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Hydrogen Storage: Modeling and Analytical Results*

Elena Bonetti,¹ Michel Frémond,² and Christian LExcellent³

¹Laboratoire Lagrange, Dipartimento di Matematica “F. Casorati”,
Università di Pavia, Via Ferrata 1, 27100 Pavia, Italy
elena.bonetti@unipv.it

²Laboratoire Lagrange, Laboratoire Central des Ponts et Chaussées,
58 boulevard Lefebvre, 75372 Paris Cedex 15, France
michel.fremond@lpc.fr

³Laboratoire Lagrange, Laboratoire de Mécanique appliquée,
24 rue de l’Épitaphe, 25000 Besançon Cedex, France
christian.lexcellent@univ-fcomte.fr

Abstract. We present a thermomechanical model describing hydrogen storage by use of metal hydrides. The problem is considered as a phase transition phenomenon. The model is recovered by continuum mechanics laws, using a generalization of the principle of virtual power accounting for microscopic movements related to the phase transition. The resulting nonlinear PDE system is investigated from the point of view of existence, uniqueness, and regularity of solutions.

Key Words. Nonlinear PDE system, Hydrogen storage, Existence and uniqueness, Microscopic movements, Dissipative phase transition.

1. Introduction

This paper deals with hydrogen storage, which is nowadays a challenging subject in energetic and industrial applications (see, e.g., [11], [15], and [18]). Traditionally, it has been stored and transported mainly as a compressed gas. Recently, an alternative tech-

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nique has been developed, exploiting the possibility of many metals to absorb hydrogen. This latter solution seems to present some advantages in terms of safety, global yield, and long-time storage. Our research develops in the direction of providing a predictive theory describing the storage of hydrogen by use of metal hydrides (i.e., the metal which has absorbed hydrogen). We deal with this problem considering both modeling and analytical aspects. This represents, in some sense, a theoretical starting point to be developed in the future, performing numerical and experimental investigations, towards applications.

As we are more interested in the mechanical and analytical aspects, we do not enter the physical details of the description of the phenomenon. However, for the sake of clarity, we recall some basic features of this phenomenon (see [15]). Some metals are able to absorb hydrogen atoms and combine with them to form solid solutions. For the sake of simplicity we assume that there are two solid solutions, the α -phase and the β -phase. The presence of one phase with respect to the other depends on the pressure of the hydrogen. It has been experimentally proved that there exists a nondecreasing relation between the pressure of hydrogen (or more exactly between the logarithm of the pressure) and the hydrogen capacity (i.e., the density of the hydrogen atoms with respect to the density of the metal atoms). Moreover, it is shown that this relation is strongly influenced by temperature. To provide a good mechanical model to be analytically treated from the point of view of existence, uniqueness, and regularity of solutions, it turns out to be useful to exploit the theory of phase transitions. This corresponds to introducing an internal parameter χ , representing, for instance, the concentration of the α -phase with respect to the β -phase. We make the macroscopic approximation that the two phases may coexist at each point so that χ is prescribed to take values in $[0, 1]$. More precisely, if $\chi = 1$ we have the α -phase, if $\chi = 0$ we have the β -phase, and if $\chi \in (0, 1)$ both phases are present with suitable proportions. The state variables of the model are the absolute temperature θ , the hydrogen pressure p , the phase parameter χ , and its gradient $\nabla\chi$ accounting for local interactions between the different phases. Hence, we introduce the hydrogen density ρ_H and the total density ρ (in what follows we take $\rho = 1$). Thus, letting c_H be the capacity of the hydrogen, we have that $c_H = \rho_H(\rho - \rho_H)^{-1}$, i.e.,

$$\rho_H = \rho \frac{c_H}{1 + c_H} = \frac{1}{\tau},$$

where τ is the hydrogen's specific volume. Then we introduce a suitable potential in terms of which we describe the equilibrium of our thermomechanical system, at each instant t of its evolution (see [13]). Constitutive relations for the involved thermomechanical quantities are chosen in such a way that the principles of thermodynamics are satisfied.

