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**LOWER BOUNDS FOR SUP + INF AND SUP * INF AND AN EXTENSION OF
CHEN-LIN RESULT IN DIMENSION 3.**

SAMY SKANDER BAHOURA

ABSTRACT. We give two results about Harnack type inequalities. First, on compact smooth Riemannian surface without boundary, we have an estimate of the type sup + inf. The second result concerns the solutions of prescribed scalar curvature equation on the unit ball of \mathbb{R}^n with Dirichlet condition.

Next, we give an inequality of the type $(\sup_K u)^{2s-1} \times \inf_{\Omega} u \leq c$ for positive solutions of $\Delta u = Vu^s$ on $\Omega \subset \mathbb{R}^3$, where K is a compact set of Ω and V is s -hölderian, $s \in]-1/2, 1]$. For the case $s = 1/2$, we prove that if $\min_{\Omega} u > m > 0$ and the hölderian constant A of V is small enough (in certain meaning), we have the uniform boundedness of the supremum of the solutions of the previous equation on any compact set of Ω .

1. INTRODUCTION AND RESULTS.

We denote $\Delta = -\nabla^j(\nabla_j)$, the geometric Laplacian.

On compact smooth Riemann surface without boundary (M, g) we consider the following equation :

$$\Delta u + k = Ve^u, \quad (E_1)$$

with, $k \in \mathbb{R}_{*,+}$ and $0 \leq V \leq b$ ($V \not\equiv 0$).

We suppose V smooth. The previous equation is of type prescribed scalar curvature. We search to know if it's possible to have a priori estimate of the type sup + inf.

Note that in dimension 2, on \mathbb{R}^2 , we have different results about sup + inf inequalities for the following equation:

$$\Delta u = Ve^u, \quad (E_2)$$

see [B-L-S], [B-M], [C-L 2], [L 2] and [S].

In [S], Shafirir proved an inequality of the type $\sup u + C \inf u < C'$ with minimal conditions on the prescribed scalar curvature. In [B-L-S], Brezis-Li-Shafirir have proved a $\sup u + \inf u$ inequality with lipschitzian assumption on prescribed curvature. Finally, [C-L 2] have proved the same result with hölderian assumption on V in the equation (E_2) .

Here, we are interested by the minoration of this sum. We can suppose that $Volume(M) = 1$. We obtain,

Theorem 1. For all $k, b > 0$, there exists a constant $c = c(k, b, M, g)$ such that, for all solution of (E_1) :

$$\frac{k - 4\pi}{4\pi} \sup_M u + \inf_M u \geq c.$$

We can remark that for $k = 8\pi$, we have the same result than in [B 1]. Here there is no restriction on k .

Now we work in dimension $n \geq 3$, we set $B = B_1(0)$ the unit ball of \mathbb{R}^n . We try to study some properties of the solutions of the following equation:

$$\Delta u = Vu^{N-1-\epsilon}, u > 0 \text{ in } B, u = 0 \text{ on } \partial B \quad (E_3)$$

with $0 \leq V(x) \leq b < +\infty$, $0 \leq \epsilon < 2/(n-2)$ and $N = \frac{2n}{n-2}$ the critical Sobolev exponent.

Equation (E_3) is the prescribed scalar curvature equation, it was studied a lot. We know, after using Pohozaev identity that, there is no solution for this equation if we assume $\epsilon = 0$ and $V \equiv 1$, see [P].

Theorem 2. For all compact K of B , there exists one positive constant $c = c(n, b, K)$ such that for all solution of (E_3) :

$$(\sup_B u)^7 \times \inf_K u \geq c.$$

Recall that estimates like in the last theorem exist, see for example [B 1] et [B 2].

Now we work on $\Omega \subset \mathbb{R}^3$ and we consider the following equation:

$$\Delta u = Vu^5, \quad u > 0, \quad (E_4)$$

with,

$$0 < a \leq V(x) \leq b \text{ and } |V(x) - V(y)| \leq A|x - y|^s, \quad s \in [\frac{1}{2}, 1], x, y \in \Omega. \quad (C)$$

Without loss of generality, we suppose $\Omega = B$ the unit ball of \mathbb{R}^3 .

