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Complete Bethe Ansatz solution of the open spin- s XXZ chain with general integrable boundary terms

Luc Frappat ¹, Rafael I. Nepomechie ² and Eric Ragoucy ¹

Abstract

We consider the open spin- s XXZ quantum spin chain with N sites and general integrable boundary terms for generic values of the bulk anisotropy parameter, and for values of the boundary parameters which satisfy a certain constraint. We derive two sets of Bethe Ansatz equations, and find numerical evidence that together they give the complete set of $(2s + 1)^N$ eigenvalues of the transfer matrix. For the case $s = 1$, we explicitly determine the Hamiltonian, and find an expression for its eigenvalues in terms of Bethe roots.

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1 Introduction

Much of the impetus for the development of the so-called Quantum Inverse Scattering Method and Quantum Groups came from the effort to formulate and solve higher-spin generalizations of the spin-1/2 XXZ (anisotropic Heisenberg) quantum spin chain with periodic boundary conditions [1]-[5]. (See also [6] and references therein.) For the open spin-1/2 XXZ chain [7], Sklyanin [8] discovered a commuting transfer matrix based on solutions of the boundary Yang-Baxter equation (BYBE) [9]. This made it possible to generalize some of the above works to open spin chains [10]-[16]. Although general (non-diagonal) solutions of the BYBE were eventually found [17, 18, 15], Bethe Ansatz solutions were known only for open spin chains with diagonal boundary terms. Further progress was made in [19, 20], where a solution was found for the open spin-1/2 XXZ chain with general integrable (i.e., including also non-diagonal) boundary terms, provided the boundary parameters satisfy a certain constraint. It was subsequently realized [21] that two sets of Bethe Ansatz equations (BAEs) were generally needed in order to obtain the complete set of eigenvalues. (Recently, the second set of BAEs was derived within the generalized algebraic Bethe Ansatz approach of [19] by constructing [22] a suitable second reference state. For other related work, see e.g. [23]-[31], and references therein.)

We present here a Bethe Ansatz solution of the N -site open spin- s XXZ quantum spin chain with general integrable boundary terms, following an approach which was developed for the spin-1/2 XXZ chain in [34], and then generalized to the spin-1/2 XYZ chain in [35]. Doikou solved a special case of this model (among others) in [32] using the method in [20], and also computed some of its thermodynamical properties. She further investigated this model in [33] using the method in [19]. We find a generalization of Doikou's constraint on the boundary parameters, as well as a second set of BAEs, which we argue is necessary to obtain the complete set of $(2s + 1)^N$ eigenvalues.

The outline of this paper is as follows. In Sec. 2 we review the construction of the model's transfer matrices, and the so-called fusion hierarchy which they satisfy. In Sec. 3, we identify (as in [34]) the Q operator with a transfer matrix whose auxiliary-space spin tends to infinity, and obtain a $T - Q$ equation for the model's fundamental (spin-1/2 auxiliary space) transfer matrix. Using some additional properties of this transfer matrix, we arrive at a pair of expressions for its eigenvalues in terms of roots of corresponding BAEs, as well as a corresponding constraint on the boundary parameters. We provide numerical evidence of completeness of this solution for small values of s and N . In Sec. 4 we explicitly determine the Hamiltonian for the case $s = 1$, and find an expression for its eigenvalues in terms of Bethe roots. We end in Sec. 5 with a brief further discussion of our results.

2 Transfer matrices and fusion hierarchy

Sklyanin [8] showed for an N -site open XXZ spin chain how to construct a commuting transfer matrix, which here we shall denote by $t^{(\frac{1}{2}, \frac{1}{2})}(u)$, whose auxiliary space as well as each of its N quantum spaces are spin-1/2 (i.e., two-dimensional). In a similar way, one can construct a transfer matrix $t^{(j,s)}(u)$ whose auxiliary space is spin- j ($(2j+1)$ -dimensional) and each of its N quantum spaces are spin- s ($(2s+1)$ -dimensional), for any $j, s \in \{\frac{1}{2}, 1, \frac{3}{2}, \dots\}$. The basic building blocks are so-called fused R and K^\mp matrices. The former are given by [36], [2]-[5] ¹

$$R_{\{a\}\{b\}}^{(j,s)}(u) = P_{\{a\}}^+ P_{\{b\}}^+ \prod_{k=1}^{2j} \prod_{l=1}^{2s} R_{a_k b_l}^{(\frac{1}{2}, \frac{1}{2})}(u + (k+l-j-s-1)\eta) P_{\{a\}}^+ P_{\{b\}}^+, \quad (2.1)$$

where $\{a\} = \{a_1, \dots, a_{2j}\}$, $\{b\} = \{b_1, \dots, b_{2s}\}$, and $R^{(\frac{1}{2}, \frac{1}{2})}(u)$ is given by

$$R^{(\frac{1}{2}, \frac{1}{2})}(u) = \begin{pmatrix} \text{sh}(u+\eta) & 0 & 0 & 0 \\ 0 & \text{sh} u & \text{sh} \eta & 0 \\ 0 & \text{sh} \eta & \text{sh} u & 0 \\ 0 & 0 & 0 & \text{sh}(u+\eta) \end{pmatrix}, \quad (2.2)$$

where η is the bulk anisotropy parameter. The R matrices in the product (2.1) are ordered in the order of increasing k and l . Moreover, $P_{\{a\}}^+$ is the symmetric projector

$$P_{\{a\}}^+ = \frac{1}{(2j)!} \prod_{k=1}^{2j} \left(\sum_{l=1}^k \mathcal{P}_{a_l, a_k} \right), \quad (2.3)$$

where \mathcal{P} is the permutation operator, with $\mathcal{P}_{a_k, a_k} \equiv 1$; and similarly for $P_{\{b\}}^+$. For example, for the simple case $(j, s) = (1, \frac{1}{2})$,

$$R_{\{a_1, a_2\}, b}^{(1, \frac{1}{2})}(u) = P_{a_1, a_2}^+ R_{a_1, b}^{(\frac{1}{2}, \frac{1}{2})}(u - \frac{1}{2}\eta) R_{a_2, b}^{(\frac{1}{2}, \frac{1}{2})}(u + \frac{1}{2}\eta) P_{a_1, a_2}^+. \quad (2.4)$$

The fused R matrices satisfy the Yang-Baxter equations

$$R_{\{a\}\{b\}}^{(j,k)}(u-v) R_{\{a\}\{c\}}^{(j,s)}(u) R_{\{b\}\{c\}}^{(k,s)}(v) = R_{\{b\}\{c\}}^{(k,s)}(v) R_{\{a\}\{c\}}^{(j,s)}(u) R_{\{a\}\{b\}}^{(j,k)}(u-v). \quad (2.5)$$

We note here for later convenience that the fundamental R matrix satisfies the unitarity relation

$$R^{(\frac{1}{2}, \frac{1}{2})}(u) R^{(\frac{1}{2}, \frac{1}{2})}(-u) = -\xi(u) 1, \quad \xi(u) = \text{sh}(u+\eta) \text{sh}(u-\eta). \quad (2.6)$$

¹Our definitions of fused R and K matrices differ from those in [34] by certain shifts of the arguments.

