Observability analysis for structured bilinear systems: a graph-theoretic approach

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Abstract

This paper is devoted to the generic observability analysis for structured bilinear systems using a graph-theoretic approach. On the basis of a digraph representation, we express in graphic terms the necessary and sufficient conditions for the generic observability of structured bilinear systems. These conditions have an intuitive interpretation and are easy to check by hand for small systems and by means of well-known combinatorial techniques for large scale systems.

Key words: Structured bilinear systems, generic observability, graph theory.

1 Introduction

The class of bilinear systems (BLS), representing particular nonlinear systems whose dynamics are jointly linear in the state and the input variables, was introduced in control theory in the 1960's. This kind of systems are simpler and better understood than most other nonlinear systems. Furthermore, industrial process control, economics and biology (switched circuits, mechanical brakes, controlled suspension systems, immunological systems, population growth, enzyme kinetics, ...) provide examples of BLS. Finally, the usual linearization of a nonlinear control system near an equilibrium point can be improved by using a bilinear approximation. For these reasons, many works deal with BLS.
The observability of BLS was tackled in many works among which we can cite [7,18]. The necessary and sufficient conditions to achieve this property are now very well known. These conditions have been established using essentially geometric or algebraic tools. However, the use of such tools assumes the exact knowledge of the state space matrices characterizing the system's model. In many modeling problems, these matrices have a number of fixed zero entries determined by the physical laws while the remaining entries are not known precisely. To study the properties of these systems in spite of the poor knowledge we have on them, the idea is that we only keep the zero/non-zero entries in the state space matrices. Thus, we consider models where the fixed zeros are conserved while the non-zero entries are replaced by free parameters. There is a huge amount of interesting works in the literature using this kind of models called structured models. These models are useful to describe the class of systems having the same structure and they capture most of the structural available information from physical laws. Moreover, their study requires a low computational burden which allows one to deal with large scale systems. Because of these features, we think that structured systems are adapted to study a property like the observability and subsequently this paper deals with this kind of systems.
Many results on structured systems are related to the graph-theoretic approach. This approach is mainly dedicated to linear systems for which many structural properties such as controllability, observability, solvability of several classical control problems including disturbance rejection, input-output decoupling, fault detection and isolation are studied. Survey paper [6] reviews the most significant results in this area. From these studies, it follows that the graph-theoretic approach provides simple and elegant solutions and so is very well-suited to analyse large scale or/and uncertain systems. Unfortunately, not so many works based on graph-theoretic methods deal with nonlinear systems. Among them, [9,10] give conditions to the input-output decoupling and linearization of a nonlinear system, [15] gives sufficient conditions to fulfill the observability of bilinear systems and recently, [2] provides sufficient conditions for the uniform observability of some nonlinear systems.
In this context, this paper is dedicated to the analysis of the observability for structured bilinear systems (SBLS). More precisely, for such class of systems, we give simple necessary and sufficient conditions to achieve the observability of the whole or of a given part of the state. These conditions, which are equivalent to the well-known observability criteria [7,14,18] are easier to check from a computational point of view.
The paper is organised as follows: after section 2, which is devoted to the problem formulation, a digraph representation of SBLS is defined in section 3. The main results are given in section 4 and are illustrated with an example in section 5. Finally, some concluding remarks are made.

2 Problem statement

In this paper, we consider SBLS in the form:

$$(\Sigma_\Lambda): \begin{cases} \dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} u_i(t) A_i x(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \quad (1)$$

where $x(t) = (x_1(t), \ldots, x_n(t))^T \in \mathbb{R}^n$, $u(t) = (u_1(t), \ldots, u_m(t))^T \in \mathbb{R}^m$ and $y(t) = \left(y_1(t), \ldots, y_p(t)\right)^T \in \mathbb{R}^p$ are respectively the state, the input and the output vectors. For $i = 0, \ldots, m$, $A_i \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are matrices which elements are either fixed to zero or assumed free non-zero parameters. We can parameterize these nonzero entries by scalar real (nonzero) parameters λ_i, $i = 1, \ldots, h$ forming a parameter vector $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)^T \in \mathbb{R}^h$. If all the parameters λ_i are fixed, we obtain an admissible realization of structured system (Σ_Λ). Theoretic properties of each realization can be studied according to the values of λ_i. We say that a property is true generically [5] if it is true for almost all the realizations of structured system (Σ_Λ). Here, " for almost all the realizations " is to be understood [6, 17] as " for all parameter values ($\Lambda \in \mathbb{R}^h$) except for those in some proper algebraic variety in the parameter space ". The proper algebraic variety for which the property is not true is the zero set of some nontrivial polynomial with real coefficients in the h system parameters, which can be written down explicitly. Recall that a proper algebraic variety is an algebraic variety which has Lebesgue measure zero.