The equation (E_4) is the scalar curvature equation in three dimensions. It was studied a lot, see for example [B 3], [C-L 1], [L 1]. In [C-L 1], Chen and Lin have proved that if $s > \frac{1}{2}$, then each sequence $(u_k)_k$ which are solutions of (E_4) (with fixed V) are in L_{loc}^∞ if we suppose $\min_B u_k > m > 0$. When $s = 1$ they prove that the $\sup \times \inf$ inequality holds. To prove those results, they use the moving-plane method.

In [L 1], Li proved (in particular) that the product $\sup \times \inf$ is bounded if we replace Ω by the three sphere \mathbb{S}_3 . He used the notion of isolated and isolated simple blow-up points.

We can see in [B 3] another proof of the boundedness of $\sup^{1/3} \times \inf$, also with the moving-plane method.

Note that, if we suppose Ω a Riemannian manifold of dimension 3 (not necessarily compact), Li and Zhang (see [L-Z]) have proved that the $\sup \times \inf$ holds when the prescribed scalar curvature is a constant.

Note that, in our work, we have no assumption on energy. There are many results, if we suppose the energy bounded.

Here, we use the moving-plane method to have $\sup \times \inf$ inequalities. This method was developed by Gidas-Ni-Nirenberg, used by Chen-Lin and Li-Zhang, see [G-N-N], [C-L 1] and [L-Z]. In our work we follow and use the technique of Li and Zhang, see [L-Z].

Theorem 3. If $s \in [\frac{1}{2}, 1]$, then, for all positive numbers a, b, A and all compact K of B , there exists a positive constant $c = c(a, b, A, s, K)$ such that:

$$(\sup_K u)^{2s-1} \times \inf_B u \leq c,$$

where u is solution of (E_4) with V satisfying (C) .

For $s = \frac{1}{2}$ and $a, b, m > 0$, there exists $\delta = \delta(a, b, m) > 0$ such that for u solution of (E_4) with $A \in]0, \delta]$ for V in (C) and $u \geq m$, we have:

$$\sup_K u \leq c = c(a, b, m, K),$$

where $K \subset\subset B_1$.

Note that in [B 3], for the dimension 4, we have a result like in the second part of the theorem 3.

About usual Harnack inequalities, we can find in [G-T] lots of those estimates. For harmonic functions ($\Delta u = -\sum_{i=1}^n \partial_{ii} u = 0$ on open set of \mathbb{R}^n), we have an estimate of the type:

$$\frac{\sup_{B_R} u}{\inf_{B_R} u} \leq 3^n,$$

on small ball B_R of radius R (see chapter 1 in [G-T]).

We have other results if we consider a general elliptic operator ($L = \partial_i(a^{ij}\partial_j) + \sum_{j=1}^n b^j \partial_j + c$ on open set of \mathbb{R}^n), we obtain:

$$\frac{\sup_{B_R} u}{\inf_{B_R} u} \leq C[n, R, c, (b^j)_j, (a^{ij})_{i,j}]$$

for a non negative function u such that $Lu = 0$ (B_R is a ball of radius R). See for example theorem 8.20 in [G-T].

For subharmonic and superharmonic functions there is another type of Harnack inequalities linking their norm L^p to their infimum or supremum. (See chapter 8 in [G-T]).

Here we follow the same idea and we try to compare the sup and the inf in a certain meaning.

2. PROOFS OF THE THEOREMS.

Proof of Theorem 1:

Consider the equation :

$$\Delta u_i + k = V_i e^{u_i},$$

Case 1: $\sup_M u_i < c < +\infty$.

