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Penalization approach of semi-linear symmetric hyperbolic problems with dissipative boundary conditions

B. Fornet, O. Guès

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Abstract

In this paper, we introduce a penalization method in order to approximate the solutions of the initial boundary value problem for a semi-linear first order symmetric hyperbolic system, with dissipative boundary conditions. The penalization is carefully chosen in order that the convergence to the wished solution is sharp, does not generate any boundary layer, and applies to fictitious domains.

1 Introduction

In this paper we consider the initial boundary value problem for a symmetric first order hyperbolic system ([F]), with maximally strictly dissipative boundary conditions, on a characteristic boundary. Typically the problem writes

$$(1.1) \quad \begin{cases} Lu = F(t, x, u) \text{ in }]0, T[\times \Omega \\ u|_{]0, T[\times \partial\Omega} \in \mathcal{N} \\ u_{t=0} = 0. \end{cases}$$

where Ω is a suitably regular open set of \mathbb{R}^d with smooth boundary, L is the first order symmetric hyperbolic system, \mathcal{N} is a smooth bundle on $\mathbb{R} \times \partial\Omega$ defining the boundary conditions, and F a smooth map that can be non linear.

The subject of the paper is mainly motivated by numerical analysis: we want to approximate the solution u of (1.1) by the solution v^ε of a well

chosen *Cauchy problem* (instead of a boundary value problem), where the complementary part of Ω will be penalized, using a large parameter:

$$(1.2) \quad \begin{cases} L^\sharp v^\varepsilon + \frac{1}{\varepsilon} \mathbf{1}_{\mathbb{R} \times \Omega^c} M v^\varepsilon = F^\sharp(t, x, v^\varepsilon) & \text{in }]0, T[\times \mathbb{R}^d \\ u|_{t=0} = 0. \end{cases}$$

where L^\sharp , F^\sharp are extensions of L and F to the whole space $\mathbb{R} \times \mathbb{R}^d$, and $M(t, x)$ is a suitable symmetric and ≥ 0 matrix. We solve this problem under general assumptions, the main point being the existence of the matrix M . We give two solutions for the matrix M .

1/ The first solution is a positive definite matrix which was introduced by J. Rauch [R1] in the study of the linear case, related to the work by J. Rauch and C. Bardos [BR] on singular perturbations. We show that for this approach v^ε converges to u , and that a boundary layer forms close to $\partial\Omega$, on the side of Ω^c . This is Theorem 2.6. In the linear non-characteristic case, the occurrence of boundary layers has been already observed in [D].

2/ The second solution contains an improvement of the previous one, and in this case the matrix M is no more invertible. In this approach the convergence of v^ε is better because there are *no boundary layers at all, at any order*. This result is stated in Theorem 2.7. In the two results, the key point is the use of the Rauch's matrix of [R1].

Let us also mention other interesting features of our results:

3/ Still motivated by concrete applications we show that one can choose the operator L^\sharp in a such a way that, instead of solving the Cauchy problem (1.2), one needs only to solve the problem

$$(1.3) \quad \begin{cases} L^\sharp v^\varepsilon + \frac{1}{\varepsilon} \mathbf{1}_{\mathbb{R} \times \Omega^c} M v^\varepsilon = F^\sharp(t, x, v^\varepsilon) & \text{in }]0, T[\times \Omega^\sharp \\ u|_{\{t=0\} \times \Omega^\sharp} = 0, \end{cases}$$

with *no boundary conditions on* $\mathbb{R} \times \partial\Omega^\sharp$, where Ω^\sharp is an open set containing Ω . No regularity is assumed on Ω^\sharp and it can be a polyhedral domain. When Ω is bounded, Ω^\sharp can be taken bounded. This is the subject of the Corollary 2.8.

4/ If $u \in H^\infty([0, T] \times \Omega)$, the convergence of v^ε towards u will hold on $[0, T] \times \Omega$.

In the paper, to simplify the proof, we treat the case where Ω is a half space $\mathbb{R}_+^d = \{x \in \mathbb{R}^d, x_d > 0\}$, but we give the extension to the general case

in a short section, without proof. We also restrict ourselves to the case where $u_{t<0} = 0$ in order to avoid the problem of compatibility conditions for the Cauchy problem. The section 2 is devoted to the precise statement of the assumptions and results in the case $\Omega = \mathbb{R}_+^n$. The section 3 describes the extension to the practical case when Ω is bounded. Section 4 is devoted to the proof of Theorem 2.7. Section 5 contains the proof of theorem 2.6.

2 Main results

Let us consider a symmetric hyperbolic operator

$$L = A_0(t, x)\partial_t + \sum_{j=1}^d A_j(t, x)\partial_j + B(t, x)$$

where A_j , $j = 0, \dots, d$ and B are $N \times N$ real matrices defined on $\mathbb{R}_+^{1+d} := \{x \in \mathbb{R}^d | x_d > 0\}$. We assume that all the entries of A_j , $j = 0, \dots, d$ and B are in $\mathcal{C}_b^\infty(\mathbb{R}^{1+d})$, the set of smooth functions bounded with bounded derivatives of all order. We also assume that all the matrices are constant out of a bounded subset of \mathbb{R}^{1+d} and that for $j = 0, \dots, d$, A_j is symmetric, A_0 being uniformly positive definite on \mathbb{R}^{1+d} .

Assumption 2.1. *The matrix $A_d(t, y, 0)$ has a constant rank on \mathbb{R}^d .*

When the rank of A_d is N , the boundary is non characteristic for L , and when $\text{rank} A_d < N$, the boundary is *characteristic of constant multiplicity*. In the sequel of the paper, if M is a symmetric $N \times N$ matrix we will note:

$$\mathbb{E}_+(M) = \sum_{\lambda > 0} \ker(M - \lambda I),$$

and

$$\mathbb{E}_-(M) = \sum_{\lambda < 0} \ker(M - \lambda I).$$

Let us call d_+ the dimension of $\mathbb{E}_+(A_d(t, y, 0))$, which is independent of (t, y) and is also the number of > 0 eigenvalues of A_d (counted with their multiplicities). For all $T > 0$ we note

$$\Omega_T :=]-1, T[\times \mathbb{R}^d, \quad \Omega_T^+ = \Omega_T \cap \{x_d > 0\}, \quad \Omega_T^- = \Omega_T \cap \{x_d < 0\}$$

and $\Gamma_T := \{(t, x) \in \mathbb{R}^{1+d} \mid -1 < t < T, x_d = 0\}$. We are interested in the initial boundary value problem in $\{x_d > 0\}$ with boundary conditions

$$(2.1) \quad u \in \mathcal{N}(t, y)$$

where $\mathcal{N}(t, y)$ defines a smooth vector bundle on the boundary. Let us denote by $\mathbb{P}_0(t, y)$ the orthogonal projector of \mathbb{R}^N onto $\text{Ker}A_d(t, y, 0)$.

Assumption 2.2. $\mathcal{N}(t, y)$ is a linear subspace of \mathbb{R}^N depending smoothly on $(t, y) \in \mathbb{R}^d$ with $\dim \mathcal{N}(t, y) = N - d_+$, and there exists a constant $c_0 > 0$ such that for all $v \in \mathbb{R}^N$ and all $(t, y) \in \mathbb{R}^d$:

$$(2.2) \quad v \in \mathcal{N}(t, y) \Rightarrow \langle A_d(t, y, 0)v, v \rangle \leq -c_0 \|(I - \mathbb{P}_0(t, y))v\|.$$

This kind of boundary conditions (2.1) were introduced by K. O. Friedrichs and are called "maximally strictly dissipative" (see [Maj]). Let us consider a smooth mapping $F \in \mathcal{C}^\infty(\mathbb{R}^{1+d+N} : \mathbb{R}^N)$, such that for all $\alpha \in \mathbb{N}^{1+d+N}$, $\partial_{t,x,u}^\alpha F$ is bounded on $\mathbb{R}^{1+d} \times K$ for any compact $K \subset \mathbb{R}^N$, such that $F(t, x, 0) \in H^\infty(\mathbb{R}^{1+d})$ and $F|_{t < 0} = 0$.

It follows from the results of [Ra] and [Gu] that there exist $T_0 > 0$ and a unique $u \in H^\infty(\Omega_{T_0}^+)$ such that

$$(2.3) \quad \begin{cases} Lu = F(t, x, u) \text{ in } \Omega_{T_0}^+, \\ u(t, y, 0) \in \mathcal{N}(t, y) \text{ on } \Gamma_{T_0}, \\ u|_{\Gamma_0} = 0. \end{cases}$$

From now on, the real $T_0 > 0$ and $u \in H^\infty(\Omega_{T_0}^+)$ are fixed once for all.

Let us now describe our main result. We want to approximate u by the solution of a singularly perturbed Cauchy problem in the domain Ω_{T_0} , where the subdomain $\Omega_{T_0}^-$ is penalized. The first step is to extend the operator L to an operator defined on $-\infty < x_d < +\infty$. Let us consider an operator

$$(2.4) \quad L^\sharp := \sum_{j=0}^{j=d} A_j^\sharp(t, x) \partial_j + B^\sharp(t, x)$$

where the matrices A_j^\sharp and B^\sharp are $N \times N$ and are defined on \mathbb{R}^{1+d} and coincides with the matrices A_j and B if $x_d > 0$. We assume that the restrictions $A_j^\sharp|_{x_d \leq 0}$ and $B^\sharp|_{x_d \leq 0}$ are in $C^\infty(\overline{\mathbb{R}^{1+d}_-})$, constant outside a bounded subset of $\overline{\mathbb{R}^{1+d}_-}$. For all $(t, x) \in \mathbb{R}^{1+d}$, the matrices $A_j^\sharp(t, x)$ $j = 0, \dots, d$ are symmetric and $A_0^\sharp(t, x)$ is uniformly positive definite on \mathbb{R}^{1+d} . An important point is that we assume continuity of A_d^\sharp :

Assumption 2.3. *The matrix A_d^\sharp is continuous on \mathbb{R}^{1+d} .*

Hence the matrices A_j^\sharp are allowed to be discontinuous across $\{x_d = 0\}$ excepted for A_d^\sharp . For example on $\{x_d < 0\}$, one can take $A_0^\sharp = I$, $A_j = 0$ for $j = 1 \cdots, d-1$, $B^\sharp = 0$, and such an extension is obtained by taking simply

$$(2.5) \quad L^\sharp := \partial_t + A_d(t, y, 0)\partial_d \text{ on } \{x_d < 0\},$$

which satisfies our assumptions.