Many studies deal with the observability of bilinear systems using generally either geometric tools or linear time-varying system theory [4]. In these studies, the notions of u-indistinguishability, u-observability and u-unobservability subspaces are defined to characterize the ability of state reconstruction using the knowledge of the input and the output. In this respect, observability definition of BLS is given in [18]. We extend this definition to SBLS:

Definition 1 *Structured bilinear system (Σ_Λ) is generically observable if, for almost all the realisations of (Σ_Λ), there exists an input $u(t)$ such that any pair of initial states $x^0(0)$ and $x^1(0)$ are distinguishable by observation of the corresponding outputs $y^0(t)$ and $y^1(t)$ for $t \geq 0$.*

Directly, from results provided in [18], the generic observability of SBLS (Σ_Λ) is necessary and sufficient to ensure, for almost all the realizations of (Σ_Λ), the existence of an observer which provides an asymptotic estimate of the state. Moreover, Theorem 1 of [7] states that for piecewise constant or continuous controls, a BLS is observable iff there exists an input $u(t)$ for which it is u-observable. The proof is based on the construction of an universal input $\bar{u}$, which distinguishes all initial states $x^1(0)$ from $x^0(0) = \mathbf{0}$ by concatenating at most $n+1$ constant inputs: $\bar{u} = u^0 \circ u^1 \circ \ldots \circ u^n$. At the k^{th} stage in the construction, the set of states indistinguishable from $x(0) = \mathbf{0}$ is reduced in dimension by well chosen input value u^k. This leads to the following observability criterion for BLS. This criterion is based on the observability matrix rank conditions proposed in [7] for BLS.

Theorem 2 *For piecewise continuous input signals $u(t)$, structured bilinear system (Σ_Λ) is generically observable iff* ***T1:*** *g_rank$\Big(O(C, A_0, A_1, \ldots, A_m)\Big) = n$, where*

$$O(C, A_0, A_1, \ldots, A_m) = col\,(C, CA_0, CA_1, \ldots, CA_m, CA_0^2, CA_0A_1, \ldots\ldots, CA_0A_m,\ CA_1A_0,\ \ldots, CA_m^{n-1})$$

is the observability matrix of system (Σ_Λ) and where g_rank(M) denotes the generic rank of matrix M [12].

The aim of this paper is to provide graphic conditions equivalent to the one of Theorem 2. It turns out that these graphic conditions are easier to check and so more adapted to tackle large scale or uncertain systems. Indeed, the computation of g_rank$(O(C, A_0, A_1, \ldots, A_m))$ is quite difficult particularly for large scale systems.

3 Graph representation of structured bilinear systems

This section is devoted, in a first stage, to the definition of a digraph which represents SBLS (Σ_Λ). Next, some useful notations and definitions are given.