The fused K^- matrices are given by [10, 11, 14]

$$K_{\{a\}}^{- (j)}(u) = P_{\{a\}}^+ \prod_{k=1}^{2j} \left\{ \left[\prod_{l=1}^{k-1} R_{a_l a_k}^{(\frac{1}{2}, \frac{1}{2})}(2u + (k+l-2j-1)\eta) \right] \right. \\ \left. \times K_{a_k}^{- (\frac{1}{2})}(u + (k-j - \frac{1}{2})\eta) \right\} P_{\{a\}}^+, \quad (2.7)$$

where $K^{- (\frac{1}{2})}(u)$ is the matrix [17, 18]

$$\begin{pmatrix} 2(\operatorname{sh} \alpha_- \operatorname{ch} \beta_- \operatorname{ch} u + \operatorname{ch} \alpha_- \operatorname{sh} \beta_- \operatorname{sh} u) & e^{\theta_-} \operatorname{sh} 2u \\ e^{-\theta_-} \operatorname{sh} 2u & 2(\operatorname{sh} \alpha_- \operatorname{ch} \beta_- \operatorname{ch} u - \operatorname{ch} \alpha_- \operatorname{sh} \beta_- \operatorname{sh} u) \end{pmatrix} \quad (2.8)$$

where α_- , β_- and θ_- are arbitrary boundary parameters. The products of braces $\{\dots\}$ in (2.7) are ordered in the order of increasing k . For example, for the case $j = 1$,

$$K_{a_1, a_2}^{- (1)}(u) = P_{a_1, a_2}^+ K_{a_1}^{- (\frac{1}{2})}(u - \frac{1}{2}\eta) R_{a_1, a_2}^{(\frac{1}{2}, \frac{1}{2})}(2u) K_{a_2}^{- (\frac{1}{2})}(u + \frac{1}{2}\eta) P_{a_1, a_2}^+. \quad (2.9)$$

The fused K^- matrices satisfy the boundary Yang-Baxter equations [9]

$$R_{\{a\}\{b\}}^{(j, s)}(u-v) K_{\{a\}}^{- (j)}(u) R_{\{a\}\{b\}}^{(j, s)}(u+v) K_{\{b\}}^{- (j)}(u) \\ = K_{\{b\}}^{- (j)}(u) R_{\{a\}\{b\}}^{(j, s)}(u+v) K_{\{a\}}^{- (j)}(u) R_{\{a\}\{b\}}^{(j, s)}(u-v). \quad (2.10)$$

The fused K^+ matrices are given by

$$K_{\{a\}}^{+ (j)}(u) = \frac{1}{f^{(j)}(u)} K_{\{a\}}^{- (j)}(-u - \eta) \Big|_{(\alpha_-, \beta_-, \theta_-) \rightarrow (-\alpha_+, -\beta_+, \theta_+)}, \quad (2.11)$$

where the normalization factor

$$f^{(j)}(u) = \prod_{l=1}^{2j-1} \prod_{k=1}^l [-\xi(2u + (l+k+1-2j)\eta)] \quad (2.12)$$

is chosen to simplify the form of the fusion hierarchy, see below (2.17).

The transfer matrix $t^{(j, s)}(u)$ is given by

$$t^{(j, s)}(u) = \operatorname{tr}_{\{a\}} K_{\{a\}}^{+ (j)}(u) T_{\{a\}}^{(j, s)}(u) K_{\{a\}}^{- (j)}(u) \hat{T}_{\{a\}}^{(j, s)}(u), \quad (2.13)$$

where the monodromy matrices are given by products of N fused R matrices,

$$T_{\{a\}}^{(j, s)}(u) = R_{\{a\}, \{b^{[N]}\}}^{(j, s)}(u) \dots R_{\{a\}, \{b^{[1]}\}}^{(j, s)}(u), \\ \hat{T}_{\{a\}}^{(j, s)}(u) = R_{\{a\}, \{b^{[1]}\}}^{(j, s)}(u) \dots R_{\{a\}, \{b^{[N]}\}}^{(j, s)}(u). \quad (2.14)$$

These transfer matrices commute for different values of spectral parameter for any $j, j' \in \{\frac{1}{2}, 1, \frac{3}{2}, \dots\}$ and any $s \in \{\frac{1}{2}, 1, \frac{3}{2}, \dots\}$,

$$[t^{(j, s)}(u), t^{(j', s)}(u')] = 0. \quad (2.15)$$

These transfer matrices also obey the fusion hierarchy [10, 11, 14]²

$$t^{(j-\frac{1}{2},s)}(u-j\eta)t^{(\frac{1}{2},s)}(u) = t^{(j,s)}(u-(j-\frac{1}{2})\eta) + \delta^{(s)}(u)t^{(j-1,s)}(u-(j+\frac{1}{2})\eta), \quad (2.16)$$

$j = 1, \frac{3}{2}, \dots$, where $t^{(0,s)} = 1$, and $\delta^{(s)}(u)$ is a product of various quantum determinants, and is given by

$$\begin{aligned} \delta^{(s)}(u) &= 2^4 \left[\prod_{k=0}^{2s-1} \xi(u + (s-k-\frac{1}{2})\eta) \right]^{2N} \frac{\text{sh}(2u-2\eta)\text{sh}(2u+2\eta)}{\text{sh}(2u-\eta)\text{sh}(2u+\eta)} \\ &\times \text{sh}(u+\alpha_-)\text{sh}(u-\alpha_-)\text{ch}(u+\beta_-)\text{ch}(u-\beta_-) \\ &\times \text{sh}(u+\alpha_+)\text{sh}(u-\alpha_+)\text{ch}(u+\beta_+)\text{ch}(u-\beta_+). \end{aligned} \quad (2.17)$$

We remark that the normalization factor $f^{(j)}(u)$ (2.12) has been chosen so that the LHS of (2.16) has coefficient 1.

In the remainder of this section we list some important further properties of the “fundamental” transfer matrix $t^{(\frac{1}{2},s)}(u)$, which we shall subsequently use to help determine its eigenvalues. However, it is more convenient to work with a rescaled transfer matrix $\tilde{t}^{(\frac{1}{2},s)}(u)$ defined by

$$\tilde{t}^{(\frac{1}{2},s)}(u) = \frac{1}{g^{(\frac{1}{2},s)}(u)^{2N}} t^{(\frac{1}{2},s)}(u), \quad (2.18)$$

where

$$g^{(\frac{1}{2},s)}(u) = \prod_{k=1}^{2s-1} \text{sh}(u + (s-k+\frac{1}{2})\eta) \quad (2.19)$$

(which has the crossing symmetry $g^{(\frac{1}{2},s)}(-u-\eta) = \pm g^{(\frac{1}{2},s)}(u)$) is an overall scalar factor of the fused R matrix $R^{(\frac{1}{2},s)}(u)$. In particular, the rescaled transfer matrix does not vanish at $u = 0$ when s is a half-odd integer.