Now the problem is to find a matrix $M(t, x)$ with C^∞ coefficients, constant out of a compact set in \mathbb{R}^{1+d} , such that the hyperbolic Cauchy problem

$$(2.6) \quad \begin{cases} L^\sharp v^\varepsilon + \frac{1}{\varepsilon} \mathbf{1}_{\{x_d < 0\}} M v^\varepsilon = \mathbf{1}_{\{x_d > 0\}} F(t, x, v^\varepsilon) \text{ in } \Omega_T \\ v^\varepsilon|_{\Gamma_0} = 0. \end{cases}$$

admits a unique piecewise smooth solution v^ε which converges to u in $\Omega_{T_0}^+$, as $\varepsilon > 0$ goes to 0. First of all, it's worth emphasizing that the problem (2.6) makes sense, although the matrices A_j^\sharp , $j = 0, \dots, d-1$, could be *discontinuous* across the hyperplane $\{x_d = 0\}$. The point is that A_d is continuous so that one can write the principal part of the differential operator in a conservative form. We refer to the appendix for a more detailed discussion of this point, with a proof of the well-posedness.

We will give two solutions to this problem of penalization. Let us begin with a preliminary lemma.

Lemma 2.4. *For all point $p = (t, y, 0) \in \mathbb{R}^d$ there exist a neighborhood $\mathcal{V}(p)$ in \mathbb{R}^{1+d} and a matrix $\Psi \in C^\infty(\mathcal{V}, \text{GL}_N(\mathbb{R}))$ satisfying*

$$0 < c \leq |\det \Psi(t, x)| < c^{-1}, \quad \forall (t, x) \in \mathcal{V}$$

for some constant c , and such that: $\mathbb{E}_+(\Psi A_d \Psi)^\perp = \Psi^{-1} \mathcal{N}$ on $\{x_d = 0\} \cap \mathcal{V}$.

Proof. Let us note $\tilde{A}_d := {}^t \Psi A_d \Psi$ and $\mathbb{E}_{\leq 0}(\tilde{A}_d) = \mathbb{E}_+(\tilde{A}_d)^\perp$, which can also be defined by

$$\mathbb{E}_{\leq 0}(\tilde{A}_d) = \sum_{\lambda \leq 0} \ker(\tilde{A}_d - \lambda I).$$

There holds $\mathbb{E}_{\leq 0}(\tilde{A}_d) = \Psi^{-1} \mathbb{E}_{\leq 0}(\Psi {}^t \Psi A_d)$. Hence the claimed result is equivalent to find Ψ which satisfies $\mathcal{N}(t, y) = \mathbb{E}_{\leq 0}(\Psi {}^t \Psi A_d)$ for all $x =$

$(t, y, 0) \in \mathbb{R}^d \times \{0\}$. Now, we know from a lemma by J. Rauch in [Ra1] that there exists a smooth symmetric definite and positive matrix $E(t, y)$ such that $\mathcal{N}(t, y) = \mathbb{E}_{\leq 0}(E(t, y)A_d(t, y, 0))$ which concludes the proof by taking $\Psi = OE^{1/2}$, where O is any orthogonal matrix. As the proof shows, the lemma 2.4 is nothing but the Rauch's result, expressed in a different way. Note that this Rauch's result has been extended recently by F. Sueur ([Su1]) to the case of general dissipative boundary conditions (not necessarily *strictly* dissipative). \square

To simplify the presentation we will make the following assumption which enables one to use *only one* matrix Ψ to reduce the problem, as we will see in section 4.1. Nevertheless, this assumption is not a real restriction since it is explained in the comment which follow.

Assumption 2.5. *We assume that one can take $\mathcal{V}(p) = \mathbb{R}^{1+d}$ in lemma 2.4.*

Let us fix for all the sequel a matrix Ψ as in the lemma with $\mathcal{V}(p) = \mathbb{R}^{1+d}$.

Example 2.1. *We show that in the general case when the Assumption 2.5 is not satisfied, one can introduce an extended system (by using a suitable partition of unity) which satisfies the assumption 2.5. By compactness there exist a finite number of open set $\mathcal{V}_1, \dots, \mathcal{V}_k$ with the corresponding functions of lemma 2.4 $\Psi_j \in C^\infty(\mathcal{V}_j, GL_N(\mathbb{R}))$, $j = 1, \dots, k$ and associated cut-off functions $\chi_j \in C_0^\infty(\mathbb{R}^{d+1})$ $j = 1, \dots, k$ such that $\text{supp}\chi_j \subset \mathcal{V}_j$ and $\sum \chi_j = 1$ in \mathbb{R}^{d+1} . For all $j \in \{1, \dots, k\}$ the function $U_j(t, x) := \chi_j(t, x)u(t, x)$ satisfies the following system where we have noted for $(t, x) \in \mathbb{R}^{1+d}$, $\xi \in \mathbb{R}^{1+d}$, $L(t, x; \xi) = \sum_0^d \xi_i A_i$:*

$$(2.7) \quad \begin{cases} LU_j = L(t, x; d\chi_j)u + \chi_j F(t, x, u) \text{ in } \Omega_{T_0}^+, \\ U_j(t, y, 0) \in \mathcal{N}(t, y) \text{ on } \Gamma_{T_0}, \\ U_j|_{\Gamma_0} = 0. \end{cases}$$

It follows that the function

$$U(t, x) := (U_1(t, x), \dots, U_k(t, x))$$

satisfies a larger $kN \times kN$ hyperbolic system

$$(2.8) \quad \begin{cases} \mathbb{L}U = \mathbb{F}(t, x, U) \text{ in } \Omega_{T_0}^+, \\ U(t, y, 0) \in \mathbb{B}(t, y) \text{ on } \Gamma_{T_0}, \\ U|_{\Gamma_0} = 0, \end{cases}$$

where $\mathbb{B} = \mathcal{N} \times \cdots \times \mathcal{N}$,

$$(2.9) \quad \mathbb{F}(t, x, U) = \begin{pmatrix} \vdots \\ L(t, x; d\chi_j) \sum_j U_j + \chi_j F(t, x, \sum_j U_j) \\ \vdots \end{pmatrix}$$

and

$$\mathbb{L}U = \begin{pmatrix} LU_1 \\ \vdots \\ LU_k \end{pmatrix}.$$

This is again an semilinear symmetric hyperbolic system with maximally dissipative boundary conditions, satisfying the assumptions 2.1, 2.2 and the assumption 2.5.

We will state two theorems. The second one is "better" than the first one, because it gives a kind of sharp result as long as one is interested the quality of the convergence of v^ε towards u . Let us begin by introducing the matrix

$$R(t, x) := {}^t\Psi(t, x) \Psi(t, x), \quad \forall (t, x) \in \mathbb{R}^{1+d},$$

which is a symmetric and uniformly positive definite matrix (it is the matrix introduced by Rauch in [Ra1], and used in the proof of lemma 2.4). The matrix R gives a good answer to the problem of penalization and our first result concerns the following problem:

$$(2.10) \quad \begin{cases} L^\sharp w^\varepsilon + \frac{1}{\varepsilon} \mathbf{1}_{\{x_d < 0\}} R w^\varepsilon = \mathbf{1}_{\{x_d > 0\}} F(t, x, w^\varepsilon) \text{ in } \Omega_T \\ w^\varepsilon|_{\Gamma_0} = 0. \end{cases}$$

Theorem 2.6. *There is $\varepsilon_0 > 0$ such that for all $\varepsilon > 0$ small enough, the problem (2.10) has a unique solution $w^\varepsilon \in L^2(\Omega_{T_0}) \cap L^\infty(\Omega_{T_0})$ on Ω_{T_0} . Moreover $w^\varepsilon|_{\Omega_{T_0}^\pm} \in H^\infty(\Omega_{T_0}^\pm)$ and the following estimates hold*

$$(2.11) \quad \|u - w^\varepsilon|_{\Omega_{T_0}^+}\|_{H^s(\Omega_{T_0}^+)} = O(\varepsilon), \quad \forall s \in \mathbb{R},$$

and

$$(2.12) \quad \|w^\varepsilon|_{\Omega_{T_0}^-}\|_{H^s(\Omega_{T_0}^-)} = O(\varepsilon^{-s+\frac{1}{2}}), \quad \forall s \in \mathbb{R},$$

Let us comment on two points. First, we insist on the fact that the solution w^ε converges to u **on the whole set** $\Omega_{T_0}^+$ which was fixed in the preliminaries and where u was defined. Second, the convergence in (2.11) holds for all s , which means that there is no singularity with respect to ε in $\Omega_{T_0}^+$, although the perturbation is singular. However, the estimates (2.12) indicates that the behavior of w^ε is singular (with respect to ε) in $\Omega_{T_0}^-$. Indeed, the proof of the theorem gives a more precise result, and shows the existence of a boundary layer in the domain $\Omega_{T_0}^-$, and more precisely an asymptotic expansion (we note $y = (x_1, \dots, x_{d-1})$):

$$(2.13) \quad w^\varepsilon(t, x) = W(t, y, x_d/\varepsilon) + o(\varepsilon)$$

where $W(t, y, z)$ is a boundary layer profile in the sense that

$$\lim_{z \rightarrow -\infty} W(t, y, z) = 0$$

(see section 5). This is very natural since the problem is a singular perturbation problem. Furthermore, This is not surprising since boundary layers already appeared in the work by J. Droniou devoted to the linear noncharacteristic case ([D]), which can be seen as a special case of our result.

Let us now state our second result. Denote by \mathbb{P} the orthogonal projector of \mathbb{R}^N onto $(\Psi^{-1}\mathcal{N})^\perp$, and note

$$(2.14) \quad M(t, x) = {}^t\Psi^{-1}(t, x)\mathbb{P}(t, x)\Psi^{-1}(t, x)$$

which is a smooth symmetric matrix valued mapping on \mathbb{R}^{1+d} .

