_0^\infty \phi(x) [\log k_c + |cx|^a + \log \phi(x)] x^{2v+1} d_q x. \end{aligned}$$

$$0 \leq \log k_c + c^a \int_0^\infty |x|^a \phi(x) x^{2v+1} d_q x + \int_0^\infty \phi(x) \log \phi(x) x^{2v+1} d_q x.$$

In the end

$$0 \leq \log k_c + c^a M_a^a(\phi) + E(\phi). \quad (1)$$

For $f \in \mathcal{L}_{q,v,2}$ with $\|f\|_{q,v,2} = 1$ we will show that

$$E(|f|^2) + E(|\mathcal{F}_{q,v}f|^2) \leq \log B_{q,v}. \quad (2)$$

Start by assuming that

$$f \in \mathcal{L}_{q,v,1} \cap \mathcal{L}_{q,v,2}.$$

Then f will be in $\mathcal{L}_{q,v,p}$ for $1 < p < 2$. With

$$\frac{1}{p} + \frac{1}{\bar{p}} = 1,$$

Hausdorff-Young's inequality (Proposition 3) tells us that $\mathcal{F}_{q,v}f$ is in $\mathcal{L}_{q,v,\bar{p}}$. So we can define the functions

$$A(p) = \int_0^\infty |f(x)|^p dx \quad \text{and} \quad B(\bar{p}) = \int_0^\infty |\mathcal{F}_{q,v}f(x)|^{\bar{p}} x^{2v+1} d_q x.$$

Now define

$$\begin{aligned} C(p) &= \log \|\mathcal{F}_{q,v}f\|_{\bar{p}} - \log \left(B_{q,v}^{\frac{2}{p}-1} \|f\|_p \right) \\ &= \frac{1}{\bar{p}} \log B(\bar{p}) - \frac{1}{p} \log A(p) - \left(\frac{2}{p} - 1 \right) \log B_{q,v}. \end{aligned}$$

By Hausdorff-Young's inequality

$$C(p) \leq 0, \quad \text{for } 1 < p < 2,$$

and by Parsevals equality (Theorem 1)

$$C(2) = 0.$$

In the other hand

$$C'(p) = \frac{\bar{p}'}{\bar{p}} \frac{B'(\bar{p})}{B(\bar{p})} - \frac{\bar{p}'}{\bar{p}^2} \log B(\bar{p}) - \frac{p'}{p} \frac{A'(p)}{A(p)} + \frac{p'}{p^2} \log A(p) + \frac{2}{p^2} \log B_{q,v}.$$

The derivative of \bar{p} with respect to p is

$$\bar{p}' = -\frac{1}{(p-1)^2}.$$

For a given $x > 0$ the function $p \mapsto \frac{x^p - x^2}{p-2}$ is increasing and

$$\lim_{p \rightarrow 2} \frac{x^p - x^2}{p-2} = x^2 \log x.$$

So

$$C'(2^-) = \lim_{p \rightarrow 2^-} \frac{C(p) - C(2)}{p-2} = -\frac{1}{2} [A'(2^-) + B'(2^+)] + \frac{1}{2} \log B_{q,v},$$

where

$$A'(2^-) = \lim_{p \rightarrow 2^-} \frac{A(p) - A(2)}{p-2} = E(|f|^2),$$

$$B'(2^+) = \lim_{\bar{p} \rightarrow 2^+} \frac{B(\bar{p}) - B(2)}{\bar{p}-2} = E(|\mathcal{F}_{q,v}f|^2).$$

Therefore

$$C'(2^-) \geq 0 \Rightarrow A'(2^-) + B'(2^+) - \log B_{q,v} \leq 0,$$

and then

$$E(|f|^2) + E(|\mathcal{F}_{q,v}f|^2) \leq \log B_{q,v}. \quad (3)$$

If $\|f\|_{q,v,2} \neq 1$ then by replacing f by $\frac{f}{\|f\|_2}$ the inequality (3) become

$$E(|f|^2) + E(|\mathcal{F}_{q,v}f|^2) \leq \|f\|_2^2 \log B_{q,v} + 2 \log \|f\|_{q,v,2}. \quad (4)$$