3.1 Digraph definition for structured bilinear system

Not so many works use the graph-theoretic approach to study nonlinear systems. The digraph we use in this paper is quite close to the one presented in [10]. The main differences are due to the fact that we adapt our representation to the context of observability analysis and to the fact that we deal with structured systems.
The digraph we associate to (Σ_Λ) is noted $\mathcal{G}(\Sigma_\Lambda)$ and is constituted by a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$ *i.e.* $\mathcal{G}(\Sigma_\Lambda) = (\mathcal{V}, \mathcal{E})$. The vertices are associated to the state and the output components of (Σ_Λ) and the directed edges represent links between these variables.
More precisely, $\mathcal{V} = \mathbf{X} \cup \mathbf{Y}$, where $\mathbf{X} = \{\mathbf{x_1}, \ldots, \mathbf{x_n}\}$ is the set of state vertices, $\mathbf{Y} = \left\{\mathbf{y_1}, \ldots, \mathbf{y_p}\right\}$ is the set of output vertices. For sake of clarity, the state and the output vertices are written in bold fonts. The edge set is $\mathcal{E} = \bigcup_{l=0}^{m} A_l\text{-edges} \cup C\text{-edges}$, where, for $l = 0, \ldots, m$, A_l-edges $= \left\{(\mathbf{x_j}, \mathbf{x_i}) \mid A_l(i,j) \neq 0\right\}$ and C-edges $= \left\{(\mathbf{x_j}, \mathbf{y_i}) \mid C(i,j) \neq 0\right\}$.
Here $M(i,j)$ is the $(i,j)th$ element of matrix M and $(\mathbf{v_1}, \mathbf{v_2})$ denotes a directed edge from vertex $\mathbf{v_1} \in \mathcal{V}$ to vertex $\mathbf{v_2} \in \mathcal{V}$. Note that number i is indicated under each A_i-edge in order to preserve the information about the belonging of the edges in the digraph representation. Moreover, for $i \in \{0, \ldots, m\}$, to each edge $e \in A_i$-edges we associate index i. Note that we associate several indexes to an edge e if it belongs to several subsets A_i-edges. The following example illustrates the proposed digraph representation.

Example 3 *In Figure 1, we represent the digraph associated to the SBLS defined by:*

$$A_0 = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 & 0 \\ \lambda_4 & 0 & \lambda_5 & 0 & 0 \\ 0 & 0 & \lambda_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_7 \\ 0 & 0 & 0 & \lambda_8 & 0 \end{pmatrix}, \; A_1 = \begin{pmatrix} 0 & \lambda_9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_{11} & \lambda_{12} & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_{13} \\ 0 & \lambda_{14} & 0 & 0 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} \lambda_{15} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{16} & 0 \end{pmatrix}.$$

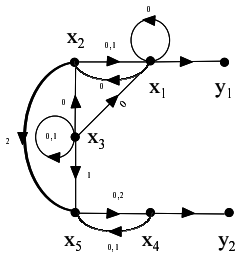

Figure 1. *Digraph for the study of observability for Example 3*

As we can see, comparatively with the digraph presented in [10], the input vertices are removed since we study the state observability. Moreover, due to this removal, the edges, which start in [10] from the input vertices, start in our digraph from state vertices and constitute the A_i-edges. Finally, as we are interested in structured systems, each edge represents a free parameter which has no numerical value. Thus, it is not necessary to indicate it on each edge of the digraph as in [10].

3.2 Definitions and notations

We present hereafter briefly some definitions concerning particular tools, we use in the paper. For the classical definitions related to the digraphs, the readers are referred to [1, 6, 10, 12, 13].

• A path P is an $\mathbf{Y}$-*topped* path if its end vertex is an element of $\mathbf{Y}$.

• Two edges $e_1 = (\mathbf{v_1}, \mathbf{v'_1})$ and $e_2 = (\mathbf{v_2}, \mathbf{v'_2})$, elements of $\mathcal{E}$, are v-disjoint if $\mathbf{v_1} \neq \mathbf{v_2}$ and $\mathbf{v'_1} \neq \mathbf{v'_2}$. Note that e_1 and e_2 can be v-disjoint even if $\mathbf{v'_1} = \mathbf{v_2}$ or $\mathbf{v_1} = \mathbf{v'_2}$. k edges are v-disjoint if they are mutually v-disjoint.

• Consider k edges $e_1 = (\mathbf{v_1}, \mathbf{v'_1})$, $e_2 = (\mathbf{v_2}, \mathbf{v'_2}), \ldots, e_k = (\mathbf{v_k}, \mathbf{v'_k})$. We define for $i = 1, \ldots, k$, I_i as the set of integers j such that $\mathbf{v'_j} = \mathbf{v'_i}$ *i.e.* $I_i \overset{def}{=} \left\{1 \leq j \leq k \mid \mathbf{v'_j} = \mathbf{v'_i}\right\}$. $e_1, e_2, \ldots, e_k$ are A-disjoint if **C1** and **C2** are satisfied, where:

C1: edges $e_1, e_2, \ldots, e_k$ have distinct begin vertices.