We deal with the enthalpy functional $G(\theta, p, \chi, \nabla\chi)$, which can be introduced using the Legendre–Fenchel transformation of the free energy $\Psi(\theta, \tau, \chi, \nabla\chi)$, with respect to the specific volume τ . By thermodynamical and duality arguments it follows that the free energy is concave with respect to θ , while we assume that it is convex with respect to τ , χ , and $\nabla\chi$. Thus, as Ψ is assumed to be convex with respect to τ and $-p = \partial\Psi/\partial\tau$, we can introduce the dual function Ψ^* of Ψ as follows:

$$\Psi^*(\theta, -p, \chi, \nabla\chi) = \sup_{\tau} \{-p\tau - \Psi(\theta, \tau, \chi, \nabla\chi)\},$$

so that letting $\tau = \partial\Psi^*/\partial(-p) = -\partial\Psi^*/\partial p$ and assuming sufficient regularity for the

functionals we get

$$\Psi^*(\theta, -p, \chi, \nabla\chi) = -p\tau - \Psi(\theta, \tau, \chi, \nabla\chi).$$

Finally, the enthalpy $G(p)$ is defined by $G(p) := -\Psi^*(-p)$, so that it results that

$$G(\theta, p, \chi, \nabla\chi) = \Psi(\theta, \tau, \chi, \nabla\chi) + p\tau. \quad (1.1)$$

In particular, we recover that G is concave with respect to p and θ , while it is convex with respect to χ and $\nabla\chi$.

Remark 1.1. The fact that the enthalpy is concave with respect to the temperature follows from (1.1) once it is observed that the free energy is concave with respect to θ , which comes by duality arguments. This property turns out to be essential to ensure the thermodynamical consistence of the model (see [13] for more details).

Then we make precise the constitutive relations holding for the entropy s , the specific volume τ , and the internal energy e (see (1.1)). We have

$$s = -\frac{\partial G}{\partial \theta}, \quad (1.2)$$

$$\tau = \frac{\partial G}{\partial p}, \quad (1.3)$$

$$e = G - p\tau + \theta s. \quad (1.4)$$

Now we recall the fundamental balance laws of continuum mechanics, written in a smooth boundary domain $\Omega \subset \mathbf{R}^n$ during a finite time interval $[0, T]$. They are: the energy balance, the momentum balance, and the mass balance. In the following we use the symbol f_t for the time derivative of the function f . The energy balance equation, in the case when macroscopic deformations are described by $-\tau_t$ (see the hydrogen mass balance written below), reads as follows:

$$e_t + \operatorname{div} \mathbf{q} = r + B\chi_t + \mathbf{H} \cdot \nabla\chi_t - p\tau_t \quad \text{in } \Omega, \quad (1.5)$$

where, in particular, B and \mathbf{H} stands for new interior forces responsible for the phase transition, χ_t and $\nabla\chi_t$ are related to microscopic velocities, r is an exterior heat source, and the vector \mathbf{q} stands for the heat flux. In particular, we observe that the right-hand side of (1.5) accounts for heat sources induced by mechanical and exterior actions. The presence of microscopic mechanically induced heat sources in (1.5) is justified by a generalization of the principle of virtual power in which interior microscopic forces and motions are also considered, as they are responsible for the phase transition (see [12]). Finally (1.5) is complemented with a boundary condition, e.g., we assume that no heat flux occurs through the boundary $\Gamma := \partial\Omega$,

$$-\mathbf{q} \cdot \mathbf{n} = 0,$$

where \mathbf{n} stands for the normal unit vector on Γ .

Here and in the remainder of the work we assume small perturbations. Hence, assuming that the mass of the hydrogen which is not in the solid solutions remains constant, the hydrogen mass balance is

$$(\rho_H)_t + \operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega, \quad (1.6)$$

where \mathbf{v} is the hydrogen mass flux. Then we combine (1.6) with the following boundary condition:

$$-\mathbf{v} \cdot \mathbf{n} + \gamma p = 0 \quad \text{on } \Gamma, \quad \gamma > 0,$$

i.e., we assume that the hydrogen flux through the boundary is proportional to the difference between the exterior and the interior pressure (here the exterior pressure is chosen equal to 0).