This transfer matrix has the following properties:

$$\tilde{t}^{(\frac{1}{2},s)}(u+i\pi) = \tilde{t}^{(\frac{1}{2},s)}(u) \quad (i\pi\text{-periodicity}) \quad (2.20)$$

$$\tilde{t}^{(\frac{1}{2},s)}(-u-\eta) = \tilde{t}^{(\frac{1}{2},s)}(u) \quad (\text{crossing}) \quad (2.21)$$

$$\tilde{t}^{(\frac{1}{2},s)}(0) = -2^3 \text{sh}^{2N}((s+\frac{1}{2})\eta) \text{ch}\eta \text{sh}\alpha_- \text{ch}\beta_- \text{sh}\alpha_+ \text{ch}\beta_+ 1 \quad (\text{initial condition}) \quad (2.22)$$

²The derivation of this hierarchy relies on some relations which, to our knowledge, have not been proved. See the Appendix for further details.

$$\begin{aligned}
\tilde{t}^{(\frac{1}{2},s)}(u)\Big|_{\eta=0} &= 2^3 \operatorname{sh}^{2N} u \left[-\operatorname{sh} \alpha_- \operatorname{ch} \beta_- \operatorname{sh} \alpha_+ \operatorname{ch} \beta_+ \operatorname{ch}^2 u \right. \\
&+ \operatorname{ch} \alpha_- \operatorname{sh} \beta_- \operatorname{ch} \alpha_+ \operatorname{sh} \beta_+ \operatorname{sh}^2 u \\
&\left. - \operatorname{ch}(\theta_- - \theta_+) \operatorname{sh}^2 u \operatorname{ch}^2 u \right] 1 \quad (\text{semi-classical limit}) \quad (2.23)
\end{aligned}$$

$$\begin{aligned}
\tilde{t}^{(\frac{1}{2},s)}(u) &\sim -\frac{1}{2^{2N+1}} e^{(2N+4)u+(N+2)\eta} \operatorname{ch}(\theta_- - \theta_+) 1 \quad \text{for } u \rightarrow +\infty \\
&(\text{asymptotic behavior}) \quad (2.24)
\end{aligned}$$

As is well known, due to the commutativity property (2.15), the corresponding simultaneous eigenvectors are independent of the spectral parameter. Hence, the above properties (2.20) - (2.24) hold also for the corresponding eigenvalues $\tilde{\Lambda}^{(\frac{1}{2},s)}(u)$.

3 Eigenvalues and Bethe Ansatz equations

We now proceed to determine the eigenvalues of the fundamental transfer matrix. Following [34], we assume that the limit

$$\bar{Q}(u) = \lim_{j \rightarrow \infty} t^{(j-\frac{1}{2},s)}(u - j\eta) \quad (3.1)$$

exists.³ We then immediately obtain from the fusion hierarchy (2.16) an equation of the $T - Q$ form for the fundamental transfer matrix $t^{(\frac{1}{2},s)}(u)$,

$$\bar{Q}(u) t^{(\frac{1}{2},s)}(u) = \bar{Q}(u + \eta) + \delta^{(s)}(u) \bar{Q}(u - \eta). \quad (3.2)$$

We further assume that the eigenvalues of $\bar{Q}(u)$ (which we denote by the same symbol) have the decomposition $\bar{Q}(u) = f(u) Q(u)$ with

$$Q(u) = \prod_{j=1}^M \operatorname{sh}(u - v_j) \operatorname{sh}(u + v_j + \eta), \quad (3.3)$$

which has the crossing symmetry $Q(-u - \eta) = Q(u)$. Here M is some nonnegative integer. It follows that the eigenvalues $\Lambda^{(\frac{1}{2},s)}(u)$ of $t^{(\frac{1}{2},s)}(u)$ are given by

$$\Lambda^{(\frac{1}{2},s)}(u) = H_1(u) \frac{Q(u + \eta)}{Q(u)} + H_2(u) \frac{Q(u - \eta)}{Q(u)}, \quad (3.4)$$

where $H_1(u) = f(u + \eta)/f(u)$ and $H_2(u) = \delta^{(s)}(u) f(u - \eta)/f(u)$; and therefore,

$$H_1(u - \eta) H_2(u) = \delta^{(s)}(u). \quad (3.5)$$

³For a more rigorous and extensive discussion of Q operators for the six- and eight-vertex models with periodic boundary conditions, as well numerous earlier references, see [37].

The crossing symmetry (2.21) together with (3.4) imply that

$$H_2(u) = H_1(-u - \eta). \quad (3.6)$$

We conclude that $H_1(u)$ must satisfy

$$H_1(u - \eta) H_1(-u - \eta) = \delta^{(s)}(u), \quad (3.7)$$

where $\delta^{(s)}(u)$ is given by (2.17).

A set of solutions of (3.7) for $H_1(u)$ is given by

$$\begin{aligned} H_1^{(\pm)}(u|\epsilon_1, \epsilon_2, \epsilon_3) &= -2^2 \epsilon_2 \left[\prod_{k=0}^{2s-1} \text{sh}(u + (s - k - \frac{1}{2})\eta) \right]^{2N} \frac{\text{sh}(2u)}{\text{sh}(2u + \eta)} \\ &\times \text{sh}(u \pm \alpha_- + \eta) \text{ch}(u \pm \epsilon_1 \beta_- + \eta) \text{sh}(u \pm \epsilon_2 \alpha_+ + \eta) \text{ch}(u \pm \epsilon_3 \beta_+ + \eta), \end{aligned} \quad (3.8)$$

where $\epsilon_1, \epsilon_2, \epsilon_3$ can independently take the values ± 1 . It follows from (3.6) that the corresponding $H_2(u)$ functions are given by

$$\begin{aligned} H_2^{(\pm)}(u|\epsilon_1, \epsilon_2, \epsilon_3) &= -2^2 \epsilon_2 \left[\prod_{k=0}^{2s-1} \text{sh}(u + (s - k + \frac{1}{2})\eta) \right]^{2N} \frac{\text{sh}(2u + 2\eta)}{\text{sh}(2u + \eta)} \\ &\times \text{sh}(u \mp \alpha_-) \text{ch}(u \mp \epsilon_1 \beta_-) \text{sh}(u \mp \epsilon_2 \alpha_+) \text{ch}(u \mp \epsilon_3 \beta_+). \end{aligned} \quad (3.9)$$

An argument from [34] (which makes use of the periodicity (2.20)) can again be used to conclude that the eigenvalues can be uniquely expressed as

$$\Lambda^{\frac{1}{2}, s}(u) = a H_1^{(\pm)}(u|\epsilon_1, \epsilon_2, \epsilon_3) \frac{Q(u + \eta)}{Q(u)} + a H_2^{(\pm)}(u|\epsilon_1, \epsilon_2, \epsilon_3) \frac{Q(u - \eta)}{Q(u)}, \quad (3.10)$$

up to an overall sign $a = \pm 1$. Noting that the functions H_1 (3.8) and H_2 (3.9) have the factor $g^{\frac{1}{2}, s}(u)^{2N}$ (2.19) in common, we conclude that the eigenvalues of $\tilde{t}^{\frac{1}{2}, s}(u)$ (2.18) are given by

$$\tilde{\Lambda}^{\frac{1}{2}, s}(u) = \tilde{H}_1^{(\pm)}(u|\epsilon_1, \epsilon_2, \epsilon_3) \frac{Q(u + \eta)}{Q(u)} + \tilde{H}_2^{(\pm)}(u|\epsilon_1, \epsilon_2, \epsilon_3) \frac{Q(u - \eta)}{Q(u)}, \quad (3.11)$$

where

$$\begin{aligned} \tilde{H}_1^{(\pm)}(u|\epsilon_1, \epsilon_2, \epsilon_3) &= -2^2 \epsilon_2 \text{sh}^{2N}(u - (s - \frac{1}{2})\eta) \frac{\text{sh}(2u)}{\text{sh}(2u + \eta)} \text{sh}(u \pm \alpha_- + \eta) \\ &\times \text{ch}(u \pm \epsilon_1 \beta_- + \eta) \text{sh}(u \pm \epsilon_2 \alpha_+ + \eta) \text{ch}(u \pm \epsilon_3 \beta_+ + \eta), \quad (3.12) \\ \tilde{H}_2^{(\pm)}(u|\epsilon_1, \epsilon_2, \epsilon_3) &= -2^2 \epsilon_2 \text{sh}^{2N}(u + (s + \frac{1}{2})\eta) \frac{\text{sh}(2u + 2\eta)}{\text{sh}(2u + \eta)} \text{sh}(u \mp \alpha_-) \\ &\times \text{ch}(u \mp \epsilon_1 \beta_-) \text{sh}(u \mp \epsilon_2 \alpha_+) \text{ch}(u \mp \epsilon_3 \beta_+). \quad (3.13) \end{aligned}$$

We have fixed the overall sign $a = +1$ in (3.11) using the initial condition (2.22).