C2: for $i = 1, \ldots, k$, $\left(I_i = \{i\}\right)$ or $\big($there exist r distinct integers $i_1, i_2, \ldots, i_r$ such that $e_{j_1} \in A_{i_1}$-edges, $e_{j_2} \in A_{i_2}$-edges, $\ldots$, $e_{j_r} \in A_{i_r}$-edges, where $j_1, j_2, \ldots, j_r$ are all the elements of $I_i\big)$.

Roughly speaking, k edges are A-disjoint if their begin vertices are all distinct and if all the edges which have the same end vertex can be associated to distinct indexes i.

To illustrate the latter definition, note that in Example 3, $e_1 = (\mathbf{x_2}, \mathbf{x_1})$ and $e_2 = (\mathbf{x_1}, \mathbf{x_1})$ are A-disjoint as well as $e_1 = (\mathbf{x_2}, \mathbf{x_1})$ and $e_3 = (\mathbf{x_3}, \mathbf{x_1})$. So, e_1, e_2 and e_3 are mutually A-disjoint but they are not A-disjoint even if condition **C1** is satisfied. Indeed, these three edges have the same end vertex *i.e.* $I_1 = I_2 = I_3 = \{1, 2, 3\}$ and there do not exist three distinct integers i_1, i_2 and i_3 such that $e_1 \in A_{i_1}$-edges, $e_2 \in A_{i_2}$-edges and $e_3 \in A_{i_3}$-edges. Thus, **C2** is not satisfied. Otherwise, edges $e_1 = (\mathbf{x_2}, \mathbf{x_5})$, $e_2 = (\mathbf{x_3}, \mathbf{x_5})$ and $e_3 = (\mathbf{x_4}, \mathbf{x_5})$ are A-disjoint. Indeed, as they have distinct begin edges, **C1** is satisfied. Moreover, even if $I_1 = I_2 = I_3 = \{1, 2, 3\}$, as $e_1 \in A_2$-edges, $e_2 \in A_1$-edges and $e_3 \in A_0$-edges, **C2** is satisfied too.

4 Main results

We give hereafter a graphical criterion to characterize the generic-observability of system (Σ_Λ).

Proposition 4 *Structured bilinear system* (Σ_Λ) *is generically observable iff in its associated digraph* $\mathcal{G}(\Sigma_\Lambda)$
i. *every state vertex is the begin vertex of an* $\mathbf{Y}$*-topped path;*
ii. *there exist n A-disjoint edges in* $\mathcal{G}(\Sigma_\Lambda)$.

Proof:

Necessity:

• Assume that Condition (**i.**) is not satisfied *i.e.* $\exists\, \mathbf{x_j}$ from which there does not exist an $\mathbf{Y}$-*topped* path. Then, we can state that, $\forall k > 0$, $\forall i_1, i_2, \ldots, i_k$ elements of $\{0, 1, \ldots, m\}$, the j^{th} column of $CA_{i_1}A_{i_2}\ldots A_{i_k}$ is equal to zero. Thus, condition **T1** of Theorem 2 cannot be satisfied and the system is not generically observable.

• If structured bilinear system (Σ_Λ) is generically observable *i.e.* $\text{g\_rank}\Big(O(C, A_0, A_1, \ldots, A_m)\Big) = n$ then $Im\Big(C^T, A_0^TC^T, A_1^TC^T, \ldots, A_m^TC^T, (A_0^T)^2C^T, A_0^TA_1^TC^T, \ldots, A_0^TA_m^TC^T, A_1^TA_0^TC^T, \ldots (A_m^T)^{n-1}C^T\Big) = \mathbb{R}^n$. Yet, since $\forall M \in \mathbb{R}^{n\times p}$, $Im(A_i^TM) \subseteq Im(A_i^T)$, we have that $Im\Big(C^T, A_0^TC^T, A_1^TC^T, \ldots, A_m^TC^T, (A_0^T)^2C^T, \ldots, A_0^TA_m^TC^T, A_1^TA_0^TC^T, \ldots (A_m^T)^{n-1}C^T\Big) \subseteq Im\Big(C^T, A_0^T, \ldots, A_m^T\Big) \subseteq \mathbb{R}^n$.