Theorem 2.7. *Let us chose the matrix M defined by (2.14). There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$, the problem (2.6) has a unique solution $v^\varepsilon \in L^2(\Omega_{T_0}) \cap L^\infty(\Omega_{T_0})$ on Ω_{T_0} . Moreover $v^\varepsilon|_{\Omega_{T_0}^\pm} \in H^\infty(\Omega_{T_0}^\pm)$ and*

$$(2.15) \quad \|u - v^\varepsilon|_{\Omega_{T_0}^\pm}\|_{H^s(\Omega_{T_0}^\pm)} = O(\varepsilon), \quad \forall s \in \mathbb{R}.$$

Moreover, the behavior of v^ε is not singular with respect to ε in the sense that

$$(2.16) \quad \|\partial^\alpha v^\varepsilon|_{\Omega_{T_0}^\pm}\|_{L^2(\Omega_{T_0}^\pm)} = O(1), \quad \forall \alpha \in \mathbb{N}^{1+d}.$$

In that case, the convergence holds on each side $\Omega_{T_0}^\pm$ in $H^s(\Omega_{T_0}^\pm)$ respectively for all $s \in \mathbb{R}$, which means that there is no singularity with respect to ε , although the perturbation is singular. In particular there are no boundary layers (at any order) and **the convergence rate is optimal**. In $\Omega_{T_0}^+$ the limit of v^ε is u , and in $\Omega_{T_0}^-$ the limit of v^ε is a smooth function in $H^\infty(\Omega_{T_0}^-)$ which is described precisely in the section 4.

The theorem 2.7 is proved in section 4. The theorem 2.11 is proved in section 5.

Application: fictive boundary and absorbing layer. In practice one has a lot of freedom in the choice of the matrices $A_j^\sharp(t, x)$ in $x_d < 0$ as we have already said. A very interesting choice for numerical applications is to chose this matrices in order that *all the eigenvalues of $A_d^\sharp(t, x)$ are < 0 when $x_d = -\delta$ for some fixed $\delta > 0$* . For example, one can take

$$(2.17) \quad A_d^\sharp(t, y, x_d) = \left(1 + \frac{x_d}{\delta}\right) A_d(t, y, 0) + \frac{x_d}{\delta} \text{Id}, \quad \text{for } -\delta < x_d < 0, t \in]-1, T_0[$$

and

$$(2.18) \quad A_j^\sharp(t, x) = 0, \quad \text{for } x_d < 0, \quad j = 1, \dots, d-1.$$

Then, instead of considering the Cauchy problem (2.6), one introduces the domain

$$\Omega_{T_0}^\sharp := \{(t, x) \in \mathbb{R}^{1+d} \mid -1 < t < T_0, -\delta < x_d\}$$

and consider the problem

$$(2.19) \quad \begin{cases} L^\sharp v^\varepsilon + \frac{1}{\varepsilon} \mathbf{1}_{\{x_d < 0\}} M v^\varepsilon = \mathbf{1}_{\{x_d > 0\}} F(t, x, v^\varepsilon) & \text{in } \Omega_{T_0}^\sharp \\ v^\varepsilon|_{\Gamma_0} = 0. \end{cases}$$

for which *no boundary condition is needed* precisely because $A_d|_{x_d=-\delta}$ is negative definite.

Corollary 2.8. *There exist $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$ the problem (2.19) has a unique solution $v^\varepsilon \in L^2(\Omega_{T_0}^\sharp) \cap L^\infty(\Omega_{T_0}^\sharp)$. Furthermore, $v^\varepsilon|_{\pm x_d > 0} \in H^\infty(\Omega_{T_0}^\sharp \cap \{\pm x_d > 0\})$, and $v^\varepsilon|_{\Omega_{T_0}^+} \rightarrow u$ in $H^s(\Omega_{T_0}^+)$ for all $s \in \mathbb{R}$ as $\varepsilon \rightarrow 0$ and*

$$\|\partial^\alpha v^\varepsilon|_{x_d < 0}\|_{L^2(\Omega_{T_0}^\sharp \cap \{x_d < 0\})} = O(\varepsilon).$$

Proof. The problem (2.19) is well posed since the boundary $\{x_d = -\delta\}$ is a maximal strictly negative subspace for $A_d^\sharp(t, y, -\delta)$. On the other hand, the restriction to $\Omega_{T_0}^\sharp$ of the solution of the Cauchy problem (2.6) is a solution of (2.19). Hence, the two solutions coincides and the result is a consequence of the Theorem 2.7. \square

This result shows that it is enough to solve the problem without any boundary condition on $\Omega_{T_0}^\sharp$ and for ε small enough this will give a good approximation (up to $0(\varepsilon)$) of the solution u of the original mixed problem. The region $-\delta < x_d < 0$ is a layer which "absorbs" the energy of the outgoing waves in such a way that the behavior of the solution in the domain $\{x_d > 0\}$ is arbitrary closed to that of u , as ε goes to 0.

3 More general domains

The result of this paper can be easily extended to the case of more general domains. For example, if Ω is a bounded connected open subset of \mathbb{R}^d , with smooth boundary $\partial\Omega$, and locally on one side of $\partial\Omega$, the two theorems can be extended to the mixed problem in $[0, T] \times \Omega$. In that case the matrix $A_d(t, y, 0)$ has to be replaced by the matrix

$$\mathcal{A}(t, x) := \sum_{i=1}^{i=d} \nu_i(x) A_i(t, x)$$

where $\nu(x) = (\nu_1(x), \dots, \nu_d(x))$ is the outgoing unitary normal at $x \in \partial\Omega$. Instead of Assumption 2.1, we assume that \mathcal{A} has a constant rank on $\mathbb{R} \times \partial\Omega$, and d_+ denotes the constant dimension of $\mathbb{E}_+(\mathcal{A}(x))$, $x \in \partial\Omega$. Instead of assumption 2.2, we assume that \mathcal{N} is a real \mathcal{C}^∞ bundle on $\partial\Omega$ of dimension $N - d_+$, and that for every $x \in \partial\Omega$, the quadratic form $v \mapsto \langle \mathcal{A}(t, x)v, v \rangle$ is definite negative on $\mathcal{N}(x) \cap \ker \mathcal{A}(x)^\perp$. In the Lemma 2.4, the conclusion is that

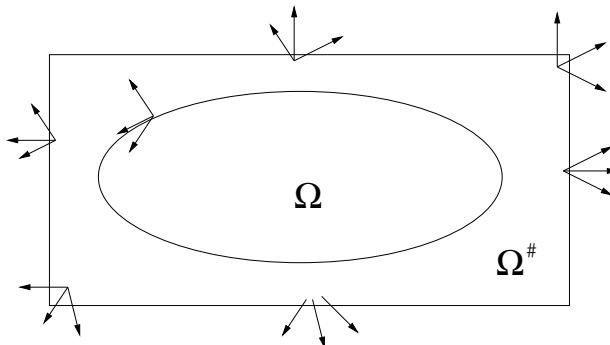
$$(3.1) \quad \mathbb{E}_+(\mathcal{A}(t, x)\Psi(t, x))^\perp = \Psi(t, x)^{-1}\mathcal{N}(x), \quad \forall (t, x) \in (\mathbb{R} \times \partial\Omega) \cap \mathcal{V}(p).$$

where $\Psi \in \mathcal{C}^\infty(\mathcal{V}(p))$, for $p \in \mathbb{R} \times \partial\Omega$. Finally, Instead of the Assumption 2.5 one assumes on can take \mathcal{V} containing $\mathbb{R} \times \Omega$ (or simply $]0, T_0[\times \Omega$ which

is enough). Concerning the extension L^\sharp of the operator to the exterior of $\mathbb{R} \times \Omega$ one requires that the extension of \mathcal{A}

$$\mathcal{A}^\sharp(t, x) = \sum_{i=1}^d \nu_i(x) A_i^\sharp(t, x)$$

is *continuous* across $\mathbb{R} \times \partial\Omega$ (and this corresponds to the Assumption 2.3). In particular, one can take an extension on a thin neighborhood of $\mathbb{R} \times \Omega$ of the form $\mathbb{R} \times \Omega^\sharp$. If one chooses Ω^\sharp to be a regular open set with completely outgoing eigenvectors for $\mathcal{A}^\sharp(t, x)$ when $x \in \partial\Omega^\sharp$, we will have again to solve a Cauchy problem in $[0, T_0] \times \Omega^\sharp$ (without boundary conditions on $\partial\Omega^\sharp$) and the solution u^ε will converges in $[0, T_0] \times \Omega$ to the solution u of the mixed problem in Ω with boundary conditions \mathcal{N} . The set $\Omega^\sharp \setminus \Omega$ plays the role of an "absorbing layer" which enables one to completely forget about the boundary conditions on $\partial\Omega$, while still solving a problem on the *bounded* domain Ω^\sharp . As a matter of fact, having in mind numerical applications, it is interesting to emphasize the fact that one can take for Ω^\sharp a polyhedral domain, with boundaries parallel to the coordinate axes for example.



The picture: on the boundary of $\partial\Omega$ the characteristic modes can be ingoing, outgoing, or tangent to $\partial\Omega$. But on the boundary of the extended domain Ω^\sharp , all the fields are outgoing: hence none boundary condition is needed.

4 Proof of the theorem 2.7

4.1 Step 1: reduction of the problem

From now on, to simplify the notations, we will forget the symbol \sharp of the extended matrices to $x_d < 0$, and simply note A_j , B , and L instead of A_j^\sharp ,

B^\sharp, L^\sharp . There is no risk of confusion since, initially, all the matrices were not defined for $x_d < 0$.

The main idea in the proof of Theorem 2.7 is to change the unknown in order to replace the problem (2.3) by a new (equivalent) one of the same type, but where the space \mathcal{N} is exactly the subspace orthogonal to $\mathbb{E}_+(A_d(t, y, 0))$ in \mathbb{R}^N .