The expression (3.11) for $\tilde{\Lambda}^{(\frac{1}{2},s)}(u)$ is also consistent with the asymptotic behavior (2.24) if the boundary parameters satisfy the constraint

$$\alpha_- + \epsilon_1 \beta_- + \epsilon_2 \alpha_+ + \epsilon_3 \beta_+ = \epsilon_0 (\theta_- - \theta_+) + \eta k + \frac{1}{2} (1 - \epsilon_2) i \pi \pmod{2i\pi}, \quad (3.14)$$

where also $\epsilon_0 = \pm 1$; and if M (appearing in the expression (3.3) for $Q(u)$) is given by

$$M = sN - \frac{1}{2} \mp \frac{k}{2}. \quad (3.15)$$

The requirement that M be an integer evidently implies that

$$sN - \frac{1}{2} \mp \frac{k}{2} = \text{integer}. \quad (3.16)$$

In particular, for s an integer, k is an odd integer; and for s a half-odd integer, k is odd (even) if N is even (odd), respectively.

Finally, the expression (3.11) for $\tilde{\Lambda}^{(\frac{1}{2},s)}(u)$ is also consistent with the semi-classical limit (2.23) if the $\{\epsilon_i\}$ satisfy the constraint

$$\epsilon_1 \epsilon_2 \epsilon_3 = +1. \quad (3.17)$$

To summarize: if the boundary parameters $(\alpha_{\pm}, \beta_{\pm}, \theta_{\pm})$ satisfy the constraints (3.14), (3.16) and (3.17) for some choice (± 1) of $\{\epsilon_i\}$ and for some appropriate value of k , then the eigenvalues of $\tilde{t}^{(\frac{1}{2},s)}(u)$ (2.18) are given by

$$\tilde{\Lambda}^{(\frac{1}{2},s)(\pm)}(u) = \tilde{H}_1^{(\pm)}(u|\epsilon_1, \epsilon_2, \epsilon_3) \frac{Q^{(\pm)}(u + \eta)}{Q^{(\pm)}(u)} + \tilde{H}_2^{(\pm)}(u|\epsilon_1, \epsilon_2, \epsilon_3) \frac{Q^{(\pm)}(u - \eta)}{Q^{(\pm)}(u)}, \quad (3.18)$$

where $\tilde{H}_1^{(\pm)}(u|\epsilon_1, \epsilon_2, \epsilon_3)$ and $\tilde{H}_2^{(\pm)}(u|\epsilon_1, \epsilon_2, \epsilon_3)$ are given by (3.12), (3.13), and

$$Q^{(\pm)}(u) = \prod_{j=1}^{M^{(\pm)}} \text{sh}(u - v_j^{(\pm)}) \text{sh}(u + v_j^{(\pm)} + \eta), \quad M^{(\pm)} = sN - \frac{1}{2} \mp \frac{k}{2}. \quad (3.19)$$

The parameters $\{v_j^{(\pm)}\}$ are roots of the corresponding Bethe Ansatz equations,

$$\frac{\tilde{H}_2^{(\pm)}(v_j^{(\pm)}|\epsilon_1, \epsilon_2, \epsilon_3)}{\tilde{H}_2^{(\pm)}(-v_j^{(\pm)} - \eta|\epsilon_1, \epsilon_2, \epsilon_3)} = - \frac{Q^{(\pm)}(v_j^{(\pm)} + \eta)}{Q^{(\pm)}(v_j^{(\pm)} - \eta)}, \quad j = 1, \dots, M^{(\pm)}; \quad (3.20)$$

or, more explicitly,

$$\begin{aligned} & \left(\frac{\text{sh}(\tilde{v}_j^{(\pm)} + s\eta)}{\text{sh}(\tilde{v}_j^{(\pm)} - s\eta)} \right)^{2N} \frac{\text{sh}(2\tilde{v}_j^{(\pm)} + \eta) \text{sh}(\tilde{v}_j^{(\pm)} \mp \alpha_- - \frac{\eta}{2}) \text{ch}(\tilde{v}_j^{(\pm)} \mp \epsilon_1 \beta_- - \frac{\eta}{2})}{\text{sh}(2\tilde{v}_j^{(\pm)} - \eta) \text{sh}(\tilde{v}_j^{(\pm)} \pm \alpha_- + \frac{\eta}{2}) \text{ch}(\tilde{v}_j^{(\pm)} \pm \epsilon_1 \beta_- + \frac{\eta}{2})} \\ & \times \frac{\text{sh}(\tilde{v}_j^{(\pm)} \mp \epsilon_2 \alpha_+ - \frac{\eta}{2}) \text{ch}(\tilde{v}_j^{(\pm)} \mp \epsilon_3 \beta_+ - \frac{\eta}{2})}{\text{sh}(\tilde{v}_j^{(\pm)} \pm \epsilon_2 \alpha_+ + \frac{\eta}{2}) \text{ch}(\tilde{v}_j^{(\pm)} \pm \epsilon_3 \beta_+ + \frac{\eta}{2})} \\ & = - \prod_{k=1}^{M^{(\pm)}} \frac{\text{sh}(\tilde{v}_j^{(\pm)} - \tilde{v}_k^{(\pm)} + \eta) \text{sh}(\tilde{v}_j^{(\pm)} + \tilde{v}_k^{(\pm)} + \eta)}{\text{sh}(\tilde{v}_j^{(\pm)} - \tilde{v}_k^{(\pm)} - \eta) \text{sh}(\tilde{v}_j^{(\pm)} + \tilde{v}_k^{(\pm)} - \eta)}, \quad j = 1, \dots, M^{(\pm)}, \quad (3.21) \end{aligned}$$

where $\tilde{v}_j^{(\pm)} \equiv v_j^{(\pm)} + \eta/2$.

We have investigated the completeness of this solution for small values of s and N numerically using a method developed by McCoy and his collaborators (see, e.g., [38]), and further explained in [21]. We find that, for $k \geq 2sN + 1$ (and therefore $M^{(+)} \leq -1$), all the eigenvalues of $\tilde{t}^{(\frac{1}{2},s)}(u)$ are given by $\tilde{\Lambda}^{(\frac{1}{2},s)(-)}(u)$. Similarly, for $k \leq -(2sN + 1)$ (and therefore $M^{(-)} \leq -1$), all the eigenvalues are given by $\tilde{\Lambda}^{(\frac{1}{2},s)(+)}(u)$. Moreover, for $|k| \leq 2sN - 1$ (and therefore both $M^{(+)}$ and $M^{(-)}$ are nonnegative), both $\tilde{\Lambda}^{(\frac{1}{2},s)(-)}(u)$ and $\tilde{\Lambda}^{(\frac{1}{2},s)(+)}(u)$ are needed to obtain all the eigenvalues. ⁴

Some sample results are summarized in Tables 1-4. For example, let us consider Table 1, which is for the case $N = 2$, $s = 1$. According to (3.16), k must be odd for this case. For such values of k , the number of eigenvalues of $\tilde{t}^{(\frac{1}{2},s)}(u)$ which we found that are given by $\tilde{\Lambda}^{(\frac{1}{2},s)(-)}(u)$ and $\tilde{\Lambda}^{(\frac{1}{2},s)(+)}(u)$ in (3.18) are listed in the second and third columns, respectively. Notice that, for each row of the table, the sum of these two entries is 9, which coincides with the total number of eigenvalues (3^2). A similar result can be readily seen in the other tables. ⁵ These results strongly support the conjecture that $\tilde{\Lambda}^{(\frac{1}{2},s)(-)}(u)$ and $\tilde{\Lambda}^{(\frac{1}{2},s)(+)}(u)$ in (3.18) together give the complete set of $(2s + 1)^N$ eigenvalues of the transfer matrix $\tilde{t}^{(\frac{1}{2},s)}(u)$ for $|k| \leq 2sN - 1$.