Thus, necessarily $\text{g\_rank}\Big(O(C, A_0, A_1, \ldots, A_m)\Big) = n$ implies that $Im\left(C^T, A_0^T, A_1^T, \ldots, A_m^T\right) = \mathbb{R}^n$ and so $\text{g\_rank}\left(C^T, A_0^T, A_1^T \ldots, A_m^T\right) = n$ which is equivalent to $\text{g\_rank}(\text{col}\,(C, A_0, A_1, \ldots, A_m)) = n$.

On the other hand, we can prove (Appendix A) using the results of [12, 13, 16], that:

Statement 1: The maximal number of A-disjoint edges in $\mathcal{G}(\Sigma_\Lambda)$ is equal to $\text{g\_rank}\Big(\text{col}\,(C, A_0, A_1, \ldots, A_m)\Big)$.

Thus, immediately, the generic-observability of SBLS (Σ_Λ) implies that condition (**ii.**) is satisfied.

Sufficiency:

At first, let us recall some facts on linear systems. Consider a structured linear system,

$$\Sigma_{L,\Lambda} : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

It is well known that the system is observable iff $g\_rank\begin{pmatrix} C \\ sI - A \end{pmatrix} = n$, $\forall s \in \mathbb{C}$. Otherwise, the PBH test [8] states that $(\Sigma_{L,\Lambda})$ is unobservable iff there exists generically a vector $q \neq 0$ such that $Aq = s_0 q$, $s_0 \in \mathbb{C}$ and $Cq = 0$.

On the one hand, if $g\_rank\begin{pmatrix} C \\ sI - A \end{pmatrix} = n$, $\forall s \in \mathbb{C} \setminus \{0\}$, then the unobservability of $(\Sigma_{L,\Lambda})$ implies necessarily that there exists generically a vector $q \neq 0$ such that $Aq = 0$ and $Cq = 0$.

On the other hand, Lemma 14.1 of [13] states that, if in the digraph associated to $(\Sigma_{L,\Lambda})$, every state vertex is a begin vertex of an **Y**-*topped* path, then $g\_rank\begin{pmatrix} C \\ sI - A \end{pmatrix} = n$, $\forall s \in \mathbb{C} \setminus \{0\}$.

Assume now that conditions of Proposition 4 are satisfied. On the one hand, using Lemma 14.1 of [13], as all the parameters of matrices $A_0, \ldots, A_m$ are assumed to be free, we can state that condition (**i.**) of Proposition 4 implies that, for almost all input values $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_m)^T$, we have $\text{g\_rank}\Big(\text{col}(C,\ sI - (A_0 + \bar{u}_1 A_1 + \ldots + \bar{u}_m A_m))\Big) = n, \quad \forall s \neq 0$. On the other hand, if SBLS (Σ_Λ) is generically unobservable, then for all input constant values, $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_m)^T$, linear systems defined by matrices $(C, \bar{A})$ are also unobservable, where $\bar{A} = A_0 + \sum_{i=1}^{m} \bar{u}_i A_i$. This is due to the fact that for all input constant values, $\bar{u}$, $\text{Im}(O(C, \bar{A}) \subseteq \text{Im}(O(C, A_0, A_1, \ldots, A_m))$.

Therefore, If (Σ_Λ) is unobservable, using the previous settings, we have that for each matrix $\bar{A}$, there exists a nonzero vector q such that $\bar{A}q = 0$ and $Cq = 0$. Since this statement is true for almost all the input values $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_m)^T$, we have that for almost $n(m+1)$-uple values of input noted $\bar{u}^j = (\bar{u}_1^j, \ldots, \bar{u}_m^j)^T$, $j = 1, \ldots, n(m+1)$, we can find nonzero vectors q_j such that the following system is satisfied:

$$\begin{cases} Cq_j = 0 \\ A_0 q_j + \sum_{i=1}^{m} \bar{u}_i^j A_i q_j = 0 \end{cases} \quad j = 1, \ldots, n(m+1) \qquad (2)$$