Let us now assume that $f \in \mathcal{L}_{q,v,2}$ and that $E(|f|^2)$ and $E(|\mathcal{F}_{q,v}f|^2)$ are defined, and then approximate f by functions in $\mathcal{L}_{q,v,1} \cap \mathcal{L}_{q,v,2}$. Let

$$h_n(x) = c_{q,v} e(-q^{2n}x^2, q^2).$$

The following holds by (4)

$$E(|h_n f|^2) + E(|\mathcal{F}_{q,v} h_n *_q \mathcal{F}_{q,v} f|^2) \leq \|h_n f\|_2^2 \log B_{q,v} + 2 \log \|h_n f\|_{q,v,2}, \quad (5)$$

and by Proposition 2 and Proposition 4

$$\lim_{n \rightarrow \infty} \|h_n f - f\|_{q,v,2} = \lim_{n \rightarrow \infty} \|\mathcal{F}_{q,v} h_n *_q \mathcal{F}_{q,v} f - \mathcal{F}_{q,v} f\|_{q,v,2} = 0,$$

which implies

$$\lim_{n \rightarrow \infty} E(|h_n f|^2) = E(|f|^2). \quad (6)$$

The functions

$$\phi_1(z) = |z|^2 \log^+ |z| \quad \text{and} \quad \phi_2(z) = |z|^2 \left(-\log^- |z| + \frac{3}{2} \right),$$

are convex on \mathbb{C} , where

$$\log^+ x = \max \{0, \log x\} \quad \text{and} \quad \log^- x = \min \{0, \log x\}.$$

Since $\mathcal{F}_{q,v}h_n \geq 0$ and by proposition 1

$$\int_0^\infty T_{q,x} \mathcal{F}_{q,v}h_n(t) t^{2v+1} d_q t = \int_0^\infty \mathcal{F}_{q,v}h_n(t) t^{2v+1} d_q t = 1,$$

for a given $x \in \mathbb{R}_q^+$ the following holds by Jensen's Inequality

$$\begin{aligned} \phi_i(\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f)(x) &\leq \int_0^\infty \phi_i(\mathcal{F}_{q,v}f)(t) T_{q,x} \mathcal{F}_{q,v}h_n(t) t^{2v+1} d_q t \\ &= \mathcal{F}_{q,v}h_n *_q \phi_i(\mathcal{F}_{q,v}f)(x). \end{aligned}$$

It follows that

$$\mathcal{F}_{q,v}h_n *_q \phi_i(\mathcal{F}_{q,v}f) \in \mathcal{L}_{q,v,1}$$

since $\phi_i(\mathcal{F}_{q,v}f)$ is in $\mathcal{L}_{q,v,1}$ and therefore

$$\phi_i(\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f) \in \mathcal{L}_{q,v,1}.$$

On the other hand

$$\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f \rightarrow \mathcal{F}_{q,v}f.$$

The Dominated convergence theorem gives

$$\lim_{n \rightarrow \infty} \int_0^\infty \phi_i(\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f)(x) x^{2v+1} d_q x = \int_0^\infty \phi_i(\mathcal{F}_{q,v}f) x^{2v+1} d_q x.$$

It also holds that

$$\begin{aligned} E(|\mathcal{F}_{q,v}f|^2) &= 2 \int_0^\infty \phi_1(\mathcal{F}_{q,v}f) x^{2v+1} d_q x \\ &\quad - 2 \int_0^\infty \phi_2(\mathcal{F}_{q,v}f) x^{2v+1} d_q x + 3 \|\mathcal{F}_{q,v}f\|_{q,v,2}^2, \end{aligned}$$

and

$$\begin{aligned} E(|\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f|^2) &= 2 \int_0^\infty \phi_1(\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f) x^{2v+1} d_q x \\ &\quad - 2 \int_0^\infty \phi_2(\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f) x^{2v+1} d_q x \\ &\quad + 3 \|\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f\|_{q,v,2}^2. \end{aligned}$$

Then

$$\lim_{n \rightarrow \infty} E(|\mathcal{F}_{q,v}h_n *_q \mathcal{F}_{q,v}f|^2) = E(|\mathcal{F}_{q,v}f|^2). \quad (7)$$

With (5) and the limits (6) and (7) we complete the proof of (2).