It may be worth noting that for $|k| = 2sN - 1$ (and, therefore, either $M^{(+)}$ or $M^{(-)}$ vanishes), the Bethe Ansatz with vanishing M still gives 1 eigenvalue. (See again Tables 1-4.) In the algebraic Bethe Ansatz approach, the corresponding eigenvector would presumably be a “bare” reference state.

We observe that for $s = 1/2$, our solution coincides with the one in [34]. Our solution is similar to the one given by Doikou [32, 33], but with a more general constraint on the boundary parameters, and with a second set of BAEs which is necessary for completeness.

4 Hamiltonian for $s = 1$

In this section we give an explicit expression for the spin-1 Hamiltonian, its relation to the transfer matrix, and its eigenvalues in terms of Bethe roots. In order to construct the integrable Hamiltonian for the case $s = 1$, we need the transfer matrix with spin-1 in *both* auxiliary and quantum spaces, i.e., $t^{(1,1)}(u)$. Since we seek an explicit expression for the Hamiltonian in terms of spin-1 generators of $su(2)$ (which are 3×3 matrices),

⁴For the conventional situation that the boundary parameters $(\alpha_{\pm}, \beta_{\pm}, \theta_{\pm})$ are finite and independent of N , the constraint (3.14) requires that also k be finite and independent of N , in which case $|k| \leq 2sN - 1$ for $N \rightarrow \infty$.

⁵It would be interesting to find a formula that would generate the entries in these tables for general values of N , s and k .

we now perform similarity transformations on the fused R and K matrices which bring them to row-reduced form, and then remove all null rows and columns. That is,

$$\begin{aligned} R_{reduced}^{(1,1)}(u) &= A_{a_1, a_2} A_{b_1, b_2} R_{\{a_1, a_2\}, \{b_1, b_2\}}^{(1,1)}(u) A_{a_1, a_2}^{-1} A_{b_1, b_2}^{-1}, \\ K_{reduced}^{\mp(1)}(u) &= A_{a_1, a_2} K_{a_1, a_2}^{\mp(1)}(u) A_{a_1, a_2}^{-1}, \end{aligned} \quad (4.1)$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (4.2)$$

The reduced R and K matrices are 9×9 (instead of 16×16) and 3×3 (instead of 4×4) matrices, respectively. It is convenient to make a further similarity (“gauge”) transformation which brings these matrices to a more symmetric form,

$$\begin{aligned} R_{reduced}^{(1,1) \text{ } gt}(u) &= (B \otimes B) R_{reduced}^{(1,1)}(u) (B^{-1} \otimes B^{-1}) \\ K_{reduced}^{\mp(1) \text{ } gt}(u) &= B K_{reduced}^{\mp(1)}(u) B^{-1}, \end{aligned} \quad (4.3)$$

where B is the 3×3 diagonal matrix

$$B = \text{diag}(1, -\sqrt{2 \text{ch } \eta}, 1). \quad (4.4)$$

We define $t^{(1,1) \text{ } gt}(u)$ to be the transfer matrix constructed with these R and K matrices, i.e.,

$$t^{(1,1) \text{ } gt}(u) = \text{tr}_0 K_{reduced}^{+(1) \text{ } gt}_0(u) T_{reduced}^{(1,1) \text{ } gt}_0(u) K_{reduced}^{-(1) \text{ } gt}_0(u) \hat{T}_{reduced}^{(1,1) \text{ } gt}_0(u), \quad (4.5)$$

where the monodromy matrices are constructed as usual from the $R_{reduced}^{(1,1) \text{ } gt}(u)$ ’s, and the auxiliary space is now denoted by “0”. Since $t^{(1,1)}(u)$ and $t^{(1,1) \text{ } gt}(u)$ are related by a similarity transformation, they have the same eigenvalues. Finally, it is again more convenient to work with a rescaled transfer matrix,

$$\tilde{t}^{(1,1) \text{ } gt}(u) = \frac{\text{sh}(2u) \text{sh}(2u + 2\eta)}{[\text{sh } u \text{sh}(u + \eta)]^{2N}} t^{(1,1) \text{ } gt}(u), \quad (4.6)$$

which, in particular, does not vanish at $u = 0$.

As noted by Sklyanin [8], the Hamiltonian H is proportional to the first derivative of the transfer matrix

$$H = c_1 \frac{d}{du} \tilde{t}^{(1,1) \text{ } gt}(u) \Big|_{u=0} + c_0 1, \quad (4.7)$$

where

$$c_1 = -\text{ch } \eta \left\{ 16 [\text{sh } 2\eta \text{ sh } \eta]^{2N} \text{sh } 3\eta \text{sh}(\alpha_- - \frac{\eta}{2}) \text{sh}(\alpha_- + \frac{\eta}{2}) \text{ch}(\beta_- - \frac{\eta}{2}) \text{ch}(\beta_- + \frac{\eta}{2}) \right. \\ \left. \times \text{sh}(\alpha_+ - \frac{\eta}{2}) \text{sh}(\alpha_+ + \frac{\eta}{2}) \text{ch}(\beta_+ - \frac{\eta}{2}) \text{ch}(\beta_+ + \frac{\eta}{2}) \right\}^{-1}. \quad (4.8)$$

We choose c_1 so that the bulk terms of the Hamiltonian have a conventional normalization (see (4.10)), and we choose c_0 so that there is no additive constant term in the expression (4.9) for the Hamiltonian,

$$H = \sum_{n=1}^{N-1} H_{n,n+1} + H_b. \quad (4.9)$$

The bulk terms $H_{n,n+1}$ are given by [1]

$$H_{n,n+1} = \sigma_n - (\sigma_n)^2 + 2 \text{sh}^2 \eta [\sigma_n^z + (S_n^z)^2 + (S_{n+1}^z)^2 - (\sigma_n^z)^2] \\ - 4 \text{sh}^2(\frac{\eta}{2}) (\sigma_n^\perp \sigma_n^z + \sigma_n^z \sigma_n^\perp), \quad (4.10)$$

where

$$\sigma_n = \vec{S}_n \cdot \vec{S}_{n+1}, \quad \sigma_n^\perp = S_n^x S_{n+1}^x + S_n^y S_{n+1}^y, \quad \sigma_n^z = S_n^z S_{n+1}^z, \quad (4.11)$$

and \vec{S} are the standard spin-1 generators of $su(2)$. The boundary terms H_b have the form [15]

$$H_b = a_1 (S_1^z)^2 + a_2 S_1^z + a_3 (S_1^+)^2 + a_4 (S_1^-)^2 + a_5 S_1^+ S_1^z + a_6 S_1^z S_1^- \\ + a_7 S_1^z S_1^+ + a_8 S_1^- S_1^z + (a_j \leftrightarrow b_j \text{ and } 1 \leftrightarrow N), \quad (4.12)$$