Finally, by the principle of virtual power written for microscopic movements, we recover an equilibrium equation for the interior forces, which formally corresponds to the balance of the momentum seen as a microscopic equilibrium equation. We have, in the absence of exterior volume and surface forces,

$$B - \operatorname{div} \mathbf{H} = 0 \quad \text{in } \Omega$$

combined with

$$\mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma.$$

Hence, to describe the thermomechanical evolution of the system, we assume that there exists a pseudo-potential of dissipation Φ depending on χ_t and $\nabla\theta$ (see [16], [12], and [14]). We recall that Φ is a convex, non-negative function such that $\varphi(\mathbf{0}) = 0$. By the above properties, it turns out that its subdifferential $\partial\Phi$ is a maximal monotone operator with $\mathbf{0} \in \partial\Phi(\mathbf{0})$. In particular, it follows that

$$\partial\Phi(\chi_t, \nabla\theta) \cdot (\chi_t, \nabla\theta) \geq 0. \quad (1.7)$$

Now we are in the position to make precise constitutive relations for B and \mathbf{H} , given in terms of G and Φ . We introduce useful notation: nd is used for nondissipative contributions and d for dissipative ones. We have (see also (1.1))

$$B = B^{\text{nd}} + B^{\text{d}} = \frac{\partial G}{\partial \chi} + \frac{\partial \Phi}{\partial \chi_t} \quad (1.8)$$

and

$$\mathbf{H} = \mathbf{H}^{\text{nd}} = \frac{\partial G}{\partial \nabla \chi}. \quad (1.9)$$

Finally, we prescribe the Fourier heat flux law

$$\mathbf{q} = -k_0 \nabla\theta, \quad (1.10)$$

where $k_0 > 0$, which by a suitable choice of Φ can be expressed by use of the pseudo-potential of dissipation Φ as follows:

$$\mathbf{q} = -\theta \frac{\partial \Phi}{\partial \nabla \theta}. \quad (1.11)$$

Then, simplifying the model, we set

$$\mathbf{v} = -\lambda \nabla p, \quad (1.12)$$

for $\lambda > 0$ (take, e.g., $\lambda = 1$). Now we point out that by use of the chain rule in (1.5) and the above constitutive relations (1.2)–(1.4), (1.8)–(1.12), we can equivalently rewrite the energy balance (1.5) in terms of s and due to (1.7) eventually infer that

$$\theta \left(s_t + \operatorname{div} \frac{\mathbf{q}}{\theta} - \frac{r}{\theta} \right) = \partial \Phi(\chi_t, \nabla \theta) \cdot (\chi_t, \nabla \theta) \geq 0. \quad (1.13)$$

Note that, as $\theta > 0$, (1.13) yields the Clausius–Duhem inequality, ensuring thermodynamical consistency.

Now, we make precise the enthalpy functional G and the pseudo-potential of dissipation Φ . We set

$$G(\theta, p, \chi, \tau) = a \log p - b\chi(\log p - \log p_0) - c_p \theta \log \theta + \frac{\nu}{2} |\nabla \chi|^2 + I_{[0,1]}(\chi),$$

where $c_p > 0$, $\nu > 0$, $I_{[0,1]}(\chi) := 0$ if $\chi \in [0, 1]$, and $I_{[0,1]} := +\infty$, otherwise. In accordance with physical experience, we let $a > 0$ and $a - b > 0$ (take, e.g., $a = 1 = -b$). Hence, experiments show that for θ sufficiently large the Van’t Hoff law holds (see [15]), i.e.,

$$\log p_0 = c_1 \frac{1}{\theta} - c_2, \quad (1.14)$$

where $c_1, c_2 < 0$ are known constants. However, as we have already pointed out, the enthalpy G has to be concave with respect to the temperature, on the whole temperature interval. Thus, we extend the above relation setting

$$\log p_0 = h(\theta),$$

where h is a sufficiently smooth function with, e.g.,

$$\begin{aligned} h(\theta) &= c_1 \theta^{-1} - c_2 && \text{for } \theta \text{ sufficiently large, say } \theta \geq \theta_c, \\ h(\theta) &= c_3 && \text{for } \theta < \theta_c. \end{aligned} \quad (1.15)$$