We set x_i the point where u_i is maximum, $u_i(x_i) = \sup_M u_i$, then:

$$0 \leq \Delta u_i(x_i) = V_i(x_i) e^{u_i(x_i)} - k \leq b e^{u_i(x_i)} - k,$$

thus,

$$\log\left(\frac{k}{b}\right) \leq u_i(x_i) \leq c'.$$

We denote G the Green function of laplacian,

$$\Delta_{y,distribution} G(x, \cdot) = 1 - \delta_x \text{ and } G(x, y) \geq 0, \int_M G(x, y) \equiv C.$$

we can write,

$$\log\left(\frac{k}{b}\right) \leq u_i(x_i) = \int_M u_i dV_g - \int_M G(x_i, y) [V_i(y) e^{u_i(y)} - k] dV_g \leq \int_M u_i + C(b e^c - k).$$

We deduce:

$$-\infty < c_2 \leq \int_M u_i \leq c_1 < +\infty, \forall i.$$

Now, we write,

$$\min_M u_i = u_i(y_i) = \int_M u_i + \int_M G(y_i, y) [V_i(y) e^{u_i(y)} - k] dV_g \geq c_2 - kC > -\infty.$$

Thus,

$$\|u_i\|_{L^\infty} \leq c' < +\infty, \forall i.$$

Case 2: $\sup_M u_i \rightarrow +\infty$.

According to T. Aubin (see [A]), we have,

$$G(x, y) = -\frac{1}{2\pi} \log d(x, y) + g(x, y),$$

where, g is a regular part of G , it is a continuous function on $M \times M$.

Let us note x_i the point where u_i is maximum, $u_i(x_i) = \max_M u_i$. We can suppose that $x_i \rightarrow x_0$ and in the conformal isothermal coordinates around x_0 we set $v_i(x) = u_i(x_i + xe^{-u_i(x_i)/2}) - u_i(x_i)$, then,

$$\Delta v_i + h_i = \tilde{V}_i e^{v_i}, \quad h_i \rightarrow 0$$

$v_i(0) = 0, v_i(x) \leq 0, 0 \leq \tilde{V}_i(x) \leq b$. We can use theorem 3 of [B-M] and we deduce after passing to the subsequence that:

$$v_i(x) \geq C > -\infty, \text{ for } |x| \leq r.$$

Thus,

$$u_i(y) \geq u_i(x_i) + C, \text{ if } d(y, x_i) \leq re^{-u_i(x_i)/2},$$

Now, we work on $M - B(x_i, re^{-u_i(x_i)/2})$,

$$G(x_i, y) \leq \frac{1}{4\pi} u_i(x_i) + C_1, \text{ on } \partial B(x_i, re^{-u_i(x_i)/2})$$

$$\Delta[u_i - kG(x_i, \cdot)] \geq 0, \text{ on } M - B(x_i, re^{-u_i(x_i)/2})$$

$$u_i(y) - kG(x_i, y) - \frac{4\pi - k}{4\pi} u_i(x_i) + kC_1 - C \geq 0, \text{ on } \partial B(x_i, re^{-u_i(x_i)/2}).$$

By maximum principle, we obtain:

$$u_i \geq kG(x_i, \cdot) + \frac{4\pi - k}{4\pi} u_i(x_i) - kC_1 + C, \text{ on } M - B(x_i, re^{-u_i(x_i)/2}),$$

We use the fact, $\int_M G(x_i, y) \equiv \text{constant}$, and by integration of the last inequality we have,

$$\inf_M u_i + \frac{k - 4\pi}{4\pi} \sup_M u_i \geq c > -\infty,$$

Example with $V_i \rightarrow 0$: we can take, $u_i \equiv \log k + \log i$ and $V_i \equiv 1/i$.

Remark: If we suppose $V_i \geq a > 0$ uniformly, then, when $k < 4\pi$ we can not have $\sup_M u_i \rightarrow +\infty$. To see this, it is sufficient to integrate the equation.

Proof of Theorem 2:

We are going to prove that each sequence has a subsequence who has the searched inequality.

Next, we use the fact that, if we have possibility to extract a subsequence we do it and we denote $(u_i)_i$ the subsequence.

