Faà di Bruno subalgebras of the Hopf algebra of planar trees from combinatorial Dyson-Schwinger equations

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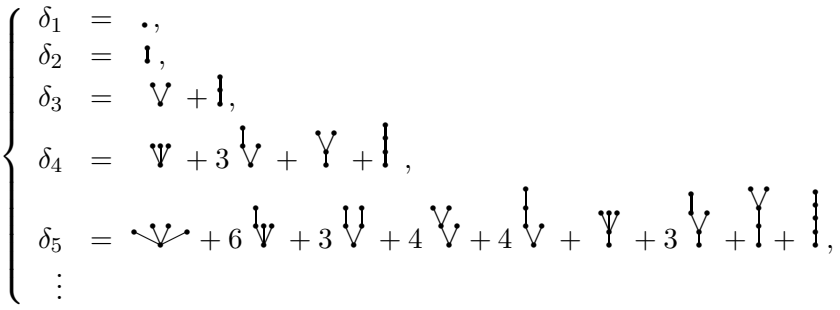
ABSTRACT. We consider the combinatorial Dyson-Schwinger equation $X = B^+(P(X))$ in the non-commutative Connes-Kreimer Hopf algebra of planar rooted trees $\mathcal{H}_{NCK}$, where B^+ is the operator of grafting on a root, and P a formal series. The unique solution X of this equation generates a graded subalgebra $\mathcal{A}_{N,P}$ of $\mathcal{H}_{NCK}$. We describe all the formal series P such that $\mathcal{A}_{N,P}$ is a Hopf subalgebra. We obtain in this way a 2-parameters family of Hopf subalgebras of $\mathcal{H}_{NCK}$, organized into three isomorphism classes: a first one, restricted to a polynomial ring in one variable; a second one, restricted to the Hopf subalgebra of ladders, isomorphic to the Hopf algebra of quasi-symmetric functions; a last (infinite) one, which gives a non-commutative version of the Faà di Bruno Hopf algebra. By taking the quotient, the last classe gives an infinite set of embeddings of the Faà di Bruno algebra into the Connes-Kreimer Hopf algebra of rooted trees. Moreover, we give an embedding of the free Faà di Bruno Hopf algebra on D variables into a Hopf algebra of decorated rooted trees, together with a non commutative version of this embedding.

Contents

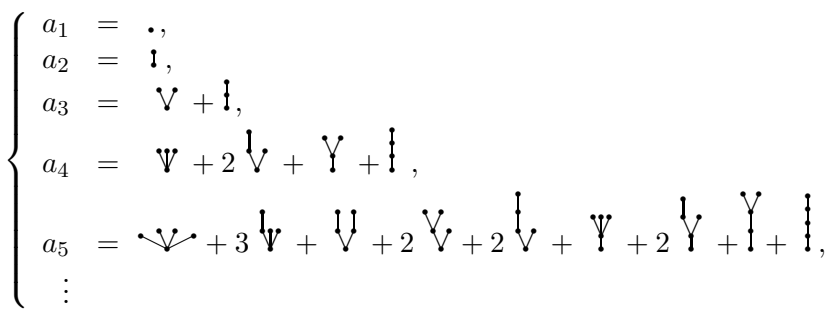
Introduction

The Connes-Kreimer Hopf algebra $\mathcal{H}_{CK}$ of rooted trees is introduced in [12]. It is commutative and not cocommutative. A particular Hopf subalgebra of $\mathcal{H}_{CK}$, namely the Connes-Moscovici subalgebra, is introduced in [5]. It is the subalgebra generated by the following elements:



The appearing coefficients, called Connes-Moscovici coefficients, are studied in [4, 7]. It is shown in [6] that the character group of this subalgebra is isomorphic to the group of formal diffeomorphisms, that is to say the group of formal series of the form $h + a_1h^2 + \ldots$, with composition. In other terms, the Connes-Moscovici subalgebra is isomorphic to the Hopf algebra of functions on the group of formal diffeomorphisms, also called the Faà di Bruno Hopf algebra.

A non commutative version $\mathcal{H}_{NCK}$ of the Connes-Kreimer Hopf algebra of trees is introduced in [9, 11]. It contains a non commutative version of the Connes-Moscovici subalgebra, described in [10]. Its abelianization can be identified with the subalgebra of $\mathcal{H}_{CK}$, here denoted by $\mathcal{A}_{1,1}$, generated by the following elements of $\mathcal{H}_{CK}$:



This subalgebra is different from the Connes-Moscovici subalgebra, but is also isomorphic to the Faà di Bruno Hopf algebra.

In this paper, we consider a family of subalgebras of $\mathcal{H}_{NCK}$, which give a non commutative version of the Faà di Bruno algebra. They are generated by a combinatorial Dyson-Schwinger equation [2, 15, 16]:

$$\mathbf{X}_P = B^+(P(\mathbf{X}_P)),$$

where B^+ is the operator of grafting on a common root, and $P = \sum p_k h^k$ is a formal series such that $p_0 = 1$. All this makes sense in a completion of $\mathcal{H}_{NCK}$, where this equation admits a unique solution $\mathbf{X}_P = \sum \mathbf{a}_k$, whose coefficients are inductively defined by:

$$\begin{cases} \mathbf{a}_1 &= \bullet, \\ \mathbf{a}_{n+1} &= \displaystyle\sum_{k=1}^{n} \sum_{\alpha_1+\ldots+\alpha_k=n} p_k B^+(\mathbf{a}_{\alpha_1} \ldots \mathbf{a}_{\alpha_k}), \end{cases}$$

For the usual Dyson-Schwinger equation, $P = \alpha(1-h)^{-1}$. We characterise the formal series P such that the associated subalgebra is Hopf: we obtain a two-parameters family $\mathcal{A}_{N,\alpha,\beta}$ of Hopf subalgebras of $\mathcal{H}_{NCK}$ and we explicitely describe the system of generator of these algebras.

We then characterise the equalities between the $\mathcal{A}_{N,\alpha,\beta}$'s and then their isomorphism classes. We obtain three classes:

1. $\mathcal{A}_{N,0,1}$, equal to $K[\bullet]$.
2. $\mathcal{A}_{N,1,-1}$, the subalgebra of ladders, isomorphic to the Hopf algebra of quasi-symmetric functions.
3. The $\mathcal{A}_{N,1,\beta}$'s, with $\beta \neq -1$, a non commutative version of the Faà di Bruno Hopf algebra.

By taking the quotient, we obtain three classes of Hopf subalgebras of $\mathcal{H}_{CK}$:

1. $\mathcal{A}_{0,1}$, equal to $K[\bullet]$.
2. $\mathcal{A}_{1,-1}$, the subalgebra of ladders, isomorphic to the Hopf algebra of symmetric functions.
3. The $\mathcal{A}_{1,\beta}$'s, with $\beta \neq -1$, isomorphic to the Faà di Bruno Hopf algebra.

We finally give an embedding of a non commutative version of the free Faà di Bruno on D variables (see [1]) in a Hopf algebra of planar rooted trees decorated by the set $\{1,\dots,D\}^3$. By taking the quotient, the free Faà di Bruno algebra appears as a subalgebra of a Hopf algebra of decorated rooted trees.

This text is organized as follows. The first section gives some recalls about the Hopf algebras of trees and the Faà di Bruno algebra. We define the subalgebras of $\mathcal{H}_{CK}$ and $\mathcal{H}_{NCK}$ associated to a formal series P in section 2 and also give here the main theorem (theorem 4), which characterizes the P's such that the associated subalgebras are Hopf. In section 3, we prove $2 \Longrightarrow 3$ of theorem 4. In section 4, we prove $4 \Longrightarrow 1$ of theorem 4. We also describe there the system of generators, and the case of equalities of the subalgebras. We describe the isomorphism classes of these subalgebras in the following section. In the last one, we consider the multivariable case.

Notations.

1. K is any field of characteristic zero.
2. Let $\lambda \in K$. We put:

$$g_\lambda(h) = (1-h)^{-\lambda} = \sum_{k=0}^{\infty} \frac{\lambda(\lambda+1)\dots(\lambda+k-1)}{k!} h^k = \sum_{k=0}^{\infty} Q_k(\lambda) h^k \in K[[h]].$$

1 Preliminaries

1.1 Valuation and n-adic topology

In this paragraph, let us consider a graded Hopf algebra $\mathcal{A}$. Let $\mathcal{A}_n$ be the homogeneous component of degree n of $\mathcal{A}$. For all $a \in \mathcal{A}$, we put:

$$val(a) = \max \left\{ n \in \mathbb{N} \,/\, a \in \bigoplus_{k \geq n} \mathcal{A}_k \right\} \in \mathbb{N} \cup \{+\infty\}.$$

For all $a, b \in \mathcal{A}$, we also put $d(a,b) = 2^{-val(a-b)}$, with the convention $2^{-\infty} = 0$. Then d is a distance on $\mathcal{A}$. The induced topology over $\mathcal{A}$ will be called the *n-adic topology*.

Let $\overline{\mathcal{A}}$ be the completion of $\mathcal{A}$ for this distance. In other terms:

$$\overline{\mathcal{A}} = \prod_{n=0}^{+\infty} \mathcal{A}_n.$$

The elements of $\overline{\mathcal{A}}$ are written in the form $\sum_{n=0}^{+\infty} a_n$, with $a_n \in \mathcal{A}_n$ for all n. Moreover, $\overline{\mathcal{A}}$ is naturally given a structure of associative algebra, by continuously extending the product of $\mathcal{A}$. The coproduct of $\mathcal{A}$ can also be extended in the following way:

$$\Delta : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}} \hat{\otimes} \overline{\mathcal{A}} = \prod_{i,j \in \mathbb{N}} \mathcal{A}_i \otimes \mathcal{A}_j.$$

For all $p = \sum_{k=0}^{+\infty} a_n h_n \in K[[h]]$, for all $a \in \overline{\mathcal{A}}$ such that $val(a) \geq 1$, we put:

$$p(a) = \sum_{k=0}^{+\infty} p_n a^n \in \overline{\mathcal{A}}.$$

Indeed, for all $n, m \in \mathbb{N}$, $val\left(\sum_{k=n}^{n+m} p_n a^n\right) \geq n$, so this series is Cauchy, and converges. It is an easy exercise to prove that for all $p, q \in K[[h]]$, such that q has no constant term, for all $a \in \overline{\mathcal{A}}$, with $val(a) \geq 1$, $(p \circ q)(a) = p(q(a))$.