In particular, we could take $c_3 = h(\theta_c)$ and, if θ_c is sufficiently small, it results that $c_3 < 0$. Moreover, the value θ_c has to be chosen in such a way that

$$\frac{\partial^2 G}{\partial \theta^2} = b\chi h''(\theta) - \frac{c_p}{\theta} < 0. \quad (1.16)$$

Indeed, (1.16) ensures that G is concave with respect to temperature, which is necessary to get the thermodynamical consistency of the model. As will be clear in what follows, (1.16) is in direct relation with some assumption on h concerning the analytical solvability of the resulting heat equation.

Concerning the pseudo-potential Φ , we let

$$\Phi = \frac{\mu}{2} |\chi_t|^2 + \frac{k_0}{2\theta} |\nabla \theta|^2.$$

Finally, the small perturbations assumption allows us to neglect higher-order dissipative terms in the energy balance. Thus, substituting in the balance equations constitutive relations (1.2)–(1.4), (1.8), (1.9), (1.11), and (1.12), specified in terms of G and

Φ (see also (1.10)), the resulting PDE system is written in $Q := \Omega \times (0, T)$ as follows (see (1.15)):

$$(-bh''(\theta)\theta\chi + c_p)\theta_t - k_0\Delta\theta = b\theta\chi_t h'(\theta), \quad (1.17)$$

$$\mu\chi_t - \nu\Delta\chi + \partial I_{[0,1]}(\chi) \ni b(-h(\theta) + \log p), \quad (1.18)$$

$$\left(\frac{p}{a - b\chi}\right)_t - \Delta p = 0. \quad (1.19)$$

Then (1.17)–(1.19) are combined with initial conditions

$$\theta(0) = \theta_0, \quad \chi(0) = \chi_0, \quad p(0) = p_0, \quad (1.20)$$

and boundary assumptions (∂_n is the normal derivative operator on the boundary)

$$k_0\partial_n\theta = 0, \quad (1.21)$$

$$\nu\partial_n\chi = 0, \quad (1.22)$$

$$\lambda\partial_n p + \gamma p = 0. \quad (1.23)$$

As far as we know, the above analytical formulation of the problem of hydrogen storage is new and there are no related results in the literature. Moreover, the solvability of the resulting PDEs system, written as a phase-field problem, turns out to be an interesting subject also from the analytical point of view. Indeed, the system (1.17)–(1.19) and (1.20)–(1.23) is highly nonlinear and, to solve it (in some suitable weak sense we specify later) requires analytical tools which are not trivial. This is mainly due to the coupling of higher-order nonlinear contributions involving the unknowns, a maximal monotone graph for the phase parameter, and a logarithm term involving the pressure. More precisely, in the parabolic equation (1.17) the specific heat is a nonlinear function: to ensure coerciveness, we need to prescribe a suitable assumption on the function h . In particular, the nonlinear character of the specific heat gives rise to some difficulties in the proof of the uniqueness result (see, e.g., [9] for a similar problem in a different framework). For the sake of completeness, we point out that if p is known (and sufficiently regular) the system (1.17)–(1.18) is a phase-field system with nonlinear heat capacity. Some related analytical results can be found for a model for shape memory alloys or binary phase transitions with nonlinear uniformly bounded latent heat in the free energy (see, e.g., [10], [6], [9], and references therein). However, the presence of the nonlinear term $\log p$ and the coupling between (1.18) and (1.19), as far as we know, is new. Hence, dealing with the equation governing the evolution of the pressure, we have to combine the regularity of the function χ and the pressure p , mainly to control the nonlinear evolution term. Finally, it is worth observing that the pressure has a major role in the evolution of the phase through the presence of its logarithm as a source in the corresponding evolution inclusion (see (1.18)). The logarithm is easily controlled for high values of the pressure, whenever we are able to control p , but it degenerates as $p \searrow 0$. Thus, our proof exploits some ad hoc estimates for (1.19) to control this nonlinearity in the phase equation as the pressure goes to zero. However, as will be clear in what follows, the regularity we can obtain on p is not sufficient to solve (1.19), even if it is written in a variational formulation. We restrict ourselves to studying a weaker formulation of the problem integrating