Let us consider a matrix $\Phi(t, x)$, $N \times N$, with entries C^∞ on \mathbb{R}^{1+d} , constant outside a compact set, and such that

$$0 < c \leq |\det \Phi(t, x)| < c^{-1}, \quad \forall (t, x) \in \mathbb{R}^{1+d}$$

for some constant c . Let us define $\tilde{u}(t, x) := \Phi^{-1}(t, x)u(t, x)$, which satisfies the new system

$$(4.1) \quad \begin{cases} \tilde{L}\tilde{u} = \tilde{F}(t, x, \tilde{u}) \text{ in } \Omega_{T_0}^+, \\ \tilde{u}|_{\Gamma_{T_0}} \in \tilde{\mathcal{N}} \text{ on } \Gamma_{T_0}, \\ \tilde{u}|_{\Omega_0^+} = 0. \end{cases}$$

where $\tilde{L} = \sum \tilde{A}_j \partial_j + \tilde{B}$ with $\tilde{A}_j := {}^t\Phi A_j \Phi$, $\tilde{B} = {}^t\Phi B \Phi + {}^t\Phi L(\Phi)$, $\tilde{F} = {}^t\Phi F$, and $\tilde{\mathcal{N}}(t, y) := \Phi^{-1}(t, y, 0)\mathcal{N}(t, y)$. The new system (4.1) is still a symmetric hyperbolic system which satisfies the same assumptions 2.1 and 2.2.

From now on, we fix a function Φ as in the lemma 2.4. We can define our penalization matrix on the formulation (4.1). Let us introduce the orthogonal projector $\tilde{\mathbb{P}}_+(t, x)$ of \mathbb{R}^N onto $\mathbb{E}_+(\tilde{A}_d(t, x))$, $\tilde{\mathbb{P}}_-(t, x)$ the orthogonal projector onto $\mathbb{E}_-(\tilde{A}_d(t, x))$, and $\tilde{\mathbb{P}}_0(t, x)$ the orthogonal projector onto $\ker(\tilde{A}_d(t, x))$:

$$(4.2) \quad I = \tilde{\mathbb{P}}_+(t, x) + \tilde{\mathbb{P}}_-(t, x) + \tilde{\mathbb{P}}_0(t, x), \quad \forall (t, x) \in \mathbb{R}^{d+1}.$$

We will show that a solution of the problem is given by considering the following Cauchy problem

$$(4.3) \quad \begin{cases} \tilde{L}\tilde{v}^\varepsilon + \varepsilon^{-1}\mathbf{1}_{\{x_d < 0\}}\tilde{\mathbb{P}}_+\tilde{v}^\varepsilon = \mathbf{1}_{\{x_d > 0\}}\tilde{F}(t, x, \tilde{v}^\varepsilon) \text{ in } \Omega_{T_0}, \\ \tilde{v}^\varepsilon|_{\Omega_0} = 0. \end{cases}$$

and that the solution \tilde{v}^ε of (4.3) exists on Ω_{T_0} , and converges to \tilde{u} . Going back to the original unknown $u = \Phi v$, this will prove the main result.

4.2 Step 2: an approximate solution

In this section we construct an approximate solution of (4.3) of the form

$$(4.4) \quad \tilde{v}_a^\varepsilon(t, x) = \sum_{j=0}^M \varepsilon^j \tilde{V}_j(t, x),$$

where the $\tilde{V}_j \in H^\infty(\Omega_{T_0})$ for all $j = 0, \dots, M$. In order to solve the problem (4.3) we solve the equation in the half space $x_d > 0$ and in $x_d < 0$ coupled with the transmission condition

$$(4.5) \quad [(Id - \tilde{\mathbb{P}}_0)\tilde{v}^\varepsilon]_{\{x_d=0\}} = 0.$$

This transmission condition (4.5) splits into the following two equations:

$$(4.6) \quad \tilde{\mathbb{P}}_+[\tilde{v}^\varepsilon]_{\{x_d=0\}} = 0 \quad , \quad \tilde{\mathbb{P}}_-[\tilde{v}^\varepsilon]_{\{x_d=0\}} = 0.$$

In general, if $v(t, x)$ is a function defined on Ω_{T_0} we will note $v^+ := v|_{x_d>0}$ and $v^- := v|_{x_d<0}$. Replacing (4.4) in (4.3) gives at the order ε^{-1} , and in $x_d < 0$,

$$(4.7) \quad \tilde{\mathbb{P}}_+ \tilde{V}_0^- = 0 \quad \text{in } \Omega_{T_0}^-$$

which implies by the first equation of (4.6) that

$$(4.8) \quad (\tilde{\mathbb{P}}_+ \tilde{V}_0^+)_{|\Gamma_{T_0}} = 0 \quad \text{on } \Gamma_{T_0}.$$

On the side $x_d > 0$, at the order ε^0 one gets the equation $\tilde{L}\tilde{V}_0^+ = \tilde{F}(t, x, \tilde{V}_0^+)$. Therefore, \tilde{V}_0^+ is defined as the unique solution of the mixed problem

$$(4.9) \quad \begin{cases} \tilde{L}\tilde{V}_0^+ = \tilde{F}(t, x, \tilde{V}_0^+) & \text{in } \Omega_{T_0}, \\ (\tilde{\mathbb{P}}_+ \tilde{V}_0^+)_{|\Gamma_{T_0}} = 0 & \text{on } \Gamma_{T_0}, \\ \tilde{V}_0^+|_{\Omega_0} = 0. \end{cases}$$

and by uniqueness, $\tilde{V}_0^+ = \tilde{u}$ as desired. On the side $x_d < 0$ the term of order ε^0 is

$$(4.10) \quad \tilde{L}\tilde{V}_0^- + \tilde{\mathbb{P}}_+ \tilde{V}_1^- = 0.$$

Let us call $\tilde{\Pi}(t, x) := \tilde{\mathbb{P}}_-(t, x) + \tilde{\mathbb{P}}_0(t, x) = (I - \tilde{\mathbb{P}}_+)$. Since $\tilde{\mathbb{P}}_+ \tilde{V}_0^- = 0$, there holds $\tilde{V}_0^- = \tilde{\Pi}\tilde{V}_0^-$, and applying $\tilde{\Pi}$ on the left to the equation (4.10) leads to

$$(4.11) \quad \tilde{\Pi} \tilde{L} \tilde{\Pi} (\tilde{\Pi}\tilde{V}_0^-) = 0 \quad \text{in } \Omega_{T_0}^-,$$

$\tilde{\Pi} \tilde{L} \tilde{\Pi}$ is a *symmetric hyperbolic operator* acting on the space

$$(4.12) \quad \mathcal{E} := \{ u \in L^2(\Omega_{T_0}^-, \mathbb{R}^N) \mid (I - \tilde{\Pi})u = 0 \},$$

that is on the space of functions polarized on $\ker \tilde{\mathbb{P}}_+$. For instance, this kind of hyperbolic operator appears naturally in the context of weakly non linear geometric optics (see [JMR]) where it is a usual tool. Now the second part of the transmission relations (4.6) can be written

$$(4.13) \quad \tilde{\mathbb{P}}_- \tilde{\Pi} \tilde{V}_0^-|_{x_d=0} = (\tilde{\mathbb{P}}_- \tilde{V}_0^+)|_{x_d=0}$$

which is a boundary condition for the unknown $\tilde{\Pi} \tilde{V}_0^-$ because \tilde{V}_0^+ in the right hand side is already known ($\tilde{V}_0^+ = \tilde{u}$). This boundary condition for $\tilde{\Pi} \tilde{V}_0^-$ is maximally dissipative for the operator $\tilde{\Pi} \tilde{L} \tilde{\Pi}$, hence $\tilde{\Pi} \tilde{V}_0^- = \tilde{V}_0^-$ is uniquely defined by (4.11), (4.13), and the initial condition $(\tilde{\Pi} \tilde{V}_0^-)|_{\Omega_0^-} = 0$. Since the problem is linear, \tilde{V}_0^- is actually defined on $\Omega_{T_0}^-$.

Going back to the equation (4.10) shows that $\tilde{\mathbb{P}}_+ \tilde{V}_1^-$ is determined (as was $\tilde{\mathbb{P}}_+ \tilde{V}_0^-$ by the ε^{-1} terms),

$$(4.14) \quad \tilde{\mathbb{P}}_+ \tilde{V}_1^- = -\tilde{\mathbb{P}}_+ \tilde{L} \tilde{V}_0^-,$$

and the equation (4.10) is now entirely satisfied.

The construction follows by induction. For example, let us continue the construction in order to determine \tilde{V}_1 completely. The equation for the ε^1 terms in the side $\{x_d > 0\}$ is

$$(4.15) \quad \tilde{L} \tilde{V}_1^+ = \tilde{F}'_u(t, x, \tilde{V}_0^+) \tilde{V}_1^+,$$

and the functions \tilde{V}_1^+ and \tilde{V}_1^- are linked by the transmission conditions 4.6 which writes at the order ε^1 :

$$(4.16) \quad \tilde{\mathbb{P}}_+ \tilde{V}_1^+|_{\Gamma_{T_0}} = \tilde{\mathbb{P}}_+ \tilde{V}_1^-|_{\Gamma_{T_0}},$$

and

$$(4.17) \quad \tilde{\mathbb{P}}_- \tilde{V}_1^-|_{\Gamma_{T_0}} = \tilde{\mathbb{P}}_- \tilde{V}_1^+|_{\Gamma_{T_0}}.$$

Since the right hand side of (4.16) is known (from (4.14)), the function \tilde{V}_1^+ is the unique solution of the well posed mixed problem (4.15), (4.16), with the initial condition $\tilde{V}_1^+|_{t<0} = 0$.