where $S^\pm = S^x \pm iS^y$. The coefficients $\{a_i\}$ of the boundary terms at site 1 are given in terms of the boundary parameters $(\alpha_-, \beta_-, \theta_-)$ by

$$a_1 = \frac{1}{4} a_0 (\text{ch } 2\alpha_- - \text{ch } 2\beta_- + \text{ch } \eta) \text{sh } 2\eta \text{ sh } \eta, \\ a_2 = \frac{1}{4} a_0 \text{sh } 2\alpha_- \text{sh } 2\beta_- \text{sh } 2\eta, \\ a_3 = -\frac{1}{8} a_0 e^{2\theta_-} \text{sh } 2\eta \text{ sh } \eta, \\ a_4 = -\frac{1}{8} a_0 e^{-2\theta_-} \text{sh } 2\eta \text{ sh } \eta, \\ a_5 = a_0 e^{\theta_-} \left(\text{ch } \beta_- \text{sh } \alpha_- \text{ch } \frac{\eta}{2} + \text{ch } \alpha_- \text{sh } \beta_- \text{sh } \frac{\eta}{2} \right) \text{sh } \eta \text{ch }^{\frac{3}{2}} \eta, \\ a_6 = a_0 e^{-\theta_-} \left(\text{ch } \beta_- \text{sh } \alpha_- \text{ch } \frac{\eta}{2} + \text{ch } \alpha_- \text{sh } \beta_- \text{sh } \frac{\eta}{2} \right) \text{sh } \eta \text{ch }^{\frac{3}{2}} \eta, \\ a_7 = -a_0 e^{\theta_-} \left(\text{ch } \beta_- \text{sh } \alpha_- \text{ch } \frac{\eta}{2} - \text{ch } \alpha_- \text{sh } \beta_- \text{sh } \frac{\eta}{2} \right) \text{sh } \eta \text{ch }^{\frac{3}{2}} \eta, \\ a_8 = -a_0 e^{-\theta_-} \left(\text{ch } \beta_- \text{sh } \alpha_- \text{ch } \frac{\eta}{2} - \text{ch } \alpha_- \text{sh } \beta_- \text{sh } \frac{\eta}{2} \right) \text{sh } \eta \text{ch }^{\frac{3}{2}} \eta, \quad (4.13)$$

where

$$a_0 = \left[\text{sh}(\alpha_- - \frac{\eta}{2}) \text{sh}(\alpha_- + \frac{\eta}{2}) \text{ch}(\beta_- - \frac{\eta}{2}) \text{ch}(\beta_- + \frac{\eta}{2}) \right]^{-1}. \quad (4.14)$$

Moreover, the coefficients $\{b_i\}$ of the boundary terms at site N are given in terms of the boundary parameters $(\alpha_+, \beta_+, \theta_+)$ by

$$b_i = a_i \Big|_{\alpha_- \rightarrow \alpha_+, \beta_- \rightarrow -\beta_+, \theta_- \rightarrow \theta_+}. \quad (4.15)$$

We now proceed to find an expression for the energies in terms of the Bethe roots. It follows from (4.7) that the eigenvalues of the Hamiltonian are given by

$$E = c_1 \frac{d}{du} \tilde{\Lambda}^{(1,1) \text{ } g^t}(u) \Big|_{u=0} + c_0. \quad (4.16)$$

Furthermore,

$$\tilde{\Lambda}^{(1,1) \text{ } g^t}(u) = \frac{\text{sh}(2u) \text{sh}(2u + 2\eta)}{[\text{sh } u \text{sh}(u + \eta)]^{2N}} \Lambda^{(1,1)}(u), \quad (4.17)$$

where we have used (4.6) and the fact that $\Lambda^{(1,1) \text{ } g^t}(u) = \Lambda^{(1,1)}(u)$. From the fusion hierarchy (2.16) with $j = s = 1$, we obtain (after performing the shift $u \rightarrow u + \eta/2$) the following relation between $\Lambda^{(1,1)}(u)$ and $\Lambda^{(\frac{1}{2},1)}(u)$:

$$\Lambda^{(1,1)}(u) = \Lambda^{(\frac{1}{2},1)}(u - \frac{\eta}{2}) \Lambda^{(\frac{1}{2},1)}(u + \frac{\eta}{2}) - \delta^{(1)}(u + \frac{\eta}{2}). \quad (4.18)$$

Recalling (2.18)

$$\Lambda^{(\frac{1}{2},1)}(u) = g^{(\frac{1}{2},1)}(u)^{2N} \tilde{\Lambda}^{(\frac{1}{2},1)}(u), \quad (4.19)$$

and also our result (3.18) for $\tilde{\Lambda}^{(\frac{1}{2},1)}(u)$, we finally arrive at an expression for the energies in terms of the Bethe roots

$$E = \text{sh}^2(2\eta) \sum_{j=1}^{M^{(\pm)}} \frac{1}{\text{sh}(\tilde{v}_j^{(\pm)} - \eta) \text{sh}(\tilde{v}_j^{(\pm)} + \eta)} + N \left(\frac{\text{sh } 3\eta}{\text{sh } \eta} - 3 \right) + c^{(\pm)}, \quad (4.20)$$

where $\tilde{v}_j^{(\pm)} \equiv v_j^{(\pm)} + \eta/2$ as in (3.21), and $c^{(\pm)}$ are constants whose cumbersome expressions we refrain from presenting here. (These constants are independent of N , but do depend on the bulk and boundary parameters and on $\{\epsilon_i\}$.)

We have verified that the energies given by the Bethe Ansatz (4.20) coincide with those obtained by direct diagonalization of the Hamiltonian (4.9) for values of N up to 4. For the case $N = 3$, some sample results are presented in Tables 5, 6, for boundary parameter values corresponding to $k = 1$. Hence, as already noted in Table 2, 17 levels are obtained with $\tilde{\Lambda}^{(\frac{1}{2},1)(-)}(u)$ and are listed in Table 5; and 10 levels are obtained with $\tilde{\Lambda}^{(\frac{1}{2},1)(+)}(u)$ and are listed in Table 6. Together, they give all 27 energies obtained by direct diagonalization of the Hamiltonian.

For $s > 1$, it is in principle possible to proceed in a similar way. However, the computations become significantly more cumbersome, and we shall not pursue them further.

5 Discussion

We have found a Bethe Ansatz solution of the open spin- s XXZ chain with general integrable boundary terms (3.18) - (3.21), which is valid for generic values of the bulk anisotropy parameter η , provided that the boundary parameters satisfy the constraints (3.14), (3.16) and (3.17). We have presented numerical evidence that this solution is complete, and we have explicitly exhibited the Hamiltonian and its relation to the transfer matrix for the case $s = 1$. The apparent correctness of this solution provides further support for the validity of the identification (3.1) of the Q operator as a transfer matrix whose auxiliary-space spin tends to infinity.

A drawback of this solution is that for $|k| \leq 2sN - 1$, one does not know a priori in which of the two “sectors” (i.e., $\tilde{\Lambda}^{(\frac{1}{2},s)(-)}(u)$ or $\tilde{\Lambda}^{(\frac{1}{2},s)(+)}(u)$) a given level – such as the ground state – will be. For the case $s = 1/2$, alternative Bethe Ansatz-type solutions have been found [31] which do not suffer from this difficulty, and for which the boundary parameters do not need to obey the constraints (3.14), (3.16) and (3.17). However, those solutions hold only for values of bulk anisotropy corresponding to roots of unity. (For yet another approach to this problem, see [28].) Perhaps such solutions can also be generalized to higher values of s .