We have,

$$\Delta u_i = V_i u_i^{N-1-\epsilon_i}, \quad u_i > 0 \text{ on } B, \quad (\tilde{E})$$

with $0 \leq V_i(x) \leq b$ ($V_i \not\equiv 0$).

Let us note G the Green function of the laplacian on unit ball with Dirichlet condition. G is of the form:

$$G(x, y) = \frac{1}{n(n-2)\omega_n|x-y|^{n-2}} - \frac{1}{n(n-2)\omega_n(|x|^2|y|^2 + 1 - 2x \cdot y)^{(n-2)/2}}.$$

Denote x_i the point where u_i is maximum. We write:

$$u_i(x_i) = \int_B G(x_i, y) V_i(x) [u_i(y)]^{N-1-\epsilon_i} dy \leq b [u_i(x_i)]^{N-1-\epsilon_i} \int_B G(x_i, y) dy.$$

Consider the function $h(x) = |x|^2 - 1$, we have:

$$\int_B G(x_i, y) dy = \frac{1 - |x_i|^2}{2n} \leq d(x_i, \partial B)/n.$$

We deduce:

$$0 < \frac{n}{b} \leq [u_i(x_i)]^{4/(n-2)-\epsilon_i} d(x_i, \partial B).$$

Case 1: $\max_B u_i \leq c$

Then, $d(x_i, \partial B) \geq c' > 0$. By elliptic estimates, $u_i \rightarrow u$, with $u > 0$. Then, $\inf_K u_i \geq \tilde{c} > 0$ with $K \subset\subset B$.

To see this, we can write (\tilde{E}) as:

$$\Delta u_i = f_i$$

with, f_i uniformly in L^p for $p > n$. We can use the elliptic estimates to have u_i uniformly in $W^{2,p}(B)$ and by the Sobolev embedding, we have u_i uniformly in $C^{1,\theta}(\bar{B})$, for some $\theta \in]0, 1[$.

Now, we can see that:

$$\int_B \nabla u_i \cdot \nabla \varphi = \left(\int_B V_i u_i^{N-1-\epsilon_i} \varphi \right) \geq 0, \text{ for all } \varphi \in C_0^\infty(B), \varphi \geq 0 \text{ (distribution).}$$

We can pass to the limit $u_i \rightarrow u \geq 0$ (subsequence) and $u \in C^1(\bar{B})$. Then, we have:

$$\int_B \nabla u \cdot \nabla \varphi \geq 0, \varphi \in C_0^\infty(B), \varphi \geq 0 \text{ (distribution).}$$

We can use the strong maximum principle for weak solutions, see for example, Gilbarg-Trudinger, theorem 8.19 (applied to $-u \leq 0$):

If, there is a point t in B such that, $u(t) = 0$ then, $u \equiv 0$. But we can see that $u_i(x_i) \geq \tilde{c}' > 0$ with \tilde{c}' do not depends on i and $x_i \not\rightarrow \partial B$ (subsequence).

Finally, $u > 0$ on B .

Remark 1. Why do we do this ? in fact, we have neither $u_i \in C^2(\bar{B})$ nor $u_i \rightarrow u$ in C^2 norm because we don't have more regularity on V_i and finally we don't have $\Delta u \geq 0$ in the strong sense. We have weakly $\Delta u \geq 0$ with a good regularity on u . Here, it is sufficient to have: C^1 regularity on u and an uniform boundedness for u_i in $C^{1,\theta}$ ($0 < \theta < 1$), to obtain a good convergence for u_i . After we can use a strong maximum principle for weak solutions.

Remark 2. If we take a sequence of functions V_i which converge uniformly to 0 (for example), the previous case 1 is not possible.