It remains to determine $(\text{Id} - \tilde{\mathbb{P}}_+) \tilde{V}_1^- = \tilde{\Pi} \tilde{V}_1^-$. The equation for the terms of order ε^1 in $\{x_d < 0\}$ is

$$(4.18) \quad \tilde{L} \tilde{V}_1^- + \tilde{\mathbb{P}}_+ \tilde{V}_2^- = 0.$$

We first apply $\tilde{\Pi}$ to the equation in order to cancel the term in \tilde{V}_2^- which is unknown, and replace $\tilde{V}_1^- = \tilde{\Pi} \tilde{V}_1^- + \tilde{\mathbb{P}}_+ \tilde{V}_1^- = \tilde{\Pi} \tilde{V}_1^- - \tilde{\mathbb{P}}_+ \tilde{L} \tilde{V}_0^-$ which leads to the equation

$$(4.19) \quad \tilde{\Pi} \tilde{L} \tilde{\Pi} (\tilde{\Pi} \tilde{V}_1^-) = \tilde{\Pi} \tilde{L} \tilde{\mathbb{P}}_+ \tilde{L} \tilde{V}_0^- \quad \text{in } \Omega_{T_0}^-.$$

This is again a symmetric hyperbolic system in the space \mathcal{E} and the equation (4.17) appear to be boundary conditions for this system since the right hand side of (4.17) is known. Hence $\tilde{\Pi} \tilde{V}_1^-$ is the unique solution of the mixed problem (4.19), (4.17), with the initial condition $\tilde{\Pi} \tilde{V}_1^-|_{t=0} = 0$. In conclusion, \tilde{V}_1^- is completely determined, and going back to the equation (4.18) we see that $\tilde{\mathbb{P}}_+ \tilde{V}_2^-$ is also determined, and that the equation (4.18) is entirely satisfied. The next steps of the construction are completely analogous.

4.3 Step3: estimations

We have constructed an approximate solution \tilde{v}_a^ε of the form (4.4) of the problem (4.3), satisfying

$$(4.20) \quad \begin{cases} \tilde{L} \tilde{v}_a^\varepsilon + \varepsilon^{-1} \mathbf{1}_{\{x_d < 0\}} \tilde{\mathbb{P}}_+ \tilde{v}_a^\varepsilon = \mathbf{1}_{\{x_d > 0\}} \tilde{F}(t, x, \tilde{v}_a^\varepsilon) + \varepsilon^k r^\varepsilon & \text{in } \Omega_{T_0}, \\ \tilde{v}_a^\varepsilon|_{\Omega_0} = 0. \end{cases},$$

where the error term r^ε is piecewise smooth:

$$(4.21) \quad \|r^\varepsilon\|_{L^2(\Omega_{T_0})} = O(1), \quad \|r^\varepsilon|_{\pm x_d > 0}\|_{H^m(\Omega_{T_0}^\pm)} = O(1), \quad (\varepsilon \rightarrow 0^+, \forall m \in \mathbb{N}).$$

We look for an exact solution \tilde{v}^ε of the form

$$(4.22) \quad \tilde{v}^\varepsilon = \tilde{v}_a^\varepsilon + \varepsilon \tilde{w}^\varepsilon$$

where \tilde{w}^ε is defined by the system

$$\begin{cases} \tilde{L} \tilde{w}^\varepsilon + \varepsilon^{-1} \mathbf{1}_{\{x_d < 0\}} \tilde{\mathbb{P}}_+ \tilde{w}^\varepsilon = \mathbf{1}_{\{x_d > 0\}} \tilde{G}(t, x, \tilde{v}_a^\varepsilon, \varepsilon \tilde{w}^\varepsilon) \tilde{w}^\varepsilon + \varepsilon^{k-1} r^\varepsilon & \text{in } \Omega_{T_0}, \\ \tilde{w}^\varepsilon|_{\Omega_0} = 0. \end{cases}$$

where \tilde{G} is the C^∞ function defined by the Taylor formula:

$$(4.23) \quad \tilde{G}(t, x, v, \varepsilon w)w = \varepsilon^{-1}(\tilde{F}(t, x, v + \varepsilon w) - \tilde{F}(t, x, v)).$$

This is a semilinear hyperbolic system, and we will solve it by using a standard Picard's iterative scheme, where we note $\mathbf{1}_- = \mathbf{1}_{\{x_d < 0\}}$ and $\mathbf{1}_+ = \mathbf{1}_{\{x_d > 0\}}$:

$$\begin{cases} \tilde{L}\tilde{w}^{\varepsilon, \nu+1} + \varepsilon^{-1}\mathbf{1}_-\tilde{\mathbb{P}}_+\tilde{w}^{\varepsilon, \nu+1} - \mathbf{1}_+\tilde{G}(t, x, \tilde{v}_a^\varepsilon, \varepsilon\tilde{w}^{\varepsilon, \nu})\tilde{w}^{\varepsilon, \nu+1} = \varepsilon^{k-1}r^\varepsilon, \\ \tilde{w}^{\varepsilon, \nu+1}|_{\Omega_0} = 0. \end{cases}$$

In order to show the convergence of the sequence $\tilde{w}^{\varepsilon, \nu}$ we need estimations for the following linear problem

$$(4.24) \quad \begin{cases} \tilde{L}\mathbf{v} + \varepsilon^{-1}\mathbf{1}_-\tilde{\mathbb{P}}_+\mathbf{v} - \mathbf{1}_+\tilde{G}(\tilde{v}_a^\varepsilon, \varepsilon\mathbf{b})\mathbf{v} = \varepsilon^{k-1}r^\varepsilon, \\ \mathbf{v}|_{\Omega_0} = 0, \end{cases}$$

where $\tilde{G}(\tilde{v}^\varepsilon, \varepsilon\mathbf{b}) = \tilde{G}(t, x, \tilde{v}^\varepsilon, \varepsilon\mathbf{b})$ where \mathbf{b} is a given function, which plays the role of $\tilde{w}^{\varepsilon, \nu}$ when solving the system for the unknown $\mathbf{v} = \tilde{w}^{\varepsilon, \nu+1}$.

Let us introduce some notations. We will denote by Z_j the vector fields $Z_j = \partial_j$ if $j = 0, \dots, d-1$ and $Z_d := \frac{x_d}{\langle x_d \rangle} \partial_d$, with $\langle x_d \rangle = (1 + x_d^2)^{1/2}$. We will note $Z^\alpha := Z_0^{\alpha_0} Z_1^{\alpha_1} \dots Z_d^{\alpha_d}$ for $\alpha = (\alpha_0, \dots, \alpha_d) \in \mathbb{N}^{1+d}$. For $\lambda > 0$ we will note

$$\|v\|_{0, \lambda} := \left(\int_{\Omega_{T_0}} e^{-2\lambda t} |v(t, x)|^2 dt dx \right)^{1/2}$$

and for $m \in \mathbb{N}$ and $\lambda > 0$

$$\|v\|_{m, \lambda, \varepsilon} := \sum_{|\alpha| \leq m} \lambda^{m-|\alpha|} \|(\sqrt{\varepsilon}Z)^\alpha v\|_{0, \lambda}.$$

We will also need the following norms, corresponding to the same definition but where the domain of integration is replaced by $\Omega_{T_0}^-$:

$$\|\mathbf{v}\|_{m, \lambda, \varepsilon, -} := \sum_{|\alpha| \leq m} \lambda^{m-|\alpha|} \|e^{-\lambda t} (\sqrt{\varepsilon}Z)^\alpha \mathbf{v}\|_{L^2(\Omega_{T_0}^-)},$$

and $\|\cdot\|_{m, \lambda, \varepsilon, +}$ the norm where $\Omega_{T_0}^-$ is replaced by $\Omega_{T_0}^+$. We denote by $H_{co}^m(\Omega_{T_0})$ the subspace of all $\mathbf{v} \in L^2(\Omega_{T_0})$ such that $\|\mathbf{v}\|_{m, \lambda, \varepsilon}$ is finite. We will also note $|u|_\infty := \|u\|_{L^\infty(\Omega_{T_0})}$.

Proposition 4.1. *Let G the function defined in (4.23). For all $\mathbf{b}, \mathbf{v} \in H_{co}^m(\Omega_{T_0}) \cap L^\infty(\Omega_{T_0})$ valued in \mathbb{R}^N , the function $\mathbf{1}_{\{x_d > 0\}} G(t, x, \tilde{v}_a^\varepsilon, \varepsilon \mathbf{b}) \mathbf{v}$ is also in $H_{co}^m(\Omega_{T_0}) \cap L^\infty(\Omega_{T_0})$. Moreover, for all $R > 0$ there exists $C(R) > 0$ such that, if $\|\mathbf{b}\|_{L^\infty(\Omega_{T_0})} \leq R$ there holds:*

$$(4.25) \quad \|\mathbf{1}_{\{x_d > 0\}} G(t, x, \tilde{v}_a^\varepsilon, \varepsilon \mathbf{b}) \mathbf{v}\|_{m, \lambda, \varepsilon} \leq C(R) (\|\mathbf{v}\|_{m, \lambda, \varepsilon} + |\mathbf{v}|_\infty \|\mathbf{b}\|_{m, \lambda, \varepsilon}),$$

for all $\varepsilon > 0$.

Proof. This is a "Moser" type estimate, which follows in a classical way from the corresponding weighted Gagliardo-Nirenberg estimates proved in the appendix of [Gu], and the Hölder estimate. \square

Let us introduce a new notation. We will denote by $H_\pm^1(\Omega_T)$ the space of functions $u \in L^2(\Omega_T)$ such that $u|_{\Omega_T^+} \in H^1(\Omega_T^+)$ and $u|_{\Omega_T^-} \in H^1(\Omega_T^-)$. We can now prove the following estimate on the linear problem:

Proposition 4.2. *Let $R > 0$ and $m \in \mathbb{N}$. There are constants $c_m(R) > 0$ and $\lambda_m(R) > 1$ such that the following holds true. For all $\mathbf{b} \in H_{co}^m \cap L^\infty(\Omega_{T_0})$ such that $|\mathbf{b}|_\infty \leq R$, for all $\mathbf{f} \in H_{co}^m(\Omega_{T_0}) \cap H_\pm^1(\Omega_{T_0})$, with $\mathbf{f}|_{t < 0} = 0$, the problem (4.24) has a unique solution $\mathbf{u} \in H_{co}^m(\Omega_{T_0}) \cap H_\pm^1(\Omega_{T_0})$. Moreover, it follows that $\mathbf{v} \in L^\infty(\Omega_{T_0})$ and the following estimate holds*

$$(4.26) \quad \lambda^{1/2} \|\mathbf{v}\|_{m, \lambda, \varepsilon} + \varepsilon^{-1/2} \|\tilde{\mathbb{P}} \mathbf{v}_-\|_{m, \lambda, \varepsilon, -} \leq c_m(R) \lambda^{-1/2} (\|\mathbf{f}\|_{m, \lambda, \varepsilon} + \|\mathbf{b}\|_{m, \lambda, \varepsilon} |\mathbf{v}|_\infty).$$

for all $\lambda \geq \lambda_m(R)$, and all $\varepsilon > 0$.