Obviously, there cannot exist more than n independent vectors q_j. Let us note $q_1, \ldots, q_n$ the vectors such that $\text{span}(q_1, q_2, \ldots, q_{n(m+1)}) \subseteq \text{span}(q_1, q_2, \ldots, q_n)$ (we can renumber the vectors if necessary). All the vectors q_j, $j = n+1, \ldots, n(m+1)$ are linear combinations of $q_1, \ldots, q_n$. Therefore, system (2) contains the following $n(m+1) + n$ equations

$$\begin{cases} Cq_k = 0 & k = 1, \ldots, n \\ \sum_{k=1}^{n} \sum_{i=0}^{m} a_{i,k}^j(\bar{u}) A_i q_k = 0 & j = 1, \ldots, n(m+1) \end{cases} \qquad (3)$$

where $a_{i,k}^j(\bar{u})$ are linear functions of $\bar{u}^j$, $j = 1, \ldots, n(m+1)$. Since system (2) is satisfied for almost all the input values, we can generically find $\bar{u}^j$, $j = 1, \ldots, n(m+1)$ such that $\det\begin{pmatrix} a_{0,1}^1(\bar{u}) & a_{0,2}^1(\bar{u}) & \ldots & a_{m,n}^1(\bar{u}) \\ a_{0,1}^2(\bar{u}) & a_{0,2}^2(\bar{u}) & \ldots & a_{m,n}^2(\bar{u}) \\ \vdots & \vdots & \vdots & \vdots \\ a_{0,1}^{n(m+1)}(\bar{u}) & a_{0,2}^{n(m+1)}(\bar{u}) & \ldots & a_{m,n}^{n(m+1)}(\bar{u}) \end{pmatrix} \neq 0$. In this case, the only solution of system (3) is

$$\begin{cases} Cq_k = 0 \\ A_0 q_k = A_1 q_k = \ldots = A_m q_k = 0 \end{cases} \quad k = 1, \ldots, n \qquad (4)$$

Obviously, if (Σ_Λ) is unobservable then at least one vector q_k, $k = 1, \ldots, n$ is nonzero. Consequently, SBLS (Σ_Λ) is unobservable only if there exists at least a nonzero vector q such that $Cq = 0$ and $A_0 q = A_1 q = \ldots = A_m q = 0$. However, if condition (**ii**.) of Proposition 4 is satisfied, then $g\_\text{rank}\Big(\text{col}\,(C,\ A_0,\ A_1,\ \ldots,\ A_m)\Big) = n$ and so there does not exist a vector $q \neq 0$ such that $Cq = 0$ and $A_0 q = A_1 q = \ldots = A_m q = 0$. Hence, for almost all the realizations of SBLS (Σ_Λ), conditions of Proposition 4 are sufficient to ensure the observability. △

In the case of system described in Example 3, Figure 2 represents 5 A-disjoint edges extracted from digraph of Figure 1. Since all the state vertices are begin vertices of **Y**-*topped* paths, this system is generically observable.

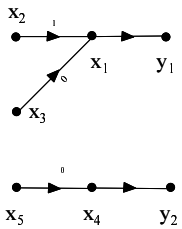

Figure 2. *A set of 5 A-disjoint edges for example 3*

As in [12] (Theorem 14.1) for the linear case, we can give simpler graphic conditions based on bipartite graphs:

Corollary 5 *Structured bilinear system* (Σ_Λ) *is generically observable iff in its associated digraph* $\mathcal{G}(\Sigma_\Lambda)$
i. *every state vertex is the begin vertex of an* **Y**-*topped path;*
ii. $g\_rank(col(C, A_0, A_1, \ldots, A_m)) = n$ *or equivalently, the maximal matching of the bipartite graph associated to matrix* $col(C, A_0, A_1, \ldots, A_m)$ *is equal to n.*