(1.19) with respect to time and we are able to prove a global existence result. Then, considering a two-dimensional setting, we are allowed to exploit some Gagliardo–Nirenberg inequalities (holding in the two-dimensional case). This provides sufficient regularity on p and allows us to solve directly the abstract version of (1.19). In this case we are also able to prove a regularity result, holding in the case when the initial data are sufficiently smooth. Thus, by the uniqueness theorem, the two-dimensional problem turns out to be well-posed. Regarding the well-posedness of the three-dimensional system (where some weaker version of (1.19) is considered), we are able to prove a uniqueness result only for solutions which are smoother than those for which we can state the existence. The problem of finding, in the three-dimensional system, the existence of solutions with the regularity required to prove uniqueness is still an open problem.

Here is the outline of the paper. In the next section we introduce the variational formulation of the n -problem and state the main existence result holding for $n \leq 3$ (Theorem 2.1). Then, improving the regularity of solutions, Theorem 2.2 shows a stronger existence result in the case when $n \leq 2$. Finally, Theorem 2.3 states the uniqueness of sufficiently regular solutions. In particular, Theorems 2.2 and 2.3 lead to the well-posedness of the two-dimensional problem.

2. Analytical Formulation and Main Results

In this section we introduce the abstract problem we deal with and state the main existence and uniqueness results of this paper. We render the physical constants to 1 (or -1) as specified in the Introduction (i.e., $a = -b = c_p = \lambda = k_0 = \nu = \mu = \gamma = 1$). For the sake of clarity, before proceeding, we introduce some useful notation. Let

$$V \hookrightarrow H \hookrightarrow V' \tag{2.1}$$

be a Hilbert triplet with

$$H := L^2(\Omega), \quad V := H^1(\Omega).$$

As usual, H is identified with its dual space H' and $\langle \cdot, \cdot \rangle$ stands for the duality pairing between V' and V . We denote the norm both in Banach space X and in X^3 by the same symbol $\|\cdot\|_X$. Hence, we introduce the following abstract operators:

$$\mathcal{A} : V \rightarrow V', \quad \langle \mathcal{A}v, u \rangle = \int_{\Omega} \nabla v \cdot \nabla u, \quad u, v \in V \tag{2.2}$$

$$\mathcal{B} : V \rightarrow V', \quad \langle \mathcal{B}v, u \rangle = \int_{\Omega} \nabla v \cdot \nabla u + \int_{\Gamma} vu, \quad u, v \in V. \tag{2.3}$$

Then, to simplify notation, we set

$$W := \{f \in H^2(\Omega) : \partial_n f = 0 \text{ on } \Gamma\}.$$

We rewrite (1.17)–(1.19), combined with (1.20) and (1.21)–(1.23), in the abstract setting of the triplet (V, H, V') . Note, in particular, that the resulting duality pairings in mathematics correspond to the variational formulation of the balance laws in continuum

mechanics (see [12]). Thus, the system is rewritten, in V' and a.e. in $(0, T)$, as follows:

$$(h''(\theta)\theta\chi + 1)\theta_t + \mathcal{A}\theta = -\theta\chi_t h'(\theta), \quad (2.4)$$