Proof. In all the proof we will note $\tilde{\mathbb{P}}$ instead of $\tilde{\mathbb{P}}_+$ to simplify.

1/ The first step is the L^2 estimate. Let us call \mathbf{f} the right hand side of the equation (4.24), and note $\tilde{\mathbf{v}} = e^{-\lambda t} \mathbf{v}$ so that \mathbf{v} satisfies

$$(4.27) \quad \begin{cases} \tilde{L} \tilde{\mathbf{v}} + \lambda \tilde{\mathbf{v}} + \varepsilon^{-1} \mathbf{1}_{\{x_d < 0\}} \tilde{\mathbb{P}} \tilde{\mathbf{v}} = e^{-\lambda t} \mathbf{f}, \\ \tilde{\mathbf{v}}|_{\Omega_0} = 0, \end{cases}$$

Taking the scalar product of the equations with $\tilde{\mathbf{v}}$ and integrating by parts in Ω_{T_0} leads to the following L^2 estimate where we note $\mathbf{v}_- = \mathbf{v}|_{\Omega_{T_0}^-}$:

$$(4.28) \quad \lambda^{1/2} \|\mathbf{v}\|_{0, \lambda} + \frac{1}{\varepsilon^{1/2}} \|\tilde{\mathbb{P}} \mathbf{v}_-\|_{0, \lambda, -} \lesssim \frac{1}{\lambda^{1/2}} \|\mathbf{f}\|_{0, \lambda},$$

for all $\lambda \geq \lambda_0$.

2/ In order to estimate higher order derivatives, we need to prepare the system and change again of unknown. Since the spaces $\mathbb{E}_+(A_d(t, y, 0))$, $\mathbb{E}_-(A_d(t, y, 0))$ and $\ker A_d(t, y, 0)$ have constant rank, there exist for all $(t^0, y^0) \in \mathbb{R}^d$ a neighborhood $\mathcal{V}(t^0, y^0)$ of (t^0, y^0) in \mathbb{R}^d and a smooth matrix $\Psi(t, y) \in \mathcal{C}^\infty(\mathcal{V}(t^0, y^0), \mathcal{M}_{N \times N}(\mathbb{R}))$, ${}^t\Psi\Psi = \text{Id}$ such that $\Psi(t, y)\mathbb{E}_+(t, y, 0)$, $\Psi(t, y)\mathbb{E}_-(t, y, 0)$ and $\Psi(t, y)\ker A_d(t, y, 0)$ are constant linear subspaces of \mathbb{R}^N (=independent of (t, y)). Let us note these spaces respectively: \mathbb{V}_- , \mathbb{V}_+ and \mathbb{V}_0 . To simplify the proof, we also assume that one can take $\mathcal{V}(0, 0) = \mathbb{R}^d$, so that one has just to work with one change of variable. (In the general case, one would have to introduce a finite number of local coordinate patches). We introduce the unknown defined by $\mathbf{u}(t, x) := \Psi(t, y)\mathbf{v}(t, x)$. the system satisfied by \mathbf{u} is

$$(4.29) \quad \partial_t \mathbf{u} + \sum_1^d \Psi A_j {}^t\Psi \partial_t \mathbf{u} + \frac{1}{\varepsilon} \mathbf{1}_{\{x_d < 0\}} \Psi \mathbb{P} {}^t\Psi \mathbf{u} + \mathcal{B}(\varepsilon \mathbf{b}) \mathbf{u} = \Psi \mathbf{f},$$

with

$$\mathcal{B}(\varepsilon \mathbf{b}) = \Psi(\partial_t {}^t\Psi + \sum A_j \partial_j {}^t\Psi) + \Psi B {}^t\Psi + \mathbf{1}_- \Psi \tilde{G}(\tilde{v}_a^\varepsilon, \varepsilon \mathbf{b}) {}^t\Psi$$

The matrix $\Psi \mathbb{P} {}^t\Psi$ is constant, and is the matrix of the orthogonal projector of \mathbb{R}^N onto \mathbb{V}_+ . We will make a first order Taylor expansion around $x_d = 0$ of the matrix $\Psi A_d {}^t\Psi$. The matrix $\Psi A_d(t, y, 0) {}^t\Psi$ has constant kernel \mathbb{V}_0 and constant range \mathbb{V}_0^\perp . We denote by Π_0 the (matrix of) the orthogonal projector of \mathbb{R}^N onto \mathbb{V}_0^\perp , and $\Pi := \Psi \mathbb{P} {}^t\Psi$. There exists a smooth symmetric matrix $S(t, y)$, uniformly regular (that is $0 < c \leq |\det S| \leq C$ on \mathbb{R}^d), such that $[\Pi_0, S] = 0$ and: $\Psi(t, y) A_d(t, y, 0) {}^t\Psi(t, y) = \Pi_0 S \Pi_0 (= S \Pi_0)$. With these notations, the system (4.29) takes the form

$$(4.30) \quad \partial_t \mathbf{u} + \sum_1^d \mathcal{A}_j Z_j \mathbf{u} + \Pi_0 S \Pi_0 \partial_d \mathbf{u} + \frac{1}{\varepsilon} \mathbf{1}_{\{x_d < 0\}} \Pi \mathbf{u} + \mathcal{B}(\varepsilon \mathbf{b}) \mathbf{u} = \Psi \mathbf{f}.$$

We note \mathbf{L}^ε the first order operator defined by the left hand side of (4.30). The function \mathbf{u} satisfies also the L^2 estimate (4.28) with $\Pi \mathbf{u}$ in place of $\tilde{\mathbb{P}} \mathbf{v}$. Let us now apply to the equation (4.30) the operator $(\sqrt{\varepsilon} Z)^\alpha = \varepsilon^{\frac{|\alpha|}{2}} Z^\alpha$, where $|\alpha| \leq m$. The energy estimate gives

$$(4.31) \quad \lambda^{m-|\alpha|+1/2} \|(\sqrt{\varepsilon} Z)^\alpha \mathbf{u}\|_{0, \lambda} + \frac{1}{\varepsilon^{1/2}} \lambda^{m-|\alpha|} \|\Pi(\sqrt{\varepsilon} Z)^\alpha \mathbf{u}_-\|_{0, \lambda, -} \lesssim \frac{1}{\lambda^{1/2}} (\|f\|_{m, \lambda, \varepsilon} + \lambda^{m-|\alpha|} \|[\mathbf{L}^\varepsilon, (\sqrt{\varepsilon} Z)^\alpha] \mathbf{u}\|_{0, \lambda}),$$

and one is lead to control the commutator in the right hand side. An important point is that Π and Π_0 commute with Z^α . Hence

$$(4.32) \quad \begin{aligned} [\mathbf{L}^\varepsilon, Z^\alpha] &= \sum [\mathcal{A}_j Z_j, Z^\alpha] + \Pi_0 [S \partial_d, Z^\alpha] \Pi_0 + [\mathcal{B}(\varepsilon \mathbf{b}), Z^\alpha] \\ &= \sum_{|\beta| \leq |\alpha|} \mathcal{B}_\beta Z^\beta + \sum_{|\gamma| \leq |\alpha| - 1} \mathcal{C}_\gamma Z^\gamma \Pi_0 \partial_d + [\mathcal{B}(\varepsilon \mathbf{b}), Z^\alpha] \end{aligned}$$

where \mathcal{B}_β and \mathcal{C}_γ are smooth $N \times N$ matrices. Hence

$$(4.33) \quad \begin{aligned} \lambda^{m-|\alpha|} \|[L^\varepsilon, (\sqrt{\varepsilon} Z)^\alpha] \mathbf{u}\|_{0,\lambda} &\lesssim \\ \|\mathbf{u}\|_{m,\lambda,\varepsilon} + \|\sqrt{\varepsilon} \Pi_0 \partial_d \mathbf{u}\|_{m-1,\lambda,\varepsilon} + \|\mathbf{b}\|_{m,\lambda,\varepsilon} \|\mathbf{u}\|_\infty. \end{aligned}$$

Expressing $\Pi_0 \partial_d \mathbf{u}$ by the equation (4.30) leads to

$$(4.34) \quad \begin{aligned} \|\sqrt{\varepsilon} \Pi_0 \partial_d \mathbf{u}\|_{m-1,\lambda} &\lesssim \sqrt{\varepsilon} \|\mathbf{u}\|_{m,\lambda,\varepsilon} + \frac{1}{\sqrt{\varepsilon}} \|\Pi \mathbf{u}_-\|_{m-1,\lambda,-} + \sqrt{\varepsilon} \|\mathbf{f}\|_{m-1,\lambda,\varepsilon} \\ &\quad + \sqrt{\varepsilon} \|\mathbf{b}\|_{m-1,\lambda,\varepsilon} \|\mathbf{u}\|_\infty. \end{aligned}$$

By replacing in (4.33) and (4.32) one gets

$$(4.35) \quad \begin{aligned} \lambda^{m-|\alpha|+1/2} \|(\sqrt{\varepsilon} Z)^\alpha \mathbf{u}\|_{0,\lambda} + \frac{1}{\sqrt{\varepsilon}} \lambda^{m-|\alpha|} \|\Pi (\sqrt{\varepsilon} Z)^\alpha \mathbf{u}_-\|_{0,\lambda,-} &\lesssim \\ \frac{1}{\lambda^{1/2}} (\|\mathbf{f}\|_{m,\lambda,\varepsilon} + \|\mathbf{u}\|_{m,\lambda,\varepsilon} + \frac{1}{\sqrt{\varepsilon}} \|\Pi \mathbf{u}_-\|_{m,\lambda,\varepsilon,-} + \|\mathbf{b}\|_{m,\lambda,\varepsilon} \|\mathbf{u}\|_\infty). \end{aligned}$$