The main advantage of the previous corollary is its computational aspect. Indeed, the first condition can be checked using depth search algorithms. These algorithms have a complexity order $O(M \cdot N)$ where M is the number of edges in the digraph and $N = n+p$ the number of vertices. For our digraphs, in the worst case $M = (m+1) \cdot n^2 + n \cdot p$. Thus, the complexity of these algorithms, in our case, is $O(m \cdot n^3)$, (assuming without loss of generality that $p \leq n$). To check the second condition of Corollary 5, we define the bipartite graph related to matrix $col(C, A_0, A_1, \ldots, A_m)$ [12]. Then, we use the Bipmatch method [11], which allows to compute the cardinal of maximal matching into a bipartite graph. The complexity order of algorithms using this method is $O(M \cdot N^{0.5})$,

where, $M = (m+1)\cdot n^2 + n\cdot p$ and $N = n+p+(m+1)\cdot n$. So, for checking the second condition of Corollary 5 we can use algorithms which have complexity order $O(m^{3/2}\cdot n^{5/2})$. The proposed solution has an acceptable computational burden and so is very well suited to large scale systems.
The previous result can be seen as a generalization of the results concerning linear structured systems recalled in [6,12]:

Proposition 6 *A structured linear system* $(\Sigma_{L,\Lambda})$: $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$ *is generically observable iff in its associated graph* $\mathcal{G}(\Sigma_{L,\Lambda})$
i. *every state vertex is the begin vertex of an* **Y***-topped path;*
ii. *the maximal matching of the bipartite graph associated to matrix col*(C, A) *is equal to n.*

The first condition of both Corollary 5 and Proposition 6 is exactly the same. Moreover, since for a linear system, there is no matrices A_i, $i = 1,\ldots,m$, the second condition of Proposition 4 is equivalent to the second one of Corollary 5 in the linear case. Thus, in the linear case, Proposition 4 is equivalent to the one presented in [12] or to the one based on paths and cycles [6].

5 Example

In this section, we illustrate the results presented above with an example. This example is intentionally not very complicated. Nevertheless, it is clear that the observability criteria presented in Proposition 4 and Corollary 5 are very well-adapted to more complex or large-scale systems using combinatorial programming techniques.
Consider the SBLS represented in the digraph of Figure 3. This system has 19 state components. The matrix representation of this system is not given because of lack of place. Nevertheless, it can be easily deduced from the digraph. Our aim is to show the simplicity of the proposed method on a relatively large example. The proposed conditions can be checked easily by hand whereas the rank test of the observability matrix is quite difficult to do by hand.
In fact, this system is observable as each state vertex is a begin vertex of an **Y***-topped* paths and there exist 19 A-disjoint edges as it is displayed in Figure 4.