For a given $b > 0$ and $c, d \in \mathbb{R}_q^+$ such that $0 < B_{q,v}k_c k_d < 1$ and by (1) we can write

$$-E(|f|^2) \leq \log k_c + c^a \|x^{a/2} f\|_{q,v,2}^2.$$

$$-E(|\mathcal{F}_{q,v}f|^2) \leq \log k_d + d^b \|x^{b/2} \mathcal{F}_{q,v}f\|_{q,v,2}^2.$$

Which implies with (2)

$$\begin{aligned} -\log B_{q,v} &\leq -E(|f|^2) - E(|\mathcal{F}_{q,v}f|^2) \\ &\leq \log k_c k_d + c^a \|x^{a/2} f\|_{q,v,2}^2 + d^b \|x^{b/2} \mathcal{F}_{q,v}f\|_{q,v,2}^2. \end{aligned}$$

By replacing f by $\frac{f}{\|f\|_{q,v,2}}$ we get

$$c^a \|x^{a/2} f\|_{q,v,2}^2 + d^b \|x^{b/2} \mathcal{F}_{q,v}f\|_{q,v,2}^2 \geq -\log(B_{q,v}k_c k_d) \|f\|_{q,v,2}^2.$$

This finish the proof. ■

Corollary 1 *There exist $k > 0$ such that for any function $f \in \mathcal{L}_{q,v,2}$ we have*

$$\|xf\|_{q,v,2}^2 \|x\mathcal{F}_{q,v}f\|_{q,v,2}^2 \geq k \|f\|_{q,v,2}^2.$$

Proof. Let $a = b = 2$ then by Theorem 3

$$\|xf\|_{q,v,2}^2 + \|x\mathcal{F}_{q,v}f\|_{q,v,2}^2 \geq -\frac{1}{c^2} \log \left(B_{q,v} \frac{\sigma_2^2}{c^{4(v+1)}} \right) \|f\|_{q,v,2}^4.$$

Now put

$$f_t(x) = f(tx), \quad t \in \mathbb{R}_q^+,$$

then

$$\mathcal{F}_{q,v} f_t(x) = \frac{1}{t^{2v+2}} \mathcal{F}_{q,v} f(x/t), \quad \|x \mathcal{F}_{q,v} f_t\|_{q,v,2}^2 = \frac{1}{t^{2v}} \|\mathcal{F}_{q,v} f\|_{q,v,2}^2,$$

and

$$\|f_t\|_{q,v,2}^2 = \frac{1}{t^{2v+2}} \|f\|_{q,v,2}^2, \quad \|x f_t\|_{q,v,2}^2 = \frac{1}{t^{2v+4}} \|x f\|_{q,v,2}^2,$$

which gives

$$t^4 \|\mathcal{F}_{q,v} f\|_{q,v,2}^2 + t^2 \frac{1}{c^2} \log \left(B_{q,v} \frac{\sigma_2^2}{c^{4(v+1)}} \right) \|f\|_{q,v,2}^2 + \|x f\|_{q,v,2}^2 \geq 0,$$

and then

$$\|x f\|_{q,v,2}^2 \|\mathcal{F}_{q,v} f\|_{q,v,2}^2 \geq \frac{1}{c^4} \log \left(B_{q,v} \frac{\sigma_2^2}{c^{4(v+1)}} \right)^2 \|f\|_{q,v,2}^4.$$

We choose k as follows

$$k = \sup_{c \in \mathbb{R}_q^+} \frac{1}{c^4} \log \left(B_{q,v} \frac{\sigma_2^2}{c^{4(v+1)}} \right)^2$$

Which leads to the result. ■

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