1.2 The commutative Connes-Kreimer Hopf algebra of trees

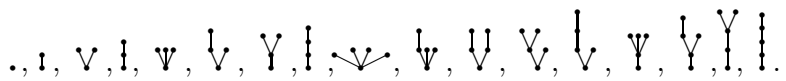
This Hopf algebra is introduced by Kreimer in [12] and studied for example in [3, 5, 7, 8, 13, 14].

Definition 1

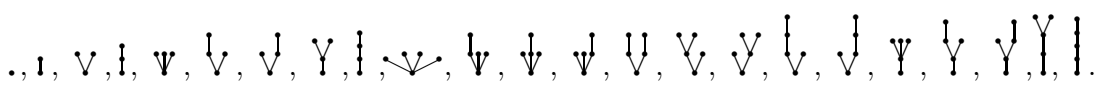
1. A *rooted tree* t is a finite, connected graph without loops, with a special vertex called *root*. The set of rooted trees will be denoted by $\mathbb{T}_{CK}$.
2. The *weight* of a rooted tree is the number of its vertices.
3. A *planar rooted tree* is a tree which is given an imbedding in the plane. The set of planar rooted trees will be denoted by $\mathbb{T}_{NCK}$.

Examples.

1. Rooted trees of weight ≤ 5:

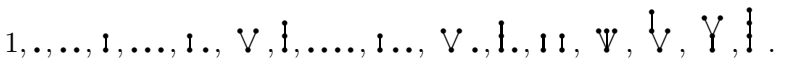


2. Planar rooted trees of weight ≤ 5:



The Connes-Kreimer Hopf algebra of rooted trees $\mathcal{H}_{CK}$ is the free associative commutative algebra freely generated over K by the elements of $\mathbb{T}_{CK}$. A linear basis of $\mathcal{H}_{CK}$ is given by *rooted forests*, that is to say monomials in rooted trees. The set of rooted forests will be denoted by $\mathbb{F}_{CK}$. The weight of a rooted forest $F = t_1 \dots t_n$ is the sum of the weights of the t_i's.

Examples. Rooted forests of weight ≤ 4:

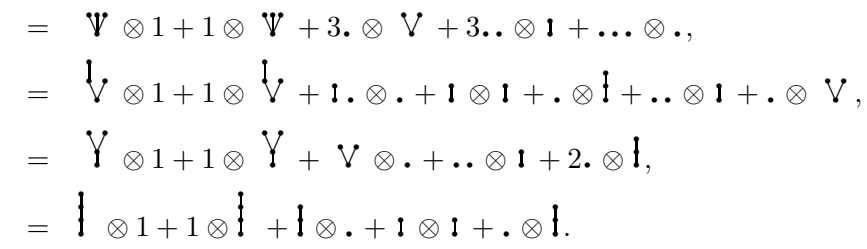


We now recall the Hopf algebra structure of $\mathcal{H}_{CK}$. An *admissible cut* of t is a non empty cut such that every path in the tree meets at most one cut edge. The set of admissible cuts of t is denoted by $\mathcal{A}dm(t)$. If c is an admissible cut of t, one of the trees obtained after the application of c contains the root of t: we shall denote it by $R^c(t)$. The product of the other trees will be denoted by $P^c(t)$. The coproduct of t is then given by :

$$\Delta(t) = t \otimes 1 + 1 \otimes t + \sum_{c \in \mathcal{A}dm(t)} P^c(t) \otimes R^c(t).$$

This coproduct is extended as an algebra morphism. Then $\mathcal{H}_{CK}$ becomes a Hopf algebra. Note that $\mathcal{H}_{CK}$ is given a gradation of Hopf algebra by the weight.

Examples.



We define the operator $B^+ : \mathcal{H}_{CK} \longrightarrow \mathcal{H}_{CK}$, that associates to a forest $F \in \mathbb{F}_{CK}$ the tree obtained by grafting the roots of the trees of F on a common root. For example, $B^+($$) =$. Then, for all $x \in \mathcal{H}_{CK}$:

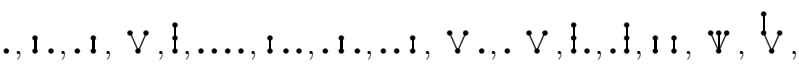
$$\Delta(B^+(x)) = B^+(x) \otimes 1 + (Id \otimes B^+) \circ \Delta(x). \tag{1}$$

This means that B^+ is a 1-cocycle for a certain cohomology of coalgebra, see [5] for more details. Moreover, this operator B^+ is homogeneous of degree 1, so is continuous. So it can be extended in an operator $B^+ : \overline{\mathcal{H}_{CK}} \longrightarrow \overline{\mathcal{H}_{CK}}$.

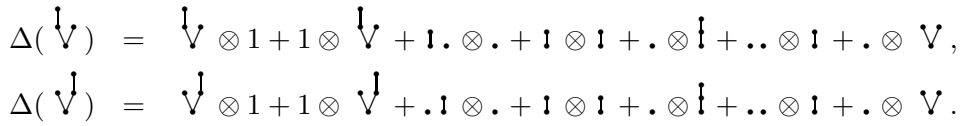
1.3 The non commutative Connes-Kreimer Hopf algebra of planar trees

This algebra is introduced simultaneously in [9, 11]. As an algebra, $\mathcal{H}_{NCK}$ is the free associative algebra generated by the elements of $\mathbb{T}_{NCK}$. A basis of $\mathcal{H}_{NCK}$ is given by planar rooted forests, that is to say words in elements of $\mathbb{T}_{NCK}$. The set of planar rooted forests will be denoted by $\mathbb{F}_{NCK}$.

Examples. Planar rooted forests of weight ≤ 4:



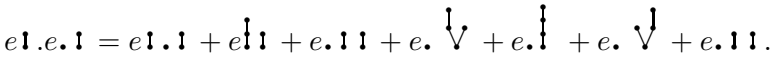
The coproduct of $\mathcal{H}_{NCK}$ is defined, as for $\mathcal{H}_{CK}$, with the help of admissible cuts. For example :



Note that $\mathcal{H}_{NCK}$ is a graded Hopf algebra, with a gradation given by the weight.

We define an operator, also denoted by $B^+ : \mathcal{H}_{NCK} \longrightarrow \mathcal{H}_{NCK}$, as for $\mathcal{H}_{CK}$. For example, $B^+(\bullet) =$  and $B^+(\bullet\bullet) =$  . Then (1) is also satisfied on $\mathcal{H}_{NCK}$. Moreover, this operator B^+ is homogeneous of degree 1, so is continuous. In consequence, it can be extended in an operator $B^+ : \overline{\mathcal{H}_{NCK}} \longrightarrow \overline{\mathcal{H}_{NCK}}$.

We proved in [9, 10] that $\mathcal{H}_{NCK}$ is a self-dual Hopf algebra: it has a non degenerate pairing denoted by $<,>$, and a dual basis $(e_F)_{F \in \mathbb{F}_{NCK}}$ of the basis of planar decorated forests. The product in the dual basis is given by graftings (in an extended sense). For example:



1.4 The Faà di Bruno Hopf algebra

Let $K[[h]]$ be the ring of formal series in one variable over K. We consider:

$$G = \left\{ h + \sum_{n \geq 1} a_n h^{n+1} \in K[[h]] \right\}.$$

This is a group for the composition of formal series. The Faà di Bruno Hopf algebra $\mathcal{H}_{FdB}$ is the Hopf algebra of functions on the opposite of the group G. More precisely, $\mathcal{H}_{FdB}$ is the polynomial ring in variables Y_i, with $i \in \mathbb{N}^*$, where Y_i is the function on G defined by:

$$Y_i : \left\{ \begin{array}{rcl} G & \longrightarrow & K \\ h + \sum_{n \geq 1} a_n h^{n+1} & \longrightarrow & a_i. \end{array} \right.$$

The coproduct is defined in the following way: for all $f \in \mathcal{H}_{FdB}$, for all $P, Q \in G$,

$$\Delta(f)(P \otimes Q) = f(Q \circ P).$$

This Hopf algebra is commutative and not cocommutative. It is also a graded, connected Hopf algebra, with Y_i homogeneous of degree i for all i. We put:

$$\mathbf{Y} = 1 + \sum_{n=1}^{\infty} Y_n \in \overline{\mathcal{H}_{FdB}}.$$

Then:

$$\Delta(\mathbf{Y}) = \sum_{n=1}^{\infty} \mathbf{Y}^{n+1} \otimes Y_n.$$

Indeed, with the convention $a_0 = b_0 = 1$:

$$\begin{aligned}\Delta(\mathbf{Y})\left(\left(\sum_{i\geq 0} a_i h^{i+1}\right)\otimes\left(\sum_{i\geq 0} b_i h^{i+1}\right)\right) &= \mathbf{Y}\left(\sum_{i\geq 0} b_i\left(\sum_{j\geq 0} a_j h^{j+1}\right)^{i+1}\right)\\ &= \sum_{i\geq 0} b_i\left(\sum_{j\geq 0} a_j\right)^{i+1},\\ \left(\sum_{n=1}^{\infty}\mathbf{Y}^{n+1}\otimes Y_n\right)\left(\left(\sum_{i\geq 0} a_i h^{i+1}\right)\otimes\left(\sum_{i\geq 0} b_i h^{i+1}\right)\right) &= \sum_{n=1}^{\infty}\left(\sum_{i\geq 0} a_i\right)^{n+1} b_n.\end{aligned}$$

The graded dual $\mathcal{H}_{FdB}^*$ is an enveloping algebra, by the Cartier-Quillen-Milnor-Moore theorem. A basis of $Prim(\mathcal{H}_{FdB}^*)$ is given by $(Z_i)_{i\in\mathbb{N}^*}$, where:

$$Z_i : \left\{\begin{array}{rcl} \mathcal{H}_{FdB} & \longrightarrow & K\\ Y_1^{\alpha_1}\dots Y_k^{\alpha_k} & \longrightarrow & 0 \text{ if } \alpha_1+\ldots+\alpha_k\neq 1,\\ Y_j & \longrightarrow & \delta_{i,j}.\end{array}\right.$$

By homogeneity, for all $i,j\in\mathbb{N}^*$, there exists a coefficient $\lambda_{i,j}\in K$ such that $[Z_i,Z_j]=\lambda_{i,j}Z_{i+j}$. Moreover:

$$\begin{aligned}[Z_i,Z_j](\mathbf{Y}) &= \lambda_{i,j}\\ &= (Z_i\otimes Z_j - Z_j\otimes Z_i)\circ\Delta(\mathbf{Y})\\ &= (Z_i\otimes Z_j - Z_j\otimes Z_i)\left(\sum_{n=1}^{\infty}\mathbf{Y}^{n+1}\otimes Y_n\right)\\ &= Z_i(\mathbf{Y}^{j+1}) - Z_j(\mathbf{Y}^{i+1})\\ &= (j+1)-(i+1)\\ &= j-i.\end{aligned}$$

So the bracket of $Prim(\mathcal{H}_{FdB}^*)$ is given by:

$$[Z_i,Z_j]=(j-i)Z_{i+j}.$$

2 Subalgebras associated to a formal series

2.1 Construction

We denote by $K[[h]]_1$ the set of formal series of $K[[h]]$ with constant term equal to 1.