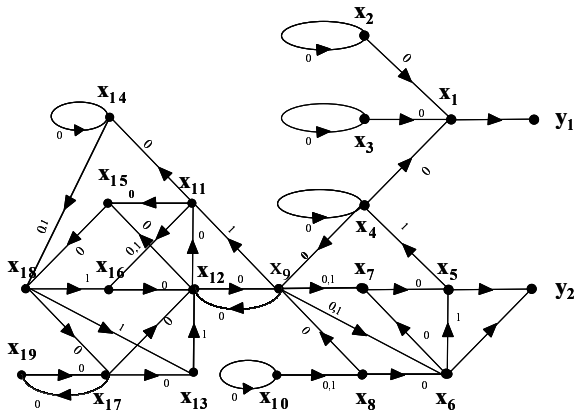

Figure 3. *Illustrative example*

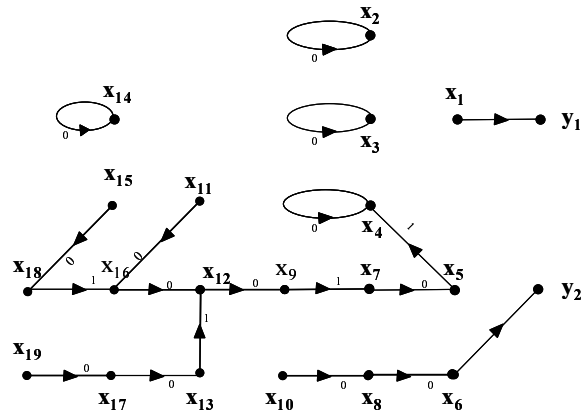

Figure 4. *A set of 19 A-disjoint edges for the illustrative example*

6 Conclusion

In this paper, we propose a new analysis tool to study the observability of structured bilinear systems. Using a new graph representation of this class of nonlinear systems, necessary and sufficient conditions for generic observability are given and expressed in graphic terms. These intrinsic conditions need few information about the system and are easy to check by means of combinatorial techniques or simply by hand for small systems. From a computational point of view, our approach is particularly suited for large-scale systems since it is free from numerical difficulties. Indeed, the proposed conditions can be easily implemented because they require simple computations based on finding edges in digraphs and on integer comparisons. Furthermore, the use of a graph-theoretic approach makes it easy to visualize the system structure. This may be very helpful for the optimisation of sensor placement to achieve the observability of the system. Indeed, starting from the above results, we can study, as in [3] for linear systems, the sensor location problem in order to achieve the generic observability.

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Appendix A: Proof of Statement 1

We prove hereafter that the maximal number of A-disjoint edges ending with vertices of $\mathbf{Y} \cup \mathbf{X}$ is equal to g_rank(M), where $M = \mathrm{col}\,(C, A_0, A_1, \ldots, A_m)$.

To compute g_rank(M), let us define bipartite graph $\mathcal{B}_{\mathcal{M}} = (\mathcal{V}^+; \mathcal{V}^-; \mathcal{E}_{\mathcal{M}})$ associated to matrix M and where $\mathcal{V}^+ \stackrel{def}{=} \{\mathbf{x_1^+}, \mathbf{x_2^+}, \ldots, \mathbf{x_n^+}\}$, $\mathcal{V}^- \stackrel{def}{=} \{\mathbf{y_1^-}, \mathbf{y_2^-}, \ldots, \mathbf{y_p^-}\} \cup \{\mathbf{x_{0,1}^-}, \mathbf{x_{0,2}^-}, \ldots, \mathbf{x_{0,n}^-}, \mathbf{x_{1,1}^-}, \mathbf{x_{1,2}^-}, \ldots, \mathbf{x_{m,n}^-}\}$ and $\mathcal{E}_{\mathcal{M}} = C^M\text{-edges} \cup \left(\bigcup_{k=0}^{m} A_k^M\text{-edges}\right)$, with $C^M\text{-edges} \stackrel{def}{=} \left\{(\mathbf{x_i^+}, \mathbf{y_j^-}) \mid C(j,i) \neq 0\right\}$ and for $k = 0, \ldots, m$, $A_k^M - \text{edges} \stackrel{def}{=} \{(\mathbf{x}_\ell^+, \mathbf{x_{k,i}^-}) \mid A_k(i,\ell) \neq 0\}$.

Note that C^M-edges are the same than C-edges in $\mathcal{G}(\Sigma_\Lambda)$.

From the results on the generic rank matrices [13], since all the parameters of matrices $A_0, \ldots, A_m$ and C are assumed to be free, we have that g_rank(M) is equal to the maximal number of disjoint edges in $\mathcal{B}_{\mathcal{M}}$, which is called the maximum matching in $\mathcal{B}_{\mathcal{M}}$.

If, for $i = 1, \ldots, n$, vertices $\mathbf{x_{0,i}^-}, \mathbf{x_{1,i}^-}, \ldots, \mathbf{x_{m,i}^-}$ are joined to form an unique vertex $\mathbf{x_i}$ and if $\mathbf{x_i^+}$ and $\mathbf{y_i^-}$ are noted respectively $\mathbf{x_i}$ and $\mathbf{y_i}$, the number of disjoint edges in $\mathcal{B}_{\mathcal{M}}$ becomes the number of A-disjoint edges in $\mathcal{G}(\Sigma_\Lambda)$. Indeed, k edges are disjoint in $\mathcal{B}_{\mathcal{M}}$ iff their corresponding edges in the original graph satisfy condition **C1** (to have distinct begin vertices in $\mathcal{B}_{\mathcal{M}}$) and condition **C2** (to have distinct end vertices in $\mathcal{B}_{\mathcal{M}}$).

Therefore, g_rank(M) is equal to the maximal number of A-disjoint edges ending with vertices in $\mathcal{G}(\Sigma_\Lambda)$ and **Statement 1** is proved. △