Summing all inequalities (4.35) for $|\alpha| \leq m$, and taking λ large enough to absorb in the left hand side the terms $\|\mathbf{u}\|_{m,\lambda,\varepsilon}$ and $\frac{1}{\sqrt{\varepsilon}} \|\Pi \mathbf{u}_-\|_{m,\lambda,\varepsilon,-}$ yields the inequality (4.26) for \mathbf{u} and hence for \mathbf{v} . \square

We need now to estimate the normal derivative of \mathbf{u} , the method is classical. We keep the notations of the proof of the previous proposition and work with the unknown \mathbf{u} and the equation (4.30). By (4.34) we already have an estimation of $\|\sqrt{\varepsilon} \partial_d \Pi_0 \mathbf{u}\|_{m-1,\lambda,\varepsilon}$. It remains to estimate $\partial_d (\text{Id} - \Pi_0) \mathbf{u}$. Let us denote $\mathbf{u}_I := (\text{Id} - \Pi_0) \mathbf{u}$, $\mathbf{u}_{II} := \Pi_0 \mathbf{u}$ and

$$\mathbb{X} := \sum_{j=0}^d (\text{Id} - \Pi_0) \mathcal{A}_j (\text{Id} - \Pi_0) Z_j.$$

By applying $(\text{Id} - \Pi_0)$ on the left to the system (4.30), since $(\text{Id} - \Pi_0)\Pi = 0$ we get the equation

$$\mathbb{X}\mathbf{u}_I = \sum_0^d C_j Z_j \mathbf{u}_{II} - (\text{Id} - \Pi_0)\mathcal{B}(\mathbf{b})\mathbf{u} + \mathbf{f}_I$$

and applying the derivation ∂_d leads to an equation of the form

$$(4.36) \quad \begin{aligned} \mathbb{X}\partial_d \mathbf{u}_I &= \sum_{|\alpha| \leq 1} M_\alpha Z^\alpha \mathbf{u} + \sum_{|\beta| \leq 1} N_\beta Z^\beta \partial_n \mathbf{u}_{II} \\ &\quad + \partial_d \mathbf{f}_I - \partial_d((\text{Id} - \Pi_0)\mathcal{B}(\mathbf{b})\mathbf{u}). \end{aligned}$$

The energy estimate for the operator \mathbb{X} , applied to equation (4.36) on $\Omega_{T_0}^+$ and on $\Omega_{T_0}^-$ respectively, implies

$$\begin{aligned} \|\partial_d \mathbf{u}_I^\pm\|_{m-2, \lambda, \varepsilon} &\lesssim \lambda^{-1} (\|\mathbf{u}\|_{m-1, \lambda, \varepsilon} + \|\partial_d \mathbf{u}_{II}^\pm\|_{m-1, \lambda, \varepsilon} + \|\partial_d \mathbf{f}^\pm\|_{m-2, \lambda, \varepsilon} \\ &\quad + c(R)(\|u\|_{m-2, \lambda, \varepsilon} + \|\mathbf{b}\|_{m-2, \lambda, \varepsilon} |\mathbf{u}|_\infty)), \end{aligned}$$

and by (4.34) we get

$$(4.37) \quad \begin{aligned} \|\partial_d \mathbf{u}_I^\pm\|_{m-2, \lambda, \varepsilon} &\lesssim \frac{c(R)}{\lambda} \left(\|\mathbf{u}\|_{m, \lambda, \varepsilon} + \frac{1}{\varepsilon} \|\Pi \mathbf{u}_-\|_{m-1, \lambda, \varepsilon, -} + \|\mathbf{b}\|_{m-1, \lambda, \varepsilon} |\mathbf{u}|_\infty \right. \\ &\quad \left. + \|\mathbf{f}\|_{m-1, \lambda, \varepsilon} + \|\partial_d \mathbf{f}^\pm\|_{m-2, \lambda, \varepsilon} \right). \end{aligned}$$

We use also this suitable version of the Sobolev embedding that we recall. There exists $\kappa > 0$ and $\rho > 0$ such that, if $u \in H_{co}^m(\Omega_{T_0})$ is such that $\partial_d u^\pm \in H_{co}^{m-2}(\Omega_{T_0}^\pm)$, the following estimate holds (see [Su2], [Gu2]):

$$(4.38) \quad |u|_\infty \leq \kappa \frac{1}{\varepsilon^\rho} e^{\lambda T} \left(\|u\|_{m, \lambda, \varepsilon} + \|\sqrt{\varepsilon} \partial_d u^+\|_{m-2, \lambda, \varepsilon} + \|\sqrt{\varepsilon} \partial_d u^-\|_{m-2, \lambda, \varepsilon} \right)$$

for all $\lambda > 0$, and all $\varepsilon > 0$. In fact, $\rho = (d+1)/4$, but this has no importance in the proof. Let us recall that k is the order of the approximate solution appearing in the right hand side of (4.20). We can now prove that the sequence $\tilde{w}^{\varepsilon, \nu}$ is bounded, under the assumption that $k-1 > \rho$.

Lemma 4.3. *There exist $\lambda > 0$, $a > 0$ and $\varepsilon_0 > 0$ such that:*

$$(4.39) \quad \|\tilde{w}^{\varepsilon, \nu}\|_{m, \lambda, \varepsilon} \leq a\varepsilon^{k-1}, \quad |\tilde{w}^{\varepsilon, \nu}|_\infty \leq 1, \quad \forall \nu \in \mathbb{N}, \forall \varepsilon \in]0, \varepsilon_0].$$

Proof. We show the lemma by induction, so we assume that $\tilde{w}^{\varepsilon,\nu}$ satisfies (4.39) (m is fixed). The proposition 4.2 gives two constants $C_m(1)$ and $\lambda_m(1)$ associated to the choice $R = 1$. Taking first λ large enough in the estimate (4.26) we obtain

$$(4.40) \quad \lambda^{1/2} \|w^{\varepsilon,\nu+1}\|_{m,\lambda,\varepsilon} + \frac{1}{\varepsilon^{1/2}} \|\Pi w_-^{\varepsilon,\nu+1}\|_{m,\lambda,\varepsilon,-} \leq \frac{a}{2} \varepsilon^{k-1} + C_m(1) \lambda^{-1/2} a \varepsilon^{k-1} |w^{\nu+1,\varepsilon}|_\infty.$$

The parameter $\lambda > 1$ is now fixed. Replacing the estimates (4.34) and (4.37) in the L^∞ imbedding (4.38) gives, for some constant κ_0 :

$$|w^{\varepsilon,\nu+1}|_\infty \leq \frac{\kappa_0}{\varepsilon^\rho} \left(\|w^{\varepsilon,\nu+1}\|_{m,\lambda,\varepsilon} + \frac{1}{\varepsilon} \|\Pi w_-^{\varepsilon,\nu+1}\|_{m-1,\lambda,\varepsilon,-} + a \varepsilon^{k-1} |w^{\varepsilon,\nu+1}|_\infty + \varepsilon^{k-1} \right).$$

and for $\varepsilon > 0$ small enough we obtain (for some constant κ_1)

$$(4.41) \quad |w^{\varepsilon,\nu+1}|_\infty \leq \frac{\kappa_1}{\varepsilon^\rho} \left(\|w^{\varepsilon,\nu+1}\|_{m,\lambda,\varepsilon} + \frac{1}{\varepsilon} \|\Pi w_-^{\varepsilon,\nu+1}\|_{m-1,\lambda,\varepsilon,-} + \varepsilon^{k-1} \right).$$

Replacing now (4.41) in (4.40), and taking again ε small enough so that all the terms $\|w^{\varepsilon,\nu+1}\|_{m,\lambda,\varepsilon}$ and $\|\Pi w_-^{\varepsilon,\nu+1}\|_{m-1,\lambda,\varepsilon,-}$ can be absorbed in the left hand side (because $k - 1 > \rho + \frac{1}{2}$), yields the estimate

$$(4.42) \quad \|w^{\varepsilon,\nu+1}\|_{m,\lambda,\varepsilon} + \frac{1}{\varepsilon^{1/2}} \|\Pi w_-^{\varepsilon,\nu+1}\|_{m,\lambda,\varepsilon,-} \leq a \varepsilon^{k-1}$$

which implies in particular the first estimate of (4.39) at the rank $\nu + 1$. We conclude the proof by replacing (4.42) in the inequality (4.41) which gives, by taking once more ε small enough, the second part of (4.39) for $w^{\varepsilon,\nu+1}$. \square

Now, using lemma 4.3, it is a classical argument to show that the sequence $w^{\varepsilon,\nu}$ converges in $L^2(\Omega_T)$ towards $w \in H_{co}^m(\Omega_T)$, satisfying the same estimates as $w^{\varepsilon,\nu}$.

5 Proof of the theorem 2.6

This section is devoted to the proof of the second main theorem of our paper. Instead of the problem (4.3), we will replace the penalization matrix $\tilde{\mathbb{P}}_+$ by

merely Id_N and consider the new simpler problem

$$(5.1) \quad \begin{cases} \tilde{L}\tilde{v}^\varepsilon + \varepsilon^{-1}\mathbf{1}_{\{x_d < 0\}}\tilde{v}^\varepsilon = \mathbf{1}_{\{x_d > 0\}}\tilde{F}(t, x, \tilde{v}^\varepsilon) \text{ in } \Omega_{T_0}, \\ \tilde{v}^\varepsilon|_{\Omega_0} = 0. \end{cases}$$

The first step is to find an approximate solution of the problem, and this section is exactly analogous to the preceding section 3.2, but for the new problem (5.1). The main difference is that the construction of the approximate solutions shows the existence of boundary layers for this problem. This is not surprising since boundary layers already appeared in the work by J. Droniou devoted to the linear, non characteristic case ([D]), which can be seen as a special case of our result.