Part of our motivation for considering this problem comes from the relation of the $s = 1$ case to the supersymmetric sine-Gordon (SSG) model [39], in particular, its boundary version [40]. Indeed, Bethe Ansatz solutions of the spin-1 XXZ chain have been used to derive nonlinear integral equations (NLIEs) for the SSG model on a circle [41] and on an interval with Dirichlet boundary conditions [42]. With our solution in hand, one can now try to derive an NLIE for the SSG model on an interval with general integrable boundary conditions, and try to make contact with previously proposed boundary actions and boundary S matrices [40].

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A Conjectured relations for the fusion hierarchy

One way to derive the fusion hierarchy (2.16) relies on the existence of the relations

$$P_{12\dots 2j-1}^+ P_{12\dots 2j}^- P_{12\dots 2j-1}^+ = P_{12\dots 2j-2}^+ P_{2j-1,2j}^- + \sum_{k=1}^{2j-1} P_{k,k+1}^+ X^{(k)} P_{k,k+1}^-, \quad j = \frac{3}{2}, 2, \dots \quad (\text{A.1})$$

where $P_{12\dots n}^- \equiv 1 - P_{12\dots n}^+$, for some set of matrices $\{X^{(1)}, \dots, X^{(2j-1)}\}$. To our knowledge, these relations have not been proved for general values of j . However, we have verified them up to $j = 3$. In particular, for $j = 3/2$,

$$X^{(1)} = \frac{1}{3}(\mathcal{P}_{23} - \mathcal{P}_{13}), \quad X^{(2)} = \frac{1}{3}(\mathcal{P}_{12} - \mathcal{P}_{13}). \quad (\text{A.2})$$

This result is equivalent to an identity found by Kulish and Sklyanin, see Eq. (4.15) in the second reference of [2]. For $j = 2$, we find

$$\begin{aligned} X^{(1)} &= \frac{1}{12}(\mathcal{P}_{23} - \mathcal{P}_{13} + \mathcal{P}_{24} - \mathcal{P}_{14} + 2\mathcal{P}_{13}\mathcal{P}_{24} - 2\mathcal{P}_{14}\mathcal{P}_{23}), \\ X^{(2)} &= \frac{1}{6}(\mathcal{P}_{13} - \mathcal{P}_{12} + \mathcal{P}_{34} - \mathcal{P}_{24} + 2\mathcal{P}_{12}\mathcal{P}_{34} - 2\mathcal{P}_{13}\mathcal{P}_{24}), \\ X^{(3)} &= \frac{1}{4}(\mathcal{P}_{13} - \mathcal{P}_{14} + \mathcal{P}_{23} - \mathcal{P}_{24}). \end{aligned} \quad (\text{A.3})$$

Moreover, for $j = 5/2$,

$$\begin{aligned} X^{(1)} &= \frac{3}{5}\mathcal{P}_{15}\mathcal{P}_{14} + \mathcal{P}_{15}\mathcal{P}_{24} - \frac{2}{5}\mathcal{P}_{15}\mathcal{P}_{34} + \frac{1}{2}\mathcal{P}_{35}\mathcal{P}_{14} \\ &\quad - \frac{1}{10}\mathcal{P}_{35}\mathcal{P}_{13} - \frac{9}{10}\mathcal{P}_{45}\mathcal{P}_{14} - \frac{1}{6}\mathcal{P}_{45}\mathcal{P}_{13} + \frac{4}{15}\mathcal{P}_{14}\mathcal{P}_{13}, \\ X^{(2)} &= \frac{1}{5}\mathcal{P}_{15}\mathcal{P}_{24} + \frac{1}{15}\mathcal{P}_{15}\mathcal{P}_{13} - \frac{2}{5}\mathcal{P}_{25}\mathcal{P}_{14} + \frac{2}{5}\mathcal{P}_{25}\mathcal{P}_{24} + \frac{4}{15}\mathcal{P}_{25}\mathcal{P}_{34} \\ &\quad - \frac{2}{15}\mathcal{P}_{25}\mathcal{P}_{13} - \frac{7}{15}\mathcal{P}_{45}\mathcal{P}_{24} - \frac{1}{15}\mathcal{P}_{45}\mathcal{P}_{13} + \frac{2}{15}\mathcal{P}_{14}\mathcal{P}_{13}, \\ X^{(3)} &= \frac{3}{10}\mathcal{P}_{15}\mathcal{P}_{14} - \frac{3}{10}\mathcal{P}_{15}\mathcal{P}_{24} + \frac{1}{10}\mathcal{P}_{25}\mathcal{P}_{14} + \frac{3}{10}\mathcal{P}_{25}\mathcal{P}_{24} \\ &\quad - \frac{2}{5}\mathcal{P}_{35}\mathcal{P}_{14} - \frac{1}{5}\mathcal{P}_{35}\mathcal{P}_{34} - \frac{2}{5}\mathcal{P}_{35}\mathcal{P}_{12}, \\ X^{(4)} &= -\frac{4}{5}\mathcal{P}_{15}\mathcal{P}_{14} + \frac{2}{5}\mathcal{P}_{15}\mathcal{P}_{24} + \frac{2}{5}\mathcal{P}_{15}\mathcal{P}_{34} - \frac{2}{5}\mathcal{P}_{15}\mathcal{P}_{23}. \end{aligned} \quad (\text{A.4})$$

We omit our lengthy results for $j = 3$. We emphasize that the matrices $\{X^{(k)}\}$ are by no means unique: for instance, only the antisymmetric part $X_{1\dots k, k+1\dots 2j}^{(k)} - X_{1\dots k+1, k\dots 2j}^{(k)}$ matters in the expression of $X^{(k)}$. In fact, for the purpose of deriving the fusion hierarchy, the explicit matrices are not important, as the terms $\sum_k P_{k,k+1}^+ X^{(k)} P_{k,k+1}^-$ in (A.1) do not contribute. What is important is only the fact that such matrices exist.

We also observe that the symmetrizer $P_{1\dots n}^+$ (2.3) can be expressed as a sum of products of *commuting* permutations:

$$P_{1\dots n}^+ = a_0^{(n)} + \sum_{\ell=1}^{[n/2]} a_\ell^{(n)} \sum_{\sigma \in \mathcal{S}_n} \mathcal{P}_{\sigma(1)\sigma(2)} \mathcal{P}_{\sigma(3)\sigma(4)} \cdots \mathcal{P}_{\sigma(2\ell-1)\sigma(2\ell)} \quad (\text{A.5})$$

where \mathcal{S}_n is the permutation group of n indices. The coefficients $a_\ell^{(n)}$ are given by the recursion relations

$$a_0^{(n+1)} = \frac{1}{n+1} \left(a_0^{(n)} - n! a_1^{(n)} \right) \quad (\text{A.6})$$

$$a_1^{(n+1)} = \frac{1}{(n+1)^2} \left(\frac{1}{(n-1)!} a_0^{(n)} + 4 a_1^{(n)} - 2 a_2^{(n)} \right) \quad (\text{A.7})$$

$$a_\ell^{(n+1)} = \frac{3\ell+1}{(n+1)^2} a_\ell^{(n)} - \frac{\ell+1}{(n+1)^2} a_{\ell+1}^{(n)} + \frac{n+2-2\ell}{(n+1)^2} a_{\ell-1}^{(n)}, \quad 2 \leq \ell \leq \left[\frac{n}{2} \right] \quad (\text{A.8})$$

$$a_{[n/2]+1}^{(n+1)} = \frac{n-2[n/2]}{(n+1)^2} a_{[n/2]}^{(n)} \quad (\text{A.9})$$

with $a_0^{(1)} = 1$ and the convention $a_\ell^{(n)} = 0$ when $\ell > [n/2]$. Above, $[]$ denotes the integer part. Note that $a_0^{(1)} = 1$ is consistent with our previous convention $\mathcal{P}_{11} = 1$, which implies $P_1^+ = 1$ and $P_1^- = 0$. From the recursion, one then finds e.g. $a_0^{(2)} = \frac{1}{2}$ and $a_1^{(2)} = \frac{1}{4}$, which reproduces the result $P_{12}^+ = \frac{1}{2}(1 + \mathcal{P}_{12})$. Let us stress that, because of the sum on all permutations in \mathcal{S}_n , a term containing ℓ permutations has a multiplicity $\ell! (n-2\ell)! 2^\ell$ with respect to a ‘reduced’ expression, where all the terms appear just once. Identity (A.5) may help in the proof of the conjecture (A.1).