Case 2: $\max_B u_i \rightarrow +\infty$

D $x_i \rightarrow x_0 \in \partial B$:

To simplify our computations, we assume $n/b > 1/2$. Then, $B(x_i, r_i) \in B$, with $r_i = \frac{1}{2[u_i(x_i)]^{4/(n-2)-\epsilon_i}}$. We consider the following functions :

$$v_i(x) = \frac{u_i[x_i + x/[u_i(x_i)]^{2/(n-2)-\epsilon_i/2}]}{u_i(x_i)},$$

Those functions v_i , exist on $\Omega_i = B(0, 5t_i)$, $t_i = 1/10[u_i(x_i)]^{2/(n-2)-\epsilon_i/2}$. We have :

$$\Delta v_i = \tilde{V}_i v_i^{N-1-\epsilon_i}, \quad 0 < v_i(x) \leq v_i(0) = 1, \quad 0 \leq \tilde{V}_i(x) \leq b.$$

with, $\tilde{V}_i(x) = V_i[x_i + x/[u_i(x_i)]^{2/(n-2)-\epsilon_i/2}]$.

We use Harnack inequality for v_i (see Theoreme 8.20 of [G-T]), we obtain:

$$\max_{B(0,t_i)} v_i \leq C \inf_{B(0,t_i)} v_i.$$

where $C = [C_0(n)]^{1+b}$ (see [G-T] and $t_i \leq 1$).

In 0, we obtain: $u_i(x) \geq C(n, b)u_i(x_i)$ for $|x| \leq s_i = 1/10[u_i(x_i)]^{4/(n-2)-\epsilon_i}$. Let us note that, $C(n, b) = C = [C_0(n)]^{1+b}$.

If we consider $B - B(x_i, s_i)$, then,

$$G(x_i, y) \leq c(n)[u_i(x_i)]^{4-(n-2)\epsilon_i}, \quad \text{for } d(x_i, y) = s_i,$$

$$\Delta G(x_i, \cdot) = 0, \quad G(x_i, \cdot)|_{\partial B} = 0,$$

$$\text{with, } c(n) = \frac{10^{n-2}}{n(n-2)\omega_n}.$$

Thus,

$$u_i(y) - \frac{C(n, b)G(x_i, y)}{c(n)[u_i(x_i)]^{3-(n-2)\epsilon_i}} \geq 0, \quad \text{for } d(y, x_i) = s_i, \text{ or, on } \partial B.$$

$$\Delta \left[u_i - \frac{C(n, b)G(x_i, \cdot)}{c(n)[u_i(x_i)]^{3-(n-2)\epsilon_i}} \right] \geq 0.$$

By maximum principle, we have:

$$u_i(y) - \frac{C(n, b)G(x_i, y)}{c(n)[u_i(x_i)]^{3-(n-2)\epsilon_i}} \geq 0, \quad \text{on } B - B(x_i, s_i).$$

In other terms,

$$u_i(y) \geq \frac{C(n, b)}{c(n)} G(x_i, y)[u_i(x_i)]^{-3+(n-2)\epsilon_i}, \quad \text{on } B - B(x_i, s_i).$$

Now, we know that,

$$G(x_i, y) \geq c'(n)(1 - |y|)^{n-2} \times (1 - |x_i|)^{n-2}.$$

$$\text{where } c'(n) = \frac{1}{2(n-2)2^{2(n-2)}\omega_n}.$$

We denote, $c'(n, b) = \frac{n}{b}$. Using the fact, $1 - |x_i| = d(x_i, \partial B) \geq c'(n, b)[u_i(x_i)]^{-4/(n-2)+\epsilon_i}$, we obtain,

$$u_i(y) \geq \frac{C(n, b)c'(n)c'(n, b)}{c(n)} (1 - |y|)^{n-2} [u_i(x_i)]^{-7+2(n-2)\epsilon_i}, \quad \text{on } B - B(x_i, s_i).$$

On $B(0, k)$ with $k < 1$, by maximum principle we have: $\inf_{B(0,k)} u_i = \inf_{\partial B(0,k)} u_i$.