Proposition 2 *Let $P\in K[[h]]_1$.*

1. *There exists a unique element $X_P=\sum_{k\geq 1} a_n\in\overline{\mathcal{H}_{CK}}$, such that $X_P=B^+\left(P(X_P)\right)$.*
2. *There exists a unique element $\mathbf{X}_P=\sum_{k\geq 1} a_n\in\overline{\mathcal{H}_{NCK}}$, such that $\mathbf{X}_P=B^+\left(P(\mathbf{X}_P)\right)$.*

Proof.

1. *Unicity.* We put $X_P = \sum_{n\geq 1} a_n$, with a_n homogeneous of degree n for all n. Then the a_n's satisfy the following equations:

$$\left\{\begin{array}{rcl} a_1 & = & \bullet, \\ a_{n+1} & = & \sum_{k=1}^{n} \sum_{\alpha_1+\ldots+\alpha_k=n} p_k B^+(a_{\alpha_1}\ldots a_{\alpha_k}). \end{array}\right. \tag{2}$$

Hence, the a_n's are uniquely defined.

Existence. The a_n's defined inductively by (2) satisfy the required condition.

2. We put $\mathbf{X}_P = \sum_{n\geq 1} \mathbf{a}_n$, $\mathbf{a}_n$ homogeneous of degree n for all n. Then the $\mathbf{a}_n$'s satisfy the following equations:

$$\left\{\begin{array}{rcl} \mathbf{a}_1 & = & \bullet, \\ \mathbf{a}_{n+1} & = & \sum_{k=1}^{n} \sum_{\alpha_1+\ldots+\alpha_k=n} p_k B^+(\mathbf{a}_{\alpha_1}\ldots \mathbf{a}_{\alpha_k}). \end{array}\right. \tag{3}$$

The end of the proof is similar. □

Definition 3 Let $P \in K[[h]]_1$.

1. The subalgebra $\mathcal{A}_P$ of $\mathcal{H}_{CK}$ is the subalgebra generated by the a_n's.
2. The subalgebra $\mathcal{A}_{N,P}$ of $\mathcal{H}_{NCK}$ is the subalgebra generated by the $\mathbf{a}_n$'s.

Remarks.

1. $\mathcal{A}_P$ is a graded subalgebra of $\mathcal{H}_{CK}$, and $\mathcal{A}_{N,P}$ is a graded subalgebra of $\mathcal{H}_{NCK}$.
2. For all $n \in \mathbb{N}^*$, a_n is an element of $Vect(\mathbb{T}_{CK})$. Hence:

$$Vect(\mathbb{T}_{CK}) \cap \mathcal{A}_P = Vect(a_n,\ n \in \mathbb{N}^*). \tag{4}$$

The same holds in the non commutative case.

2.2 Main theorem

One of the aim of this paper is to prove the following theorem:

Theorem 4 *Let $P \in K[[h]]_1$. The following assertions are equivalent:*

1. *$\mathcal{A}_{N,P}$ is a Hopf subalgebra of $\mathcal{H}_{NCK}$.*
2. *$\mathcal{A}_P$ is a Hopf subalgebra of $\mathcal{H}_{CK}$.*
3. *There exists $(\alpha, \beta) \in K^2$, such that P satisfies the following differential system:*

$$S_{\alpha,\beta} : \left\{\begin{array}{rcl} (1-\alpha\beta h)P'(h) & = & \alpha P(h) \\ P(0) & = & 1. \end{array}\right.$$

4. *There exists $(\alpha, \beta) \in K^2$, such that:*
   (a) *$P(h) = 1$ if $\alpha = 0$.*
   (b) *$P(h) = e^{\alpha h}$ if $\beta = 0$.*
   (c) *$P(h) = (1-\alpha\beta h)^{-\frac{1}{\beta}}$ if $\alpha\beta \neq 0$.*

An easy computation proves the equivalence between assertions 3 and 4. Moreover, using the Hopf algebra morphism $\varpi : \mathcal{H}_{NCK} \longrightarrow \mathcal{H}_{CK}$, defined by forgetting the planar data, it is clear that $\mathcal{A}_P = \varpi(\mathcal{A}_{N,P})$. So, assertion 1 implies assertion 2.

3 When is $\mathcal{A}_P$ a Hopf subalgebra?

3.1 Preliminary results

The aim of this section is to show $2 \Longrightarrow 3$ in theorem 4.

Lemma 5 *Suppose that $\mathcal{A}_P$ is a Hopf subalgebra. Then two cases are possible:*

1. *$P = 1$. In this case, $X_P = \bullet$ and $\mathcal{A}_P = K[\bullet]$.*

2. *$p_1 \neq 0$. In this case, $a_n \neq 0$ for all $n \geq 1$.*

Proof. Suppose first that $p_1 = 0$, and suppose that there exists $n \geq 2$ such that $p_n \neq 0$. Let us choose n minimal. Then, by (2), $a_2 = \ldots = a_n = 0$ and $a_{n+1} = p_n B^+(\bullet^n)$. Then:

$$\Delta(B^+(\bullet^n)) = B^+(\bullet^n) \otimes 1 + \sum_{k=0}^{n} \binom{n}{k} \bullet^k \otimes B^+(\bullet^{n-k}) \in \mathcal{A}_P \otimes \mathcal{A}_P,$$

which implies that $B^+(\bullet) = \mathbb{1} \in \mathcal{A}_P \cap Vect(\mathbb{T}_{CK}) = Vect(a_n)$, so $a_2 \neq 0$: contradiction. So $P = 1$ and $X_P = \bullet$.

Suppose $p_1 \neq 0$. By (2), the canonical projection of a_{n+1} on $Im((B^+)^2)$ (vector space of trees such that the root has only one child) is $p_1 B^+(a_n)$ for all $n \geq 1$. Hence, for all $n \geq 1$, $a_{n+1} = 0 \Rightarrow a_n = 0$. So for all $n \geq 1$, $a_n \neq 0$. $\square$

We put:

$$Z : \left\{ \begin{array}{rcl} \mathcal{H}_{CK} & \longrightarrow & K \\ F \in \mathbb{F}_{CK} & \longrightarrow & \delta_{\bullet,F}. \end{array} \right.$$

Note that Z is an element of the graded dual $\mathcal{H}_{CK}^*$. Moreover, Z can be extended to $\overline{\mathcal{H}_{CK}}$, and satisfies, for all $a, b \in \overline{\mathcal{H}_{CK}}$:

$$Z(ab) = Z(a)\varepsilon(b) + \varepsilon(a)Z(b).$$

Lemma 6 *Let $P \in K[[h]]_1$. If $\mathcal{A}_P$ is a Hopf subalgebra of $\mathcal{H}_{CK}$, then:*

$$(Z \otimes Id) \circ \Delta(X_P) \in \overline{\mathcal{A}_P}.$$

Proof. As $\mathcal{A}_P$ is a Hopf subalgebra, for all $n \in \mathbb{N}^*$, $\Delta(a_n) \in \mathcal{A}_P \otimes \mathcal{A}_P$. Hence, $(Z \otimes Id) \circ \Delta(a_n) \in \mathcal{A}_P$. $\square$

Lemma 7 *We consider the following continuous applications:*

$$\left\{ \begin{array}{rcccccc} \tilde{Z} & = & Z\hat{\otimes}Id & : & \overline{\mathcal{H}_{CK}}\hat{\otimes}\overline{\mathcal{H}_{CK}} & \longrightarrow & \overline{\mathcal{H}_{CK}}, \\ \tilde{\varepsilon} & = & \varepsilon\hat{\otimes}Id & : & \overline{\mathcal{H}_{CK}}\hat{\otimes}\overline{\mathcal{H}_{CK}} & \longrightarrow & \overline{\mathcal{H}_{CK}}. \end{array} \right.$$

Then $\tilde{\varepsilon}$ is an algebra morphism and $\tilde{Z}$ is a $\tilde{\varepsilon}$-derivation, i.e. satisfies:

$$\tilde{Z}(ab) = \tilde{Z}(a)\tilde{\varepsilon}(b) + \tilde{\varepsilon}(a)\tilde{Z}(b).$$

Proof. Immediate. $\square$

Let us fix $t \in \mathbb{T}_{CK}$. We put $P(h) = \sum_{n=0}^{+\infty} p_n h^n$. As $X_P = B^+(P(X_P))$, we have:

$$\begin{aligned}
\tilde{Z} \circ \Delta(X_P) &= \sum_{n=0}^{+\infty} p_n (Z \otimes Id) \circ \Delta \circ B^+(X_P^n) \\
&= \sum_{n=0}^{+\infty} p_n Z(B^+(X_P^n))1 + \sum_{n=0}^{+\infty} p_n (Z \otimes B^+)(\Delta(X_P))^n \\
&= Z(X_P)1 + B^+\left(\sum_{n=0}^{+\infty} p_n \tilde{Z}(\Delta(X_P)^n)\right) \\
&= Z(X_P)1 + B^+\left(\sum_{n=0}^{+\infty} n p_n \tilde{\varepsilon}(\Delta(X_P))^{n-1} \tilde{Z}(\Delta(X_P))\right) \\
&= Z(X_P)1 + B^+\left(\sum_{n=0}^{+\infty} n p_n X_P^{n-1} \tilde{Z}(\Delta(X_P))\right) \\
&= Z(X_P)1 + B^+\left(P'(X_P).(Z \otimes Id) \circ \Delta(X_P))\right)
\end{aligned}$$