We look for an approximate solution of the form

$$(5.2) \quad v_a^\varepsilon(t, x) = \sum_{j=0}^M \varepsilon^j V_j(t, x, x_d/\varepsilon),$$

where the $V_j(t, x, z)$ for all $j = 0, \dots, M$ are functions which writes $V_j(t, x, z) = V_j^+(t, x, z)$ if $x_d > 0$ and $z > 0$, and $V_j(t, x, z) = V_j^-(t, x, z)$ if $x_d < 0$ and $z < 0$ with respectively

$$(5.3) \quad V_j^\pm(t, x, z) = \bar{V}_j^\pm(t, x) + V_j^{*,\pm}(t, y, z)$$

with $\bar{V}_j^\pm \in H^\infty(\Omega_{T_0}^\pm)$ and $V_j^{*,\pm} \in e^{-\delta_j|z|}H^\infty(\Gamma_{T_0}^\pm \times \mathbb{R}_\pm)$ for some $\delta_j > 0$ depending on V_j^\pm . Hence, after substitution of z with x_d/ε the terms in $V_j^{*,\pm}(t, y, x_d/\varepsilon)$ are "boundary layer terms" which go to 0 in $L^2(\Omega_{T_0}^\pm)$ as $\varepsilon \rightarrow 0$, and are exponentially decaying to zero as $|x_d| \rightarrow \infty$.

We solve the equation in the half space $x_d > 0$ and in $x_d < 0$ coupled with the transmission condition

$$[(\text{Id} - \tilde{\mathbb{P}}_0)\tilde{v}^\varepsilon]_{\{x_d=0\}} = 0.$$

which can be also written

$$(5.4) \quad \tilde{\mathbb{P}}_-[\tilde{v}^\varepsilon]_{\{x_d=0\}} = 0, \text{ and } \tilde{\mathbb{P}}_+[\tilde{v}^\varepsilon]_{\{x_d=0\}} = 0.$$

In fact the study of the equations shows that all the \bar{V}_j terms vanish when $x_d < 0$ and all the V_j^* terms vanish when $z > 0$. Hence, in order to

simplify the reduction we will directly look for an approximate solution of the form

$$(5.5) \quad v_a^\varepsilon(t, x) = \sum_{j=0}^M \varepsilon^j \bar{V}_j^+(t, x), \text{ on } x_d > 0,$$

and

$$(5.6) \quad v_a^\varepsilon(t, x) = \sum_{j=0}^M \varepsilon^j V_j^{*, -}(t, y, x_d/\varepsilon), \text{ on } x_d < 0.$$

The first profile V^0 is now determined by the following three steps.

Step 1. Order ε^{-1} , size < 0 . The equation for the terms in ε^{-1} on the side $x_d < 0$ is

$$(5.7) \quad \tilde{A}_d(t, y, 0) \partial_z V_0^{*, -} + V_0^{*, -} = 0,$$

which requires the polarization condition

$$(5.8) \quad V_0^{*, -} \in \mathbb{E}_+(\tilde{A}_d(t, y, 0))$$

in order to get the exponential decay as $z \rightarrow -\infty$.

Step 2. Order ε^0 , size > 0 . The equation for the terms of order $O(1)$ on the side $x_d > 0$ is just

$$(5.9) \quad \tilde{L} \bar{V}_0^+ = F(\bar{V}_0^+).$$

Step 3. The boundary condition (5.4) at the order ε^0 gives the two conditions (taking (5.8) into account):

$$(5.10) \quad \tilde{\mathbb{P}}_+ \bar{V}_0^+|_{x_d=0} = 0,$$

and

$$(5.11) \quad \tilde{\mathbb{P}}_- \bar{V}_0^{*, -}|_{z=0} = -\tilde{\mathbb{P}}_- \bar{V}_0^+|_{x_d=0}.$$

Now, the system (5.9) together with the boundary condition (5.10) (and the understood conditions that $\bar{V}_0^+|_{t>0} = 0$) is exactly the desired original mixed problem (after the reduction of section 4.1), which is well posed, and

so $\overline{V}_0^+ = \tilde{u}$. Then, the second condition (5.11) together with the ODE (5.7) determines completely $V_0^{*, -}$.

The construction can be continued by induction and all the terms $V_j^{*, -}$, \overline{V}_j^+ are determined, for all $j \in \mathbb{N}$. The equation for $V_1^{*, -}$ (in the side $z < 0$) is:

$$(5.12) \quad \tilde{A}_d(t, y, 0) \partial_z V_1^{*, -} + V_1^{*, -} + L V_0^{*, -} + \partial_d \tilde{A}_d(t, y, 0) z \partial_z V_0^{*, -} = 0,$$

and the equation for \overline{V}_1^+ (in the side $x_d > 0$) is:

$$(5.13) \quad \tilde{L} \overline{V}_1^+ = F'(\overline{V}_0^+) \overline{V}_1^+.$$

More generally for $V_j^{*, -}$, $j \geq 1$

$$(5.14) \quad \tilde{A}_d(t, y, 0) \partial_z V_j^{*, -} + V_j^{*, -} = q_{j-1}^*,$$

where the function $q_k^*(t, y, z) \in e^{\delta z} H^\infty(\Gamma_{T_0} \times \mathbb{R}_-)$ depends only of the $V_i^{*, -}$ for $0 \leq i \leq k$. Let us consider this equations as an ODE in z , the coordinates (t, y) being parameters. For all (t, y) , this equation admits at least a solution in the space $e^{\delta z} H^\infty(\Gamma_{T_0} \times \mathbb{R}_-)$ and two solutions in this space differ from a solution of the homogenous equation (5.7). Let us fix a particular solution of the equation $Y_0(z) \in e^{\delta z} H^\infty(\Gamma_{T_0} \times \mathbb{R}_-)$, then all the solutions are of the form

$$V_j^{*, -} = Y_0 + w$$

where w is a solution of (5.7) with $w(0) = \tilde{\mathbb{P}}_+ w(0)$. Consequently, w will be completely determined by its initial value $w(z = 0)$. Now the profile equation for the terms of order ε^j is

$$(5.15) \quad \tilde{L} \overline{V}_j^+ = \overline{Q}_{j-1},$$

where the functions $\overline{Q}_k(t, x) \in H^\infty(\Omega_{T_0}^+)$ depend only on the functions \overline{V}_i^+ for $i \leq k$. The transmission condition which links $V_j^{*, -}$ and \overline{V}_j^+ writes

$$(I - \tilde{\mathbb{P}}_0) V_j^{*, -} |_{z=0} = (I - \tilde{\mathbb{P}}_0) \overline{V}_j^+ |_{x_d=0}$$

which splits into

$$(5.16) \quad \tilde{\mathbb{P}}_- \overline{V}_j^+ |_{x_d=0} = \tilde{\mathbb{P}}_- Y_0 |_{z=0}$$

and

$$(5.17) \quad \tilde{\mathbb{P}}_+ w|_{z=0} = \tilde{\mathbb{P}}_+ \bar{V}_j^+|_{x_d=0} - \tilde{\mathbb{P}}_+ Y_0|_{z=0}.$$

As for the step of order 0, the equation (5.15) and the boundary condition (5.16) determines uniquely \bar{V}_j^+ . Then, the equation (5.14) and the initial condition (5.17) determines uniquely w and hence $V_j^{*, -}$.

This shows that one can construct an approximate solution v_a^ε of the problem at any given order of approximation, in the sense that v_a^ε satisfies an equations like (4.20), but with Id in place of $\tilde{\mathbb{P}}_+$. Then the estimations of the exact solution and the justification of the asymptotic behavior are proved exactly as in the case of Theorem 2.7, in **Subsection 4.3**.

6 Appendix: about hyperbolic systems with discontinuous coefficients.

Let us consider the system (2.6) with a fixed $\varepsilon > 0$. The system can be written

$$(6.1) \quad \sum_0^d \partial_j (A_j^\sharp v) + \varepsilon^{-1} \mathbf{1}_{\{x_d < 0\}} M v - \sum_{j=0}^{j=d-1} (\partial_j A_j^\sharp) v = \mathbf{1}_{\{x_d < 0\}} F(t, x, v)$$

which shows that the equation has a sense for $u \in L^2(\Omega_T)$. To see that the system is well posed, one shows that it is equivalent to a well posed transmission problem (or initial boundary value problem). Let us note $v^+ = v|_{x_d > 0}$ and $v^- = v|_{x_d < 0}$, and let us denote by $B(t, y)$ the matrix such that

$$(A_d^\sharp v^+)|_{x_d=+0} - (A_d^\sharp v^-)|_{x_d=-0} =: B \begin{pmatrix} v^- \\ v^+ \end{pmatrix} \Big|_{x_d=0}$$

Lemma 6.1. *Let $v \in L^2(\Omega_T)$, v satisfies the system (2.6) if and only if (v^+, v^-) satisfies the transmission problem*

$$(6.2) \quad \begin{cases} L^\sharp v^- + \varepsilon^{-1} M v^- & = & 0 \text{ in } \Omega_T^- \\ L^\sharp v^+ & = & F(t, x, v^+) \text{ in } \Omega_T^+ \\ B \begin{pmatrix} v^- \\ v^+ \end{pmatrix} \Big|_{\Gamma_T} = 0, & v_{t < 0}^\pm = 0. \end{cases}$$

Proof. If v is solution of (6.1), then by restriction on Ω_T^+ , v^+ satisfies the corresponding equation in the distributional sense, and that $L^\sharp v^+ \in L^2(\Omega_T^+)$. It follows that the trace $(A_d^\sharp v^+)_{|x_d=0}$ exists in $H_{loc}^{-1/2}(\Gamma_T)$. The same is true for v^- in Ω_T^- . Finally, since there is no dirac measure in the right hand side of the equation, the transmission condition follows.

Conversely assume that $(v^-, v^+) \in L^2(\Omega_T^-) \times L^2(\Omega_T^+)$ satisfies (6.2) and introduce the function $v \in L^2(\Omega_T)$ equal to v^\pm on Ω_T^\pm . The transmission conditions imply that v satisfies the equation (6.1) and the lemma is proved. In fact the point is that $A_d^\sharp v$ is *continuous* across $\{x_d = 0\}$:

$$A_d^\sharp v \in C(\mathbb{R}_{x_d} : H_{loc}^{-1/2}(\Gamma_T)).$$

Hence there is no Dirac measure which appears when applying ∂_d to $A_d^\sharp v$. \square

The problem (6.2) is an initial boundary value problem with maximally dissipative boundary conditions (see [A],[Su3]). Hence, because of this lemma, the results on the non linear mixed hyperbolic problem with characteristic boundary and maximally dissipative boundary conditions can be applied ([Ra2], [Gu1], [Su2]), and they show the existence of the solution to the semi-linear problem (2.6).

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