Then,

$$u_i(y) \geq C(n, b)(1 - k)^{n-2} [u_i(x_i)]^{-7+2(n-2)\epsilon_i}, \quad \text{on } B(0, k) - B(x_i, s_i),$$

but, $x_i \rightarrow x_0 \in \partial B$, and for i large we can conclude that $B(x_i, s_i) \cap B(0, k) = \emptyset$ and thus,

$$\inf_{B(0,k)} u_i \times [u_i(x_i)]^7 \geq C(n, b, k).$$

We can remark that:

$$C(n, b, k) = \frac{C(n, b)c'(n)c'(n, b)}{c(n)}(1 - k)^{n-2}.$$

with, $C(n, b) = C_0(n)^{1+b}$, $c(n) = \frac{10^{n-2}}{n(n-2)\omega_n}$, $c'(n) = \frac{1}{2(n-2)2^{2(n-2)}\omega_n}$ and $c'(n, b) = \frac{n}{b}$.

Then,

$$C(n, b, k) = \frac{[C_0(n)]^{1+b}2n^2(n-2)2^{2(n-2)}\omega_n}{bn(n-2)\omega_n}(1 - k)^{n-2} = \frac{[C_0(n)]^{1+b}2n2^{2(n-2)}}{b}(1 - k)^{n-2}.$$

II) $x_i \rightarrow x_0 \in B$:

Our computations are the same as in the previous case I), there are some modifications.

We take, $t_i = 1$ and $s_i = [u_i(x_i)]^{2/(n-2)-\epsilon_i/2}$. We have:

$$G(x_i, y) \leq C(n)[u_i(x_i)]^{2-(n-2)\epsilon_i/2}.$$

After,

$$u_i(y) \geq C'(n, b)G(x_i, y)[u_i(x_i)]^{-1+(n-2)\epsilon_i/2},$$

we use the fact, $x_i \rightarrow x_0 \in B$, $G(x_i, y) \geq C''(n, b, x_0)(1 - k)^{n-2}$,

then,

$$\inf_{B(0, k)} u_i \times [u_i(x_i)]^{1-(n-2)\epsilon_i/2} \geq c(n, b, k, x_0) > 0.$$

Proof of the Theorem 3

Step 1: blow-up technique

We are going to prove the following assertion:

$$\exists c, R > 0 \text{ such that, } R \left[\sup_{B(0, R)} u \right]^{2s-1} \times \inf_B u \leq c \text{ if } \frac{1}{2} < s \leq 1,$$

and,

$$\exists c, R > 0 \text{ such that, } R \sup_{B(0, R)} u \leq c \text{ if } s = \frac{1}{2}.$$

We argue by contradiction (and after passing to a subsequence) and we suppose that for $R_k \rightarrow 0$ we have:

$$R_k \left[\sup_{B(0, R_k)} u_k \right]^{2s-1} \times \inf_B u_k \rightarrow +\infty, \text{ for } s \in]1/2, 1].$$

$$R_k \sup_{B(0, R_k)} u_k \rightarrow +\infty, \text{ for } s = 1/2.$$

Let x_k be the point such that $u_k(x_k) = \sup_{B(0, R_k)} u_k$ and consider the following function:

$$s_k(x) = \sqrt{(R_k - |x - x_k|)}u_k(x).$$

Let a_k be the point such that: $s_k(a_k) = \sup_{B(x_k, R_k)} s_k$. We set $M_k = u_k(a_k)$ and $l_k = R_k - |a_k - x_k|$. We have:

$$M_k^{-1}u_k(x) \leq \sqrt{2}, \text{ for } |x - a_k| \leq \frac{l_k}{2}M_k^2.$$

We have:

$$\frac{l_k}{2} M_k \rightarrow +\infty, v_k(y) = M_k^{-1} u_k(a_k + M_k^{-2} y) \text{ for } |y| \leq \frac{l_k}{2} M_k^2,$$

$$\Delta v_k = V_k v_k^5, v_k(0) = 1, 0 < v_k \leq \sqrt{2}.$$

We know, after passing to a subsequence, that:

$$v_k \rightarrow U, \text{ with } \Delta U = V(0)U^5, U > 0, \text{ on } \mathbb{R}^3.$$

It easy to see that we can suppose $V(0) = 1$. The result of Caffarelli-Gidas-Spruck (see [C-G-S]) assures that U has an explicit form and is radially symmetric about some point.