We consider the following linear application:

$$L_P : \left\{ \begin{array}{rcl} \overline{\mathcal{H}_{CK}} & \longrightarrow & \overline{\mathcal{H}_{CK}} \\ a & \longrightarrow & B^+(P'(X_P)a). \end{array} \right.$$

Then, immediately, for all $a \in \overline{\mathcal{H}_{CK}}$, $val(L_P(a)) \geq val(a)+1$, so $Id - L_P$ is invertible. Moreover, by the preceding computation:

$$\begin{aligned}
& \tilde{Z} \circ \Delta(X_P) = Z(X_P)1 + L_P((Z \otimes Id) \circ \Delta(X_P)) \\
\iff\ & (Id - L_P)((Z \otimes Id) \circ \Delta(X_P)) = Z(X_P)1 \\
\iff\ & \tilde{Z} \circ \Delta(X_P) = Z(X_P)(Id - L_P)^{-1}(1).
\end{aligned}$$

Hence, as $Z(X_P) = 1$, lemma 6 induces the following result:

Proposition 8 *Let $P \in K[[h]]_1$. If $\mathcal{A}_P$ is a Hopf subalgebra of $\mathcal{H}_{CK}$, then:*

$$(Id - L_P)^{-1}(1) \in \mathcal{A}_P.$$

3.2 Proof of $2 \Longrightarrow 3$

We put $Y = \sum_{k=0}^{+\infty} b_n = (Id - L_P)^{-1}(1)$. Then b_n can be inductively computed in the following way:

$$\left\{ \begin{array}{rcl} b_0 &=& 1, \\ b_{n+1} &=& \displaystyle\sum_{k=1}^{n} \sum_{\alpha_1 + \ldots + \alpha_k = n} (k+1) p_{k+1} B^+(a_{\alpha_1} \ldots a_{\alpha_k}) \\ && \displaystyle + \sum_{k=1}^{n} \sum_{\alpha_1 + \ldots + \alpha_k = n} k p_k B^+(b_{\alpha_1} a_{\alpha_2} \ldots a_{\alpha_k}). \end{array} \right. \tag{5}$$

In particular, $b_1 = p_1 \bullet$.

Suppose that $\mathcal{A}_P$ is a Hopf subalgebra. Then $b_n \in (\mathcal{A}_P \cap Vect(\mathbb{T}_{CK}))_n = Vect(a_n)$ for all $n \geq 1$, so there exists $\alpha_n \in K$, such that $b_n = \alpha_n a_n$. Let us compare the projection on $Im((B^+)^2)$ of a_{n+1} and b_{n+1}:

$$\left\{ \begin{array}{rl} p_1 B^+(a_n) & \text{for } a_{n+1}, \\ 2p_2 B^+(a_n) + p_1 B^+(b_n) = (2p_2 + p_1\alpha_n) B^+(a_n) & \text{for } b_{n+1}. \end{array} \right.$$

Suppose that $p_1 \neq 0$. Then the a_n's are all non zero by lemma 5, so α_n is uniquely determined for all $n \in \mathbb{N}^*$. We then obtain, by comparing the projections of a_{n+1} and b_{n+1} over $Im((B^+)^2)$:

$$\left\{ \begin{array}{rcl} \alpha_1 & = & p_1, \\ \alpha_{n+1} & = & 2\dfrac{p_2}{p_1} + \alpha_n. \end{array} \right.$$

Hence, for all $n \in \mathbb{N}^*$, $\alpha_n = p_1 + 2\dfrac{p_2}{p_1}(n-1)$.

Let us compare the coefficient of $B^+(\bullet^n)$ in a_{n+1} and in b_{n+1} with (2) and (5). we obtain:

$$\left\{ \begin{array}{rl} p_n & \text{for } a_{n+1}, \\ (n+1)p_{n+1} + np_n p_1 & \text{for } b_{n+1}. \end{array} \right.$$

Hence, $\alpha_{n+1} p_n = (n+1)p_{n+1} + np_n p_1$ for all $n \geq 1$. As a consequence:

$$(n+1)p_{n+1} + \left(p_1 - 2\frac{p_2}{p_1}\right) np_n = p_1 p_n.$$

This property is still true for $n = 0$, as $p_0 = 1$. By multiplying by h^n and taking the sum:

$$P'(h) + \left(p_1 - 2\frac{p_2}{p_1}\right) hP'(h) = p_1 P(h).$$

We then put $\alpha = p_1$ and $\beta = 2\dfrac{p_2}{p_1^2} - 1$. Hence:

$$(1 - \alpha\beta h)P'(h) = \alpha P(h).$$

This equality is still true if $p_1 = 0$, with $\alpha = 0$ and any β. Hence, we have shown:

Proposition 9 *If $\mathcal{A}_P$ is a Hopf subalgebra of $\mathcal{H}_{CK}$, then there exists $(\alpha, \beta) \in K^2$, such that P satisfies the following differential system:*

$$S_{\alpha,\beta} : \left\{ \begin{array}{rcl} (1 - \alpha\beta h)P'(h) & = & \alpha P(h) \\ P(0) & = & 1. \end{array} \right.$$

This implies:

1. $P(h) = 1$ if $\alpha = 0$.
2. $P(h) = e^{\alpha h}$ if $\beta = 0$.
3. $P(h) = (1 - \alpha\beta h)^{-\frac{1}{\beta}}$ if $\alpha\beta \neq 0$.

4 Is $\mathcal{A}_{N,\alpha,\beta}$ a Hopf subalgebra?

The aim of this section is to prove in $4 \Longrightarrow 1$ in theorem 4.

4.1 Generators of $\mathcal{A}_{N,\alpha,\beta}$.

We denote by $P_{\alpha,\beta}$ the solution of $S_{\alpha,\beta}$ and we put $\mathcal{A}_{\alpha,\beta} = \mathcal{A}_{P_{\alpha,\beta}}$ and $\mathcal{A}_{N,\alpha,\beta} = \mathcal{A}_{N,P_{\alpha,\beta}}$, in order to simplify the notations. We also put $\mathbf{X}_{P_{\alpha,\beta}} = \sum_{n\in\mathbb{N}} \mathbf{a}_n(\alpha,\beta)$ and $P_{\alpha,\beta} = \sum_{n=0}^{+\infty} p_n(\alpha,\beta)h^n$. The system $S_{\alpha,\beta}$ is equivalent to:

$$S'_{\alpha,\beta} : \left\{ \begin{array}{rcl} p_0(\alpha,\beta) & = & 1, \\ p_{n+1}(\alpha,\beta) & = & \alpha\dfrac{1+n\beta}{n+1} p_n(\alpha,\beta) \text{ for all } n \in \mathbb{N}. \end{array} \right.$$

Definition 10

1. For all $i \in \mathbb{N}^*$, we put $[i]_\beta = (1 + \beta(i-1))$. In particular, $[i]_1 = i$ et $[i]_0 = 1$.
2. We put $[i]_\beta! = [1]_\beta \dots [i]_\beta$. In particular, $[i]_1! = i!$ et $[i]_0! = 1$. We also put $[0]_\beta! = 1$.

Immediately, for all $n \in \mathbb{N}$:

$$p_n(\alpha,\beta) = \alpha^n \frac{[n]_\beta!}{n!}.$$

For all $F \in \mathbb{F}_{CK}$, we define the coefficient $F!$ by:

$$F! = \prod_{s \text{ vertex of } F} (\text{fertility of } s)!.$$

Note that these are not the coefficients $F!$ defined in [4, 7, 19]. They can be inductively defined by:

$$\left\{ \begin{array}{rcl} \bullet! & = & 1, \\ (t_1 \dots t_k)! & = & t_1! \dots t_k!, \\ B^+(F)! & = & k!F! \end{array} \right.$$

In a similar way, we define the following coefficients:

$$[F]_\beta! = \prod_{s \text{ vertex of } F} [\text{fertility of } s]_\beta!.$$

They can also be inductively defined:

$$\left\{ \begin{array}{rcl} [\bullet]_\beta! & = & 1, \\ [t_1 \dots t_k]_\beta! & = & [t_1]_\beta! \dots [t_k]_\beta!, \\ [B^+(F)]_\beta! & = & [k]_\beta! [F]_\beta! \end{array} \right.$$

In particular, for all forest F, $[F]_1! = F!$ and $[F]_0! = 1$.

Finally, for all $n \in \mathbb{N}^*$, we put $\mathbf{a}_n(\alpha,\beta) = \sum_{t\in\mathbb{T}_{NCK},\, |t|=n} \mathbf{a}_t(\alpha,\beta)t$.

Theorem 11 *For any tree t,*

$$\mathbf{a}_t(\alpha,\beta) = \alpha^{|t|-1} \frac{[t]_\beta!}{t!}.$$

In particular, $\mathbf{a}_t(1,0) = \frac{1}{t!}$, $\mathbf{a}_t(1,1) = 1$ *and* $\mathbf{a}_t(0,\beta) = \delta_{t,\bullet}$ *for all* β. *Moreover:*

$$\mathbf{a}_t(1,-1) = \left\{ \begin{array}{ll} 0 & \textit{if } t \textit{ is not a ladder,} \\ 1 & \textit{if } t \textit{ is a ladder.} \end{array} \right.$$

Proof. Induction on $|t|$. If $|t|=1$, then $t=\bullet$ and $\mathbf{a}_t(\alpha,\beta)=1$. Suppose the result true for all tree of weight strictly smaller than $|t|$. Then, with $t=B^+(t_1\dots t_k)$, by (3):

$$\begin{aligned}\mathbf{a}_t(\alpha,\beta) &= \alpha^{|t_1|-1+\dots+|t_k|-1}\frac{[t_1]_\beta!\dots[t_k]_\beta!}{t_1!\dots t_k!}\alpha^k\frac{[k]_\beta!}{k!}\\ &= \alpha^{|t_1|+\dots+|t_k|}\frac{[t]_\beta!}{t!}\\ &= \alpha^{|t|-1}\frac{[t]_\beta!}{t!}.\end{aligned}$$

The formulas for $(\alpha,\beta)=(1,0)$, $(1,1)$ and $(0,\beta)$ are easily deduced. Finally, for $(\alpha,\beta)=(1,-1)$, it is enough to observe that $[1]_\beta!=1$ and $[k]_\beta!=0$ if $k\geq 2$. $\square$

Examples.