$$\chi_t + \mathcal{A}\chi + \xi = h(\theta) - y, \quad (2.5)$$

$$\xi \in \partial I_{V',V}(\chi) \quad (2.6)$$

$$\left(\frac{p}{1+\chi}\right)_t + \mathcal{B}p = 0, \quad (2.7)$$

$$y = \log p, \quad (2.8)$$

where, actually, relation (2.8) is intended to hold a.e. in Q . By $\partial I_{V',V}$ in (2.6) we denote the subdifferential operator $V \rightarrow V'$ of the indicator function of the convex

$$[0, 1]_V := \{v \in V, v \in [0, 1] \text{ a.e. in } \Omega\}. \quad (2.9)$$

More precisely, we have (see (2.9))

$$\begin{aligned} \xi \in \partial I_{V',V}(\chi) \quad & \text{if and only if} \quad \chi \in [0, 1]_V, \\ \langle \xi, v - \chi \rangle \leq 0, \quad & \forall v \in [0, 1]_V. \end{aligned} \quad (2.10)$$

Remark 2.1. In the following we will show that (2.4)–(2.5) are solved in H . In particular, concerning the abstract subdifferential operator in (2.5), we will prove that $\xi \in H$ a.e. in $(0, T)$. Thus, we may infer that actually $\xi \in \partial I_{[0,1]_H}(\chi)$ a.e. in $(0, T)$ ($[0, 1]_H := \{v \in H : v \in [0, 1] \text{ a.e. in } \Omega\}$) from which one can deduce that $\xi \in \partial I_{[0,1]}(\chi)$ a.e. in Q (see [1]).

In spite of the regularity we can obtain for the solutions to (2.4)–(2.5) (see Remark 2.1), dealing with (2.7) it seems very difficult to find a solution in the variational setting we have introduced above. Thus, at a first instance, we restrict ourselves to considering the equation obtained integrating with respect to time (2.7) and to looking for a corresponding solution $\tilde{p} := 1 * p$, where

$$(1 * p)(t) = \int_0^t p(s) ds. \quad (2.11)$$

Remark 2.2. We point out that solving with respect to $1 * p$ the integrated version of (2.7) corresponds to applying the so-called Baiocchi transformation [3] (see also the Stefan problem, where the so-called freezing index, i.e., $1 * \theta$, is introduced as an unknown in place of the temperature).

Thus, the new equation is (see also (1.20))

$$\frac{1}{1+\chi} \tilde{p}_t + \mathcal{B}\tilde{p} = \frac{p_0}{1+\chi_0}, \quad \tilde{p}(0) = 0,$$

or analogously

$$\frac{p}{1+\chi} + 1 * \mathcal{B}p = \frac{p_0}{1+\chi_0}. \quad (2.12)$$

Now, we make precise the assumptions on the data of the problem. We prescribe

$$\theta_0 \in V, \quad (2.13)$$

$$\chi_0 \in [0, 1] \cap V, \quad (2.14)$$

$$p_0 \in L^2(\Omega), \quad \log p_0 \in L^1(\Omega). \quad (2.15)$$

Note that (2.15) yields $p_0 > 0$ a.e. in Ω . Hence, we ask for a suitable regularity of the thermal expansion coefficient $h(\theta)$, in agreement with the assumptions leading to the physical consistence of the model (see (1.16)). We require

$$h \in W^{2,\infty}(\mathbf{R}) \cap C^2(\mathbf{R}), \quad (2.16)$$

$$\|h\|_{W^{2,\infty}(\mathbf{R})} + |h'(\zeta)\zeta| \leq c_h, \quad |h''(\zeta)\zeta| \leq c'_h, \quad \forall \zeta \in \mathbf{R}, \quad (2.17)$$

for some positive constants c_h, c'_h . In addition, let $c_s > 0$ such that (recall that $\chi \in [0, 1]$ and (2.17))

$$1 + \chi h''(\zeta)\zeta \geq 1 - |\zeta h''(\zeta)| \geq 1 - c'_h \geq c_s > 0, \quad \forall \zeta \in \mathbf{R}. \quad (2.18)$$

Then the following theorem is proved.