One can prove relation (A.5) by recursion. Starting from the relation at level n , the recursion relation

$$(n+1) P_{1\dots n+1}^+ = \left(1 + \sum_{\ell=1}^n \mathcal{P}_{\ell, n+1} \right) P_{1\dots n}^+ = P_{1\dots n}^+ \left(1 + \sum_{\ell=1}^n \mathcal{P}_{\ell, n+1} \right)$$

and the identity (valid for any spaces a, b, c)

$$\mathcal{P}_{ab} (\mathcal{P}_{ab} + \mathcal{P}_{ac} + \mathcal{P}_{bc}) = \mathcal{P}_{ab} + \mathcal{P}_{ac} + \mathcal{P}_{bc}$$

show that the relation is also obeyed at level $n+1$. A careful analysis of the different terms leads to the relations (A.6-A.9).

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k	no. given by $\tilde{\Lambda}^{(\frac{1}{2},1)(-)}(u)$	no. given by $\tilde{\Lambda}^{(\frac{1}{2},1)(+)}(u)$
5	9	0
3	8	1
1	6	3
-1	3	6
-3	1	8
-5	0	9

Table 1: Number of eigenvalues of $\tilde{t}^{(\frac{1}{2},s)}(u)$ given by $\tilde{\Lambda}^{(\frac{1}{2},s)(\pm)}(u)$ for $N = 2$, $s = 1$. The total number of eigenvalues is $(2s + 1)^N = 3^2 = 9$.

k	no. given by $\tilde{\Lambda}^{(\frac{1}{2},1)(-)}(u)$	no. given by $\tilde{\Lambda}^{(\frac{1}{2},1)(+)}(u)$
7	27	0
5	26	1
3	23	4
1	17	10
-1	10	17
-3	4	23
-5	1	26
-7	0	27

Table 2: Number of eigenvalues of $\tilde{t}^{(\frac{1}{2},s)}(u)$ given by $\tilde{\Lambda}^{(\frac{1}{2},s)(\pm)}(u)$ for $N = 3$, $s = 1$. The total number of eigenvalues is $(2s + 1)^N = 3^3 = 27$.

k	no. given by $\tilde{\Lambda}^{(\frac{1}{2}, \frac{3}{2})^{(-)}}(u)$	no. given by $\tilde{\Lambda}^{(\frac{1}{2}, \frac{3}{2})^{(+)}}(u)$
7	16	0
5	15	1
3	13	3
1	10	6
-1	6	10
-3	3	13
-5	1	15
-7	0	16

Table 3: Number of eigenvalues of $\tilde{t}^{(\frac{1}{2}, s)}(u)$ given by $\tilde{\Lambda}^{(\frac{1}{2}, s)^{(\pm)}}(u)$ for $N = 2$, $s = 3/2$. The total number of eigenvalues is $(2s + 1)^N = 4^2 = 16$.

k	no. given by $\tilde{\Lambda}^{(\frac{1}{2}, \frac{3}{2})^{(-)}}(u)$	no. given by $\tilde{\Lambda}^{(\frac{1}{2}, \frac{3}{2})^{(+)}}(u)$
10	64	0
8	63	1
6	60	4
4	54	10
2	44	20
0	32	32
-2	20	44
-4	10	54
-6	4	60
-8	1	63
-10	0	64

Table 4: Number of eigenvalues of $\tilde{t}^{(\frac{1}{2}, s)}(u)$ given by $\tilde{\Lambda}^{(\frac{1}{2}, s)^{(\pm)}}(u)$ for $N = 3$, $s = 3/2$. The total number of eigenvalues is $(2s + 1)^N = 4^3 = 64$.

E	Bethe roots $\tilde{v}_j^{(-)}$
-8.78796	$0.0781924 \pm 0.150582 i$, 0.573709
-7.99601	$0.377364+1.5708 i$, $0.0718753 \pm 0.150316 i$
-5.5443	$0.191917 \pm 0.145165 i$, 0.529223
-5.07143	$0.375505+1.5708 i$, $0.164075 \pm 0.148506 i$
-4.31229	$0.158193 \pm 0.299905 i$, 0.158448
-3.36195	0.166008 , $0.717455 \pm 0.259354 i$
-2.87198	$0.358903+1.5708 i$, 0.156172 , 0.784233
-2.86245	$0.33466 \pm 0.286332 i$, 0.337633
-2.69332	$0.371101+1.5708 i$, $0.292713 \pm 0.157296 i$
-2.31742	$0.290731+1.5708 i$, $0.617492+1.5708 i$, 0.146609
-1.52379	0.484424 , $0.621449 \pm 0.318594 i$
-1.18428	$0.356639+1.5708i$, 0.464322 , 0.659312
-0.780678	$0.288176+1.5708 i$, $0.610874+1.5708 i$, 0.368261
-0.379026	$0.879352 \pm 0.483137 i$, 0.944398
-0.0249248	$0.337585+1.5708 i$, $0.934391 \pm 0.266345 i$
0.389221	$0.277491+1.5708i$, $0.580415+1.5708 i$, 0.97389
0.838091	$0.245314+1.5708 i$, $0.477481+1.5708 i$, $0.814847+1.5708 i$

Table 5: The 17 energies and corresponding Bethe roots given by $\tilde{\Lambda}^{(\frac{1}{2},1)(-)}(u)$ for $N = 3, s = 1, k = 1, \eta = 0.3i, \alpha_- = 0.7i, \beta_- = 0.2, \theta_- = 0.5i, \alpha_+ = 1.2i, \beta_+ = -0.2, \theta_+ = -1.1i, \{\epsilon_i\} = +1$.

E	Bethe roots $\tilde{v}_j^{(+)}$
-9.55066	$0.0900396 \pm 0.151265 i$
-5.71507	$0.244797 \pm 0.132886 i$
-4.09573	$1.50013 i$, 0.182899
-3.74447	$0.786256 i$, 0.174481
-3.02558	$0.514399 \pm 0.220939 i$
-2.07231	$1.48515 i$, 0.620007
-1.79241	$0.80571 i$, 0.568604
-1.1462	$0.350408 +1.5708 i$, $1.38463 i$
-0.791024	$0.093628+1.5708 i$, $0.842728 i$
-0.634944	$0.979155 i$, $0.863041 i$

Table 6: The 10 energies and corresponding Bethe roots given by $\tilde{\Lambda}^{(\frac{1}{2},1)(+)}(u)$ for $N = 3, s = 1, k = 1, \eta = 0.3i, \alpha_- = 0.7i, \beta_- = 0.2, \theta_- = 0.5i, \alpha_+ = 1.2i, \beta_+ = -0.2, \theta_+ = -1.1i, \{\epsilon_i\} = +1$.