$\mathbf{a}_1(\alpha,\beta) = \bullet$

$\mathbf{a}_2(\alpha,\beta) = \alpha$

$\mathbf{a}_3(\alpha,\beta) = \alpha^2\left(\frac{(1+\beta)}{2}\right.$ $+$ $\left.\right)$

$\mathbf{a}_4(\alpha,\beta) = \alpha^3\left(\frac{(1+2\beta)(1+\beta)}{6}\right.$ $+\frac{(1+\beta)}{2}$  $+\frac{(1+\beta)}{2}$  $+\frac{(1+\beta)}{2}$  $+$  $\left.\right)$

$\mathbf{a}_5(\alpha,\beta) = \alpha^4\left(\right.$ $\frac{(1+3\beta)(1+2\beta)(1+\beta)}{24}$  $+\frac{(1+2\beta)(1+\beta)}{6}$  $+\frac{(1+2\beta)(1+\beta)}{6}$  $+\frac{(1+2\beta)(1+\beta)}{6}$ $+\frac{(1+\beta)^2}{4}$  $+\frac{(1+\beta)^2}{4}$  $+\frac{(1+\beta)}{2}$  $+\frac{(1+\beta)}{2}$  $+\frac{(1+2\beta)(1+\beta)}{6}$ $+\frac{(1+\beta)}{2}$ $+\frac{(1+\beta)}{2}$ $+\frac{(1+\beta)}{2}$ $+\frac{(1+\beta)}{2}$ $+$  $\left.\right)$

In particular, $\mathbf{a}(1,1)$ is the sum of all planar trees of weight n, so $\mathcal{A}_{N,1,1}$ is the subalgebra of formal diffeomorphisms described in [10]. Moreover, $\mathbf{a}(1,-1)$ is the ladder of weight n, so $\mathcal{A}_{N,1,-1}$ is the subalgebra of ladders of $\mathcal{H}_{NCK}$.

4.2 Equalities of the subalgebras $\mathcal{A}_{N,P}$

Lemma 12 *Let $P,Q\in K[[h]]_1$. Suppose that $Q(h)=P(\gamma h)$ for a certain γ. We denote $\mathbf{X}_P=\sum_{n\geq 1}\mathbf{a}_n$. Then $\mathbf{X}_Q=\sum_{n\geq 1}\gamma^{n-1}\mathbf{a}_n$. In particular, if $\gamma\neq 0$, $\mathcal{A}_{N,P}=\mathcal{A}_{N,Q}$.*

Proof. We put $\mathbf{Y}=\sum_{n\geq 1}\gamma^{n-1}a_n$. Then:

$$\begin{aligned}B^+(Q(\mathbf{Y})) &= \sum_{n\in\mathbb{N}}\sum_{k=1}^{n}\sum_{n_1+\dots+n_k=n}\gamma^k p_k B^+(\gamma^{n_1-1}\mathbf{a}_{n_1}\dots\gamma^{n_k-1}\mathbf{a}_{n_k})\\ &= \sum_{n\in\mathbb{N}}\sum_{k=1}^{n}\sum_{n_1+\dots+n_k=n}\gamma^{k+n-k} p_k B^+(\mathbf{a}_{n_1}\dots\mathbf{a}_{n_k})\\ &= \sum_{n\in\mathbb{N}}\gamma^n\sum_{k=1}^{n}\sum_{n_1+\dots+n_k=n} p_k B^+(\mathbf{a}_{n_1}\dots\mathbf{a}_{n_k})\\ &= \sum_{n\in\mathbb{N}}\gamma^n\mathbf{a}_{n+1}\\ &= \mathbf{Y}.\end{aligned}$$

By unicity, $\mathbf{Y}=\mathbf{X}_Q$. $\square$

Theorem 13 *Let (α, β) and $(\alpha', \beta') \in K^2$. The following assertions are equivalent:*

1. $\mathcal{A}_{N,\alpha,\beta} = \mathcal{A}_{N,\alpha',\beta'}$.

2. $\mathcal{A}_{\alpha,\beta} = A_{\alpha',\beta'}$.

3. *($\beta = \beta'$ and $\alpha\alpha' \neq 0$) or ($\alpha = \alpha' = 0$).*

Proof.
$1 \Longrightarrow 2$. Obvious.

$2 \Longrightarrow 3$. By theorem 11:

$$\begin{cases} a_1 = \bullet, & a_2 = \alpha\, \overset{\bullet}{|}, & a_3 = \alpha^2 \overset{\bullet}{|}\overset{\bullet}{|} + \alpha^2 \frac{(1+\beta)}{2}\, \overset{\bullet}{\vee}; \\ a'_1 = \bullet, & a'_2 = \alpha'\, \overset{\bullet}{|}, & a'_3 = \alpha'^2 \overset{\bullet}{|}\overset{\bullet}{|} + \alpha'^2 \frac{(1+\beta')}{2}\, \overset{\bullet}{\vee}. \end{cases}$$

As $\mathcal{A}_{\alpha,\beta} = \mathcal{A}_{\alpha',\beta'}$, there exists $\gamma \neq 0$, sucht that $\alpha' \overset{\bullet}{|} = \gamma\alpha \overset{\bullet}{|}$. Hence, $\alpha' = \gamma\alpha$. In particular, if $\alpha = 0$, then $\alpha' = 0$. Suppose that $\alpha \neq 0$. As a_3 and a'_3 are colinear, the following determinant is zero:

$$\begin{vmatrix} \alpha^2 & \alpha^2 \frac{(1+\beta)}{2} \\ \alpha'^2 & \alpha'^2 \frac{(1+\beta')}{2} \end{vmatrix} = \frac{1}{2}\alpha^2\alpha'^2(\beta' - \beta) = 0.$$

As α and α' are non zero, $\beta = \beta'$.

$3 \Longrightarrow 1$. Suppose first $\alpha = \alpha' = 0$. Then $P_{\alpha,\beta} = P_{\alpha',\beta'} = 1$, so $\mathcal{A}_{N,\alpha,\beta} = \mathcal{A}_{N,\alpha',\beta'}$. Suppose $\beta = \beta'$ and $\alpha\alpha' \neq 0$. Then there exists $\gamma \in K - \{0\}$, such that $\alpha = \gamma\alpha'$. Then, immediately, $P_{\alpha,\beta}(\gamma h) = P_{\alpha',\beta'}(h)$. By the preceding lemma, $\mathcal{A}_{N,\alpha,\beta} = \mathcal{A}_{N,\alpha',\beta'}$. □

4.3 The $\mathcal{A}_{N,\alpha,\beta}$'s are Hopf subalgebras

We now prove $4 \Longrightarrow 1$ in theorem 4. If $\alpha = 0$, then $\mathcal{A}_{N,\alpha,\beta} = K[\bullet]$ and it is obvious. We take $\alpha \neq 0$. By theorem 13, we can suppose that $\alpha = 1$.

Lemma 14 *Let $k, n \in \mathbb{N}^*$. We consider the following element of $K[X_1, \ldots, X_n]$:*

$$P_k(X_1, \ldots, X_n) = \sum_{\alpha_1 + \ldots + \alpha_n = k} \frac{X_1(X_1+1)\ldots(X_1+\alpha_1-1)}{\alpha_1!} \ldots \frac{X_n(X_n+1)\ldots(X_n+\alpha_n-1)}{\alpha_n!}.$$

By putting $S = X_1 + \ldots + X_n$:

$$P_k(X_1, \ldots, X_n) = \frac{S(S+1)\ldots(S+k-1)}{k!}.$$

Proof. Induction on k. This is obvious for $k=1$. Suppose the result true at rank k. Then:

$$\begin{aligned}
& P_k(X_1,\dots,X_n)(X_1+\dots+X_n+k) \\
= & \sum_{\alpha_1+\dots+\alpha_n=k} \sum_{i=1}^{n} \begin{pmatrix} \frac{X_1(X_1+1)\dots(X_1+\alpha_1-1)}{\alpha_1!} \\ \vdots \\ \frac{X_i(X_i+1)\dots(X_i+\alpha_i)}{\alpha_i!} \\ \vdots \\ \frac{X_n(X_n+1)\dots(X_n+\alpha_n-1)}{\alpha_n!} \end{pmatrix} \\
= & \sum_{\alpha'_1+\dots+\alpha'_n=k+1} \sum_{i=1}^{n} \alpha'_i \begin{pmatrix} \frac{X_1(X_1+1)\dots(X_1+\alpha'_1-1)}{\alpha'_1!} \\ \vdots \\ \frac{X_i(X_i+1)\dots(X_i+\alpha'_i-1)}{\alpha'_i!} \\ \vdots \\ \frac{X_n(X_n+1)\dots(X_n+\alpha'_n-1)}{\alpha'_n!} \end{pmatrix} \\
= & (k+1) \sum_{\alpha'_1+\dots+\alpha'_n=k+1} \begin{pmatrix} \frac{X_1(X_1+1)\dots(X_1+\alpha'_1-1)}{\alpha'_1!} \\ \vdots \\ \frac{X_i(X_i+1)\dots(X_i+\alpha'_i-1)}{\alpha'_i!} \\ \vdots \\ \frac{X_n(X_n+1)\dots(X_n+\alpha'_n-1)}{\alpha'_n!} \end{pmatrix} \\
= & (k+1)P_{k+1}(X_1,\dots,X_n).
\end{aligned}$$

This implies the announced result. □

Let $F=t_1\dots t_k$ be a forest and t be a tree. Using the dual basis $(e_F)_{F\in\mathbb{F}_{NCK}}$:

$$\begin{aligned}
\text{coefficient of } F\otimes t \text{ in } \Delta(\mathbf{X}_{1,\beta}) &= < e_F\otimes e_t, \Delta(\mathbf{X}_{1,\beta}) > \\
&= < e_F e_t, \mathbf{X}_{1,\beta} > \\
&= \sum_{s \text{ grafting of } F \text{ on } t} < e_s, \mathbf{X}_{1,\beta} > \\
&= \sum_{s \text{ tree, grafting of } F \text{ on } t} \frac{[s]_\beta!}{s!}.
\end{aligned}$$

Let n be the weight of t and $s_1,\dots,s_n$ its vertices. Let f_i be the fertility of s_i. Let $(\alpha_1,\dots,\alpha_n)$ such that $\alpha_1+\dots+\alpha_n=k$ and consider the graftings of F on t such that α_i trees of F are grafted on s_i for all i. Then:

1. If s is such a grafting, we have:

$$\begin{cases}
[s]_\beta! &= [t]_\beta![t_1]_\beta!\dots[t_k]_\beta!\dfrac{[f_1+\alpha_1]_\beta!}{[f_1]_\beta!}\dots\dfrac{[f_n+\alpha_n]_\beta!}{[f_n]_\beta!}, \\
s! &= t!t_1!\dots t_k!\dfrac{(f_1+\alpha_1)!}{f_1!}\dots\dfrac{(f_n+\alpha_n)!}{f_n!}.
\end{cases}$$

2. The number of such graftings is:

$$\binom{f_1+\alpha_1}{\alpha_1}\dots\binom{f_n+\alpha_n}{\alpha_n}.$$

Hence, by putting $x_i = f_i + 1/\beta$ and $s = x_1 + \ldots + x_n$, by lemma 14:

$$\begin{aligned}
& \text{coefficient of } F \otimes t \text{ in } \Delta(\mathbf{X}_{1,\beta}) \\
= & \sum_{\alpha_1+\ldots+\alpha_n=k} \frac{[t]_\beta!}{t!}\frac{[t_1]_\beta!}{t_1!}\cdots\frac{[t_k]_\beta!}{t_k!}\frac{[f_1+\alpha_1]_\beta!}{[f_1]_\beta!\alpha_1!}\cdots\frac{[f_n+\alpha_n]_\beta!}{[f_n]_\beta!\alpha_n!} \\
= & \frac{[t]_\beta!}{t!}\frac{[t_1]_\beta!}{t_1!}\cdots\frac{[t_k]_\beta!}{t_k!}\sum_{\alpha_1+\ldots+\alpha_n=k}\prod_{i=1}^{n}\frac{(1+f_i\beta)\ldots(1+(f_i+\alpha_i-1)\beta)}{\alpha_i!} \\
= & \frac{[t]_\beta!}{t!}\frac{[t_1]_\beta!}{t_1!}\cdots\frac{[t_k]_\beta!}{t_k!}\sum_{\alpha_1+\ldots+\alpha_n=k}\prod_{i=1}^{n}\beta^{\alpha_i}\frac{x_i(x_i+1)\ldots(x_i+\alpha_i-1)}{\alpha_i} \\
= & \frac{[t]_\beta!}{t!}\frac{[t_1]_\beta!}{t_1!}\cdots\frac{[t_k]_\beta!}{t_k!}\beta^k P_k(x_1,\ldots,x_n) \\
= & \frac{[t]_\beta!}{t!}\frac{[t_1]_\beta!}{t_1!}\cdots\frac{[t_k]_\beta!}{t_k!}\beta^k\frac{s(s+1)\ldots(s+k-1)}{k!}.
\end{aligned}$$

Moreover, as t is a tree:

$$s = f_1 + \ldots + f_n + n/\beta = \text{number of edges of } t + n/\beta = n - 1 + n/\beta = n(1+1/\beta) - 1.$$

So, as $Q_k(S) = \dfrac{S(S+1)\ldots(S+k-1)}{k!}$:

$$\begin{aligned}
\Delta(\mathbf{X}_{1,\beta}) &= \mathbf{X}_{1,\beta}\otimes 1 + \sum_{k=0}^{\infty}\sum_{F=t_1\ldots t_k,\, t}\frac{[t]_\beta!}{t!}\frac{[t_1]_\beta!}{t_1!}\cdots\frac{[t_k]_\beta!}{t_k!}\beta^k Q_k(|t|(1+1/\beta)-1)F\otimes t \\
&= \mathbf{X}_{1,\beta}\otimes 1 + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} Q_k(n(1+1/\beta)-1)\beta^k\mathbf{X}_{1,\beta}^k\otimes\mathbf{a}_n(1,\beta) \\
&= \mathbf{X}_{1,\beta}\otimes 1 + \sum_{n=1}^{\infty}(1-\beta\mathbf{X}_{1,\beta})^{-n(1/\beta+1)+1}\otimes\mathbf{a}_n(1,\beta).
\end{aligned}$$

Proposition 15 *The coproduct of the $\mathbf{a}_n(1,\beta)$'s is given by:*

$$\Delta(\mathbf{X}_{1,\beta}) = \mathbf{X}_{1,\beta}\otimes 1 + \sum_{n=1}^{\infty}(1-\beta\mathbf{X}_{1,\beta})^{-n(1/\beta+1)+1}\otimes\mathbf{a}_n(1,\beta).$$

As a consequence, $\mathcal{A}_{N,1,\beta}$ is a Hopf subalgebra of $\mathcal{H}_{NCK}$.

Remark. By taking the abelianization of $\mathcal{A}_{N,1,\beta}$, the same holds in $\mathcal{A}_{1,\beta}$:

$$\Delta(X_{1,\beta}) = X_{1,\beta}\otimes 1 + \sum_{n=1}^{\infty}(1-\beta X_{1,\beta})^{-n(1/\beta+1)+1}\otimes a_n(1,\beta).$$

5 Isomorphisms between the $\mathcal{A}_{N,\alpha,\beta}$'s

5.1 Another system of generators of $\mathcal{A}_{N,1,\beta}$

Notation. We denote by B^- the inverse of $B^+ : \mathcal{H}_{NCK} \longrightarrow Vect(\mathbb{T}_{NCK})$, that is to say the application defined on a tree by deleting the root.

We define $\mathbf{b}_n(\alpha,\beta) = B^-(\mathbf{a}_{n+1}(\alpha,\beta))$ for all $n \in \mathbb{N}$, and:

$$\mathbf{Y}(\alpha,\beta) = \sum_{n=0}^{\infty}\mathbf{b}(\alpha,\beta).$$

We have:

$$\begin{aligned}
\mathbf{Y}(\alpha,\beta) &= \sum_{F\in\mathbb{F}_{CK}} \alpha^{|B^+(F)|-1}\frac{[B^+(F)]_\beta!}{B^+(F)!}F \\
&= \sum_{k=0}^{\infty}\sum_{t_1,\ldots,t_k\in\mathbb{T}_{CK}} \alpha^{|t_1|+\ldots+|t_k|}\frac{[k]_\beta![t_1]_\beta!\ldots[t_k]_\beta!}{k!t_1!\ldots t_k!}t_1\ldots t_k \\
&= \sum_{k=0}\frac{[k]_\beta!}{k!}\mathbf{X}^k_{\alpha,\beta} \\
&= \sum_{k=0}^{\infty}\frac{1(1+\beta)\ldots(1+\beta(k-1))}{k!}\mathbf{X}^k_{\alpha,\beta} \\
&= \sum_{k=0}^{\infty}\frac{1/\beta(1/\beta+1)\ldots(1/\beta+k-1))}{k!}\beta^k\mathbf{X}^k_{\alpha,\beta} \\
&= \sum_{k=0}^{\infty}Q_k(1/\beta)\beta^k\mathbf{X}^k_{\alpha,\beta} \\
&= (1-\beta\mathbf{X}_{\alpha,\beta})^{-1/\beta}.
\end{aligned}$$

By the last equality, $\mathbf{b}_n(\alpha,\beta)\in\mathcal{A}_{N,\alpha,\beta}$ for all $n\in\mathbb{N}$. Moreover, by the second equality, if $n\geq 1$:

$$\begin{aligned}
\mathbf{b}_n(\alpha,\beta) &= \sum_{t\in\mathbb{T}_{CK},\,|t|=n}\alpha^n\frac{[t]_\beta!}{t!}t + \text{forests with more than two trees} \\
&= \alpha\mathbf{a}_n(\alpha,\beta) + \text{forests with more than two trees.}
\end{aligned}$$

So $(\mathbf{b}_n(\alpha,\beta))_{n\geq 1}$ is a set of generators of $\mathcal{A}_{N,\alpha,\beta}$ if $\alpha\neq 0$.

Proposition 16 *Suppose $\alpha=1$. Then:*

$$\Delta(\mathbf{Y}(1,\beta)) = \sum_{n=0}^{\infty}\mathbf{Y}(1,\beta)^{n(\beta+1)+1}\otimes\mathbf{b}_n(1,\beta).$$

Proof. As $\mathbf{Y}(1,\beta)=B^-(\mathbf{X}(\alpha,\beta))$, by (1):

$$\begin{aligned}
\Delta(\mathbf{Y}(1,\beta)) &= (Id\otimes B^-)\left(\Delta(\mathbf{X}(1,\beta))-\mathbf{X}(1,\beta)\otimes 1\right) \\
&= \sum_{n=1}^{\infty}(1-\beta\mathbf{X}_{1,\beta})^{-n(1/\beta+1)+1}\otimes B^-(\mathbf{a}_n(1,\beta)) \\
&= \sum_{n=1}^{\infty}\mathbf{Y}_{1,\beta}^{n(1+\beta)-\beta}\otimes\mathbf{b}_{n-1}(1,\beta) \\
&= \sum_{n=0}^{\infty}\mathbf{Y}_{1,\beta}^{(n+1)(1+\beta)-\beta}\otimes\mathbf{b}_n(1,\beta) \\
&= \sum_{n=0}^{\infty}\mathbf{Y}_{1,\beta}^{n(1+\beta)+1}\otimes\mathbf{b}_n(1,\beta).\ \square
\end{aligned}$$

5.2 Isomorphisms between the $\mathcal{A}_{N,\alpha,\beta}$'s

Proposition 17 *If $\beta\neq -1$ and $\beta'\neq -1$, then $\mathcal{A}_{N,1,\beta}$ and $\mathcal{A}_{N,1,\beta'}$ are isomorphic.*