Step 2: The moving plane method

Now, we use the Kelvin transform and we set for $\lambda > 0$:

$$v_k^\lambda(y) = \frac{\lambda}{|y|} v_k(y^\lambda) \text{ with } y^\lambda = \frac{\lambda^2 y}{|y|^2}.$$

We denote Σ_λ by:

$$\Sigma_\lambda = B(0, R_k M_k^{2s}) - \bar{B}(0, \lambda).$$

We have the following boundary condition:

$$\lim_{k \rightarrow +\infty} \min_{|y|=R_k M_k^{2s}} (v_k(y)|y|) \rightarrow +\infty.$$

We have:

$$\Delta v_k^\lambda = V_k^\lambda (v_k^\lambda)^5.$$

We set:

$$w_\lambda = v_k - v_k^\lambda.$$

Then,

$$\Delta w_\lambda + \frac{n+2}{n-2} \xi^4 V_k w_\lambda = E_\lambda,$$

$$\text{with } E_\lambda = (V_k - V_k^\lambda)(v_k^\lambda)^5.$$

Clearly, we have the following lemma.

Lemma 1:

We have:

$$|E_\lambda| \leq A_k \times C(\lambda_1) M_k^{-2s} \lambda^5 |y|^{s-5} \leq C(\lambda_1) \lambda_1^s \times A_k M_k^{-2s} \lambda^{5-s} |y|^{s-5}.$$

Let

$$h_\lambda = -C(s, \lambda_1) A_k M_k^{-2s} \left[1 - \left(\frac{\lambda}{|y|} \right)^{4-s} \right].$$

Lemma 2:

$$\exists \lambda_0^k > 0 \text{ such that } w_\lambda + h_\lambda > 0 \text{ in } \Sigma_\lambda \forall 0 < \lambda \leq \lambda_0^k.$$

The proof of the lemma 2 is like the proof of the step 1 of the lemma 2 in [L-Z], we omit it here.

We set:

$$\lambda^k = \sup\{\lambda \leq \lambda_1, \text{ such that } w_\mu + h_\mu > 0 \text{ in } \Sigma_\mu \text{ for all } 0 < \mu \leq \lambda\}.$$

We have:

If $s \in]1/2, 1]$ then $|h_{\lambda^k}| R_k M_k^{2s} \leq C(s, \lambda_1) \sup_k A_k$, and thus,

$$w_{\lambda^k} + h_{\lambda^k} > 0 \quad \forall |y| = R_k M_k^{2s}.$$

If $s = \frac{1}{2}$, $\min_M u_k \geq m > 0$ and $A_k \rightarrow 0$, we obtain,

$$\min_{|y|=(2\lambda_1 M_k)/m} [v_k(y)|y|] \geq 2\lambda_1 > 0,$$

thus, for $|y| = \frac{2\lambda_1 M_k}{m}$ and k large we have:

$$w_{\lambda^k} + h_{\lambda^k} \geq \frac{[-\lambda^k v_k(y^\lambda) + 2\lambda_1 - C(\lambda_1, s)A_k]}{(2\lambda_1 M_k)/m} \geq \frac{m}{2\lambda_1 M_k} [-(1 + \epsilon)\lambda_1 - \epsilon\lambda_1 + 2\lambda_1] > 0,$$

where $\epsilon > 0$ is very small and $v_k(y^\lambda) \rightarrow U(0) = 1$.

For the case $s = \frac{1}{2}$, we work in $\Sigma_\lambda = B\left(0, \frac{2\lambda_1 M_k}{m}\right) - \bar{B}(0, \lambda)$. It is easy to see that, $\frac{2\lambda_1 M_k}{m} \ll R_k M_k^2$. We define λ^k as in the case $1/2 < s \leq 1$.

If we use the Hopf maximum principle, we prove that $\lambda^k = \lambda_1$ like in [L-Z]. We have the same contradiction as in [L-Z].

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