Proof. Let $\gamma\in K-\{0\}$. We put:

$$\mathbf{Z}(1,\beta)=\mathbf{Y}(1,\beta)^\gamma=\sum_{k=0}^{\infty}\mathbf{c}_n(1,\beta),$$

with $\mathbf{c}_n(1,\beta) \in \mathcal{A}(1,\beta)$, homogeneous of degree n. This makes sense, because $\mathbf{b}_0(1,\beta) = 1$. Moreover, for all $n \geq 1$:

$$\begin{aligned}\mathbf{c}_n(1,\beta) &= Q_1(\gamma)\mathbf{b}_n(1,\gamma) + \text{forests with more than two trees} \\ &= \gamma\mathbf{b}_n(1,\gamma) + \text{forests with more than two trees},\end{aligned}$$

so $(\mathbf{c}_n(1,\beta))_{n\geq 1}$ is a set of generators of $\mathcal{A}(1,\beta)$. Moreover:

$$\begin{aligned}\Delta(\mathbf{Z}(1,\beta)) &= \sum_{k=0}^{\infty} Q_k(\gamma)\left(\sum_{n=1}^{\infty}\mathbf{Y}(1,\beta)^{n(\beta+1)+1}\otimes\mathbf{b}_n(1,\beta)\right)^k \\ &= \sum_{k=0}^{\infty} Q_k(\gamma)\sum_{a_1,\ldots,a_k\geq 1}\mathbf{Y}(1,\beta)^{(a_1+\ldots+a_k)(\beta+1)+k}\otimes\mathbf{b}_{a_1}(1,\beta)\ldots\mathbf{b}_{a_k}(1,\beta) \\ &= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} Q_k(\gamma)\sum_{a_1+\ldots+a_k=l}\mathbf{Y}(1,\beta)^{l(\beta+1)+k}\otimes\mathbf{b}_{a_1}(1,\beta)\ldots\mathbf{b}_{a_k}(1,\beta) \\ &= \sum_{l=0}^{\infty}\mathbf{Y}(1,\beta)^{l(\beta+1)}\left(\sum_{k=0}^{\infty}\sum_{a_1+\ldots+a_k=l} Q_k(\gamma)\mathbf{Y}(1,\beta)\otimes\mathbf{b}_{a_1}(1,\beta)\ldots\mathbf{Y}(1,\beta)\otimes\mathbf{b}_{a_k}(1,\beta)\right) \\ &= \sum_{l=0}^{\infty}\mathbf{Y}(1,\beta)^{l(\beta+1)}\mathbf{Y}(1,\beta)^{\gamma}\otimes\mathbf{c}_l(1,\beta) \\ &= \sum_{l=0}^{\infty}\mathbf{Z}(1,\beta)^{l\left(\frac{\beta+1}{\gamma}\right)+1}\otimes\mathbf{c}_l(1,\beta).\end{aligned}$$

We now chose $\gamma = \frac{\beta+1}{\beta'+1}$. As $\beta' \neq -1$, this is well defined; as $\beta \neq -1$, this is non zero. Then:

$$\Delta(\mathbf{Z}(1,\beta)) = \sum_{l=0}^{\infty}\mathbf{Z}(1,\beta)^{l(\beta'+1)+1}\otimes\mathbf{c}_l(1,\beta).$$

So the unique isomorphism of algebras defined by:

$$\left\{\begin{array}{rcl}\mathcal{A}_{1,\beta'} & \longrightarrow & \mathcal{A}_{1,\beta} \\ \mathbf{b}_n(1,\beta') & \longrightarrow & \mathbf{c}_n(1,\beta)\end{array}\right.$$

is a Hopf algebra isomorphism. □

In the non commutative case, the following result holds:

Corollary 18 *There are three isomorphism classes of $\mathcal{A}_{N,\alpha,\beta}$'s:*

1. *the $\mathcal{A}_{N,1,\beta}$'s, with $\beta \neq -1$. These are not commutative and not cocommutative.*
2. *$\mathcal{A}_{N,1,-1}$, isomorphic to* **QSym***, the Hopf algebra of quasi-symmetric functions ([17, 20]) This one is not commutative and cocommutative.*
3. *$\mathcal{A}_{N,0,1} = K[\bullet]$. This one is commutative and cocommutative.*

Consequently, in the commutative case:

Corollary 19 *There are three isomorphism classes of $\mathcal{A}_{\alpha,\beta}$'s:*

1. *the $\mathcal{A}_{1,\beta}$'s, with $\beta \neq -1$. These are isomorphic to the Faà di Bruno algebra on one variable.*
2. *$\mathcal{A}_{1,-1}$, isomorphic to Sym, the Hopf algebra of symmetric functions. This one is commutative and cocommutative.*

3. $\mathcal{A}_{0,1} = K[\bullet]$.

Proof. As $\mathcal{A}_{\alpha,\beta}$ is the abelianization of $\mathcal{A}_{N,\alpha,\beta}$, if $\mathcal{A}_{N,\alpha,\beta} \approx \mathcal{A}_{N,\alpha',\beta'}$, then $\mathcal{A}_{\alpha,\beta} \approx \mathcal{A}_{\alpha',\beta'}$. Moreover, $\mathcal{A}_{1,\beta}$ is not cocommutative if $\beta \neq -1$, whereas $\mathcal{A}_{1,-1}$ is. So $A_{1,\beta}$ and $A_{1,-1}$ are not isomorphic if $\beta \neq -1$. It remains to show that $\mathcal{A}_{1,\beta}$ is isomorphic to the Faà di Bruno Hopf algebra on one variable if $\beta \neq -1$. Let us consider the dual Hopf algebra of $\mathcal{A}_{1,\beta}$. By Cartier-Quillen-Milnor-Moore's theorem ([18]), this is an enveloping algebra $\mathcal{U}(\mathcal{L}_{1,\beta})$. Moreover, $\mathcal{L}_{1,\beta}$ has for basis $(T_n)_{n\in\mathbb{N}^*}$ defined by:

$$T_n : \left\{ \begin{array}{rcl} \mathcal{L}_{1,\beta} & \longrightarrow & K \\ a_1^{\alpha_1} \dots a_k^{\alpha_k} & \longrightarrow & 0 \text{ if } \alpha_1 + \ldots + \alpha_k \neq 1, \\ a_m & \longrightarrow & \delta_{m,n}. \end{array} \right.$$

Moreover, T_n is homogeneous of degree n. By proposition 15, for all $i, j \geq 1$:

$$\begin{array}{rcl} (T_i \otimes T_j) \circ \Delta(X_{1,\beta}) & = & T_i(X_{1,\beta})T_j(1) + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} Q_k(n(1/\beta+1)-1)\beta^k T_i(X_{1,\beta}^k)T_j(a_n(1,\beta)) \\ & = & 0 + Q_1(j(1/\beta+1)-1)\beta T_i(X_{1,\beta}) \\ & = & j(1+\beta)-\beta. \end{array}$$

By homogeneity, there exists $\lambda_{i,j} \in K$ such that $[T_i, T_j] = \lambda_{i,j} T_{i+j}$. Then:

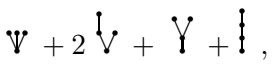
$$\begin{array}{rcl} \lambda_{i,j} & = & [T_i, T_j](X_{1,\beta}) \\ & = & (T_i \otimes T_j) \circ \Delta(X_{1,\beta}) - (T_j \otimes T_i) \circ \Delta(X_{1,\beta}) \\ & = & j(1+\beta) - \beta - i(1+\beta) + \beta \\ & = & (i-j)(1+\beta). \end{array}$$

Then, there exists a Lie algebra morphism:

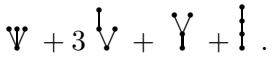
$$\left\{ \begin{array}{rcl} \mathcal{L}_{\alpha,\beta} & \longrightarrow & Prim(\mathcal{H}_{FdB}^*) \\ T_n & \longrightarrow & (1+\beta)Z_n. \end{array} \right.$$

In particular, if $\beta \neq -1$, this is an isomorphism. Hence, $A_{1,\beta}^*$ is isomorphic to $\mathcal{H}_{FdB}^*$, so $A_{1,\beta}$ is isomorphic to the Faà di Bruno Hopf algebra on one variable. □

Remark. The Connes-Moscovici subalgebra $\mathcal{H}_{CM}$ of $\mathcal{H}_{CK}$ (see [5, 6]) does not appear here: as it is generated by $\bullet, \uparrow, \curlyvee + \uparrow, \ldots$, it would be $A_{(1,1)}$. The fourth generator of $A_{(1,1)}$ is:



whereas the fourth generator of $\mathcal{H}_{CM}$ is:



So they are different.

6 The case of the free Faà di Bruno algebra with D variables

We here fix an integer $D \geq 1$. We denote by W the set of non empty words in letters $\{1, \ldots, D\}$.

6.1 Construction

We now recall the construction of the free Faà di Bruno algebra in D variables (see [1]). Consider the ring of non commutative formal series $K\langle\langle h_1, \ldots, h_D\rangle\rangle$ on D variables. We consider:

$$G_D = \left\{ \left(\sum_{w \in W} a_w^{(i)} h^w \right)_{1 \leq i \leq D} / \, a_j^{(i)} = \delta_{i,j} \right\}.$$

We use the following convention: if $u_1 \ldots u_k \in W$, then $h^w = h_{u_1} \ldots h_{u_k}$. In other terms, G_D is the set of formal diffeomorphisms on K^D which are tangent to the identity at the origin. This is a group for the composition of formal series.

Then $\mathcal{H}_{FdB,D}$ is the Hopf algebra of functions on the opposite of the group G_D. Hence, it is the polynomial ring in variable Y_w^i, $1 \leq i \leq D$, with the convention that if w has only one letter j, then $Y_j^i = \delta_{i,j}$. The coproduct is given in the following way: for all $f \in \mathcal{H}_{FdB,D}$, for all $P, Q \in G_D$,

$$\Delta(f)(P \otimes Q) = f(Q \circ P).$$

In particular, if:

$$P = \left(\sum_{w \in W} a_w^{(i)} h^w \right)_{1 \leq i \leq D} \text{ and } Q = \left(\sum_{w \in W} b_w^{(i)} h^w \right)_{1 \leq i \leq D},$$

then:

$$\begin{aligned}
\Delta(Y_w^i)(P \otimes Q) &= Y_w^i(Q \circ P) \\
&= Y_w^i \left(\left(\sum_{u_1 \ldots u_k \in W} b^j_{u_1 \ldots u_k} \sum_{w_1, \ldots, w_k \in W} a^{u_1}_{w_1} \ldots a^{u_k}_{w_k} h^{w_1 \ldots w_k} \right)_{1 \leq j \leq D} \right) \\
&= \sum_{u_1 \ldots u_k \in W} \sum_{w_1 \ldots w_k = w} b^j_{u_1 \ldots u_k} a^{u_1}_{w_1} \ldots a^{u_k}_{w_k}.
\end{aligned}$$

Hence:

$$\Delta\left(Y_w^i\right) = \sum_{k=1}^{n} \sum_{1 \leq u_i \leq D} \sum_{\substack{w_1, \ldots, w_k \in W, \\ w_1 \ldots w_k = w}} Y^{u_1}_{w_1} \ldots Y^{u_k}_{w_k} \otimes Y^i_{u_1 \ldots u_k}.$$

For $D = 1$, we recover $\mathcal{H}_{FdB}$.

6.2 Subalgebras of $\mathcal{H}_{NCK}^{\mathcal{D}}$

We now put $\mathcal{D} = \{1, \ldots, D\}^3$. The elements of $\mathcal{D}$ will be denoted in the following way: $i, (u_1, u_2)$. In the same way, it is possible to construct a commutative Hopf algebra $\mathcal{H}_{CK}^{\mathcal{D}}$ of rooted trees decorated by $\mathcal{D}$, and a non commutative Hopf algebra $\mathcal{H}_{NCK}^{\mathcal{D}}$ of planar rooted trees decorated by $\mathcal{D}$. In both cases, we define, for all $i, (u_1, u_2) \in \mathcal{D}$, a linear endomorphism $B^+_{i,(u_1,u_2)}$, which sends forest F on the tree obtained by grafting all the trees of F on a common root decorated by $i, (u_1, u_2)$.

Definition 20 Let $i \in \{1, \ldots, D\}$ and $w = u_1 \ldots u_n \in W$. We define an element $Y_w^i \in \mathcal{H}_{NCK}^{\mathcal{D}}$ inductively on n in the following way:

$$\left\{ \begin{array}{lcl}
\mathbf{Y}_w^i &=& \delta_{i,w} \text{ if } n = 1, \\
\mathbf{Y}_w^i &=& \displaystyle\sum_{1 \leq \alpha, \beta \leq D} \sum_{\substack{w_1, w_2 \in W, \\ w_1 w_2 = w}} B^+_{i,(\alpha,\beta)} \left(Y^\alpha_{w_1} Y^\beta_{w_2} \right) \text{ if } n \geq 2.
\end{array} \right.$$

Examples. For i, u_1, u_2, u_3 and u_4 elements of $\{1,\dots,D\}$:

$$\begin{aligned}
Y^i_{u_1} &= \delta_{i,u_1},\\
Y^i_{u_1u_2} &= \bullet_{i,(u_1,u_2)},\\
Y^i_{u_1u_2u_3} &= \sum_{1\le\alpha\le D} \left(\curlywedge^{\alpha,(u_1,u_2)}_{i,(\alpha,u_3)} + \curlywedge^{\alpha,(u_2,u_3)}_{i,(u_1,\alpha)} \right),\\
Y^i_{u_1u_2u_3u_4} &= \sum_{1\le\alpha,\beta\le D} \left(\curlyvee^{\alpha,(u_1,u_2)}_{i,(\alpha,\beta)} \curlyvee^{\beta,(u_3,u_4)} + \curlywedge^{\beta,(u_2,u_3)}_{\alpha,(u_1,\beta)} \curlywedge^{\beta,(u_1,u_2)}_{i,(\alpha,u_4)} + \curlywedge^{\beta,(u_3,u_4)}_{\alpha,(u_2,\beta)} \curlywedge^{\beta,(u_2,u_3)}_{i,(u_1,\alpha)} \right).
\end{aligned}$$

An easy induction shows that $Y^i_{u_1\dots u_n}$ is homogeneous of degree $n-1$.

Theorem 21 *For all $i\in\{1,\dots,D\}$, $w\in W$ of length n:*

$$\Delta\left(Y^i_w\right) = \sum_{k=1}^{n} \sum_{1\le\alpha_i\le D} \sum_{\substack{w_1,\dots,w_k\in W,\\ w_1\dots w_k=w}} Y^{\alpha_1}_{w_1}\dots Y^{\alpha_k}_{w_k}\otimes Y^i_{\alpha_1\dots\alpha_k}.$$

Proof. By induction on n. It is obvious if $n=1$ or 2. Suppose it is true for all rank $<n$. Then:

$$\begin{aligned}
\Delta(Y^i_w) &= \sum_{\alpha,\beta}\sum_{w_1w_2=w} \Delta\circ B^+_{i,(\alpha,\beta)}\left(Y^\alpha_{w_1}Y^\beta_{w_2}\right)\\
&= Y^i_w\otimes 1 + \sum_{\alpha,\beta}\sum_{w_1w_2=w}\ \sum_{w_{1,1}\dots w_{1,k}=w_1}\\
&\qquad \sum_{\alpha_1,\dots,\alpha_k}\ \sum_{w_{2,1}\dots w_{2,l}=w_2}\ \sum_{\beta_1,\dots,\beta_l} Y^{\alpha^1}_{w_{1,1}}\dots Y^{\alpha^k}_{w_{1,k}}Y^{\beta^1}_{w_{2,1}}\dots Y^{\beta^l}_{w_{2,l}}\otimes B^+_{i,(\alpha,\beta)}\left(Y^\alpha_{\alpha_1\dots\alpha_k}Y^\beta_{\beta_1\dots\beta_l}\right)\\
&= Y^i_w\otimes 1 + \sum_{k\ge2}\sum_{w_1\dots w_k=w}\ \sum_{\alpha_1,\dots,\alpha_k} Y^{\alpha_1}_{w_1}\dots Y^{\alpha_k}_{w_k}\otimes\left(\sum_{\alpha,\beta}\sum_{w'_1w'_2=\alpha_1\dots\alpha_k} B^+_{i,(\alpha,\beta)}\left(Y^\alpha_{w'_1}Y^\beta_{w'_2}\right)\right)\\
&= Y^i_w\otimes 1 + \sum_{k\ge2}\sum_{w_1\dots w_k=w}\ \sum_{\alpha_1,\dots,\alpha_k} Y^{\alpha_1}_{w_1}\dots Y^{\alpha_k}_{w_k}\otimes Y^i_{\alpha_1\dots\alpha_k}\\
&= \sum_{k\ge1}\sum_{w_1\dots w_k=w}\ \sum_{\alpha_1,\dots,\alpha_k} Y^{\alpha_1}_{w_1}\dots Y^{\alpha_k}_{w_k}\otimes Y^i_{\alpha_1\dots\alpha_k}.\ \square
\end{aligned}$$

Hence, the subalgebra of $\mathcal{H}^{\mathcal{D}}_{NCK}$ generated by the Y^i_w's is a Hopf subalgebra. Its abelianization can be seen as a subalgebra of $\mathcal{H}^{\mathcal{D}}_{CK}$, and is isomorphic to $\mathcal{H}_{FdB,D}$.

Remark. In the case where $D=1$, we put $Y^1_{\underbrace{1\dots1}_{n+1\ times}} = Y_n$. Then, by definition:

$$\begin{cases}
Y_0 &= 1,\\
Y_1 &= \bullet,\\
Y_n &= \displaystyle\sum_{k=0}^{n-1} B^+(Y_kY_{n-1-k}) = 2B^+(Y_{n-1}) + \sum_{k=1}^{n-2} B^+(\mathbf{Y}_kY_{n-1-k}) \text{ if } n\ge2.
\end{cases}$$

Hence, by (4), this is the subalgebra associated to $1+2h+h^2=(1+h)^2=P_{4,-\frac12}(h)$, namely $\mathcal{A}_{D,4,-\frac12}$.

6.3 Description of the Y_w^i's in the generic case

Definition 22 A tree $t \in \mathbb{T}_{CK}^{\mathcal{D}}$ is *admissible* if:

1. Every vertex is of fertility less than 2.
2. For each vertex of fertility 1, the decorations are set in this way:

$$\overset{a,\,(c,\,d)}{\underset{i,\,(a,\,b)}{\bullet}} \quad \text{or} \quad \overset{b,\,(c,\,d)}{\underset{i,\,(a,\,b)}{\bullet}}\,.$$

3. For each vertex of fertility 2, the decorations are set in this way:

$$\overset{a,\,(c,\,d)}{\bullet}\!\!\bigvee\!\!\underset{i,\,(a,\,b)}{\overset{b,\,(e,\,f)}{\bullet}}\,.$$

Let t be an admissible tree. We associate to it a word in W in the following inductive way:

1. $w(\bullet_{i,\,(a,\,b)}) = ab$.
2. If the root of t has fertility 1, with decorations set as $\overset{a,\,(c,\,d)}{\underset{i,\,(a,\,b)}{\bullet}}$, then, if we denote $t' = B^-(t)$, $w(t) = w(t')b$.
3. If the root of t has fertility 1, with decorations set as $\overset{b,\,(c,\,d)}{\underset{i,\,(a,\,b)}{\bullet}}$, then, if we denote $t' = B^-(t)$, $w(t) = aw(t')$.
4. If the root of t has fertility 2, then, if we denote $t't'' = B^-(t)$, $w(t) = w(t')w(t'')$.

Remark. The cases 1 and 2 are not incompatible, so $w(t)$ is not well defined. For example, for $t = \overset{a,\,(c,\,d)}{\underset{i,\,(a,\,a)}{\bullet}}$, two results are possible: *cda* and *acd*.

An easy induction shows that:

Proposition 23 *Suppose that $w \in W$ is generic, that is to say all his letters are distinct. Then Y_w^i is the sum of admissible trees t such that:*

1. *$w(t) = w$.*
2. *The decoration of the root of t is of the form $i, (a, b)$, with $1 \leq, a, b \leq D$.*

If the word is not generic, we obtain Y_w^i by specializing the generic case. For example, if $w = aaa$, we have:

$$\begin{aligned} Y_{abc}^i &= \sum_{1\leq\alpha\leq D} \overset{\alpha,\,(a,\,b)}{\underset{i,\,(\alpha,\,c)}{\bullet}} + \sum_{1\leq\alpha\leq D} \overset{\alpha,\,(b,\,c)}{\underset{i,\,(a,\,\alpha)}{\bullet}} \\ \Longrightarrow Y_{aaa}^i &= \sum_{1\leq\alpha\leq D} \overset{\alpha,\,(a,\,a)}{\underset{i,\,(\alpha,\,a)}{\bullet}} + \sum_{1\leq\alpha\leq D} \overset{\alpha,\,(a,\,a)}{\underset{i,\,(a,\,\alpha)}{\bullet}}. \end{aligned}$$

In particular, $\overset{a,\,(a,\,a)}{\underset{i,\,(a,\,a)}{\bullet}}$ appears with multiplicity 2.

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