Theorem 2.1. *Let (2.13)–(2.15) and (2.16)–(2.18) hold and fix $T > 0$, $\Omega \subset \mathbf{R}^N$, with $N \leq 3$. Then there exist $(\theta, \chi, \xi, p, y)$ with*

$$\theta \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \quad (2.19)$$

$$\chi \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \cap L^\infty(Q), \quad (2.20)$$

$$\xi \in L^2(0, T; H), \quad (2.21)$$

$$p \in L^2(0, T; H), \quad 1 * p \in L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)), \quad (2.22)$$

$$y \in L^2(0, T; V), \quad (2.23)$$

fulfilling initial conditions (1.20) and, a.e. in $(0, T)$,

$$(1 + h''(\theta)\theta\chi)\theta_t + \mathcal{A}\theta = -\theta\chi_t h'(\theta) \quad \text{in } H, \quad (2.24)$$

$$\chi_t + \mathcal{A}\chi + \xi = h(\theta) - y \quad \text{in } H, \quad (2.25)$$

$$\xi \in \partial I_{[0,1]}(\chi) \quad \text{a.e. in } \Omega, \quad (2.26)$$

$$\frac{p}{1 + \chi} + 1 * \mathcal{B}p = \frac{p_0}{1 + \chi_0} \quad \text{in } H, \quad (2.27)$$

$$y = \log p \quad \text{a.e. in } \Omega. \quad (2.28)$$

The proof of Theorem 2.1 is carried out in Section 3 by using a fixed point argument.

Now, observe that the regularities of the solutions specified by (2.19)–(2.23) are not sufficient to solve the original problem, where (2.7) is considered in place of its integrated version (2.27), even if it is written in the duality between V' and V . Moreover, we are not

able to prove the uniqueness of such solutions, mainly due to the weak regularity obtained on p . However, some further regularity results can be obtained for lower-dimensional systems. Thus, as a first step, we investigate the system (2.24)–(2.26), (2.28), and (2.7) in the two-dimensional case. Indeed, for $\Omega \subset \mathbf{R}^2$ some Gagliardo–Nirenberg inequalities may help us improve the regularity of p (see (2.22)). As a consequence, we are able to deal with (2.7) (at least in V') and, secondly, to prove uniqueness of the solution for the resulting complete PDE system.

Theorem 2.2. *Let (2.13)–(2.15) and (2.16)–(2.18) hold and fix $T > 0$, $\Omega \subset \mathbf{R}^N$, with $N \leq 2$. Then there exist $(\theta, \chi, \xi, p, y)$ solving (2.24)–(2.26), (2.28), (2.7), and (1.20), fulfilling the regularity prescribed by (2.19)–(2.23). In addition, it holds that*

$$p \in L^\infty(0, T; H) \cap L^2(0, T; V), \quad (2.29)$$

$$u_t \in L^2(0, T; V'), \quad (2.30)$$

where $u = p/(1 + \chi)$. Moreover, if

$$p_0^{-1} \in H, \quad p_0 \in V, \quad \log p_0 \in H \quad (2.31)$$

$$\chi_0 \in W, \quad (2.32)$$

then the following improved regularities are obtained:

$$p \in H^1(0, T; H) \cap L^\infty(0, T; V), \quad (2.33)$$

$$u^{-1} \in L^\infty(0, T; H) \cap L^2(0, T; V), \quad (2.34)$$

$$\chi \in W^{1,\infty}(0, T; H) \cap H^1(0, T; V). \quad (2.35)$$

The proof of Theorem 2.2 is exploited in Section 4 performing some suitable a priori estimates on the solutions of the problem whose existence is stated by Theorem 2.1.

Finally, we aim to establish a uniqueness result for the original problem (2.24)–(2.26), (2.28), (2.7), combined with (1.20), at least for sufficiently regular solutions. In particular, for $N \leq 2$ we will show the uniqueness of the solutions whose existence is stated by Theorem 2.2 (see Remark 2.3). The proof is presented in Section 5 by the use of a contradiction argument and some contraction estimates.

Theorem 2.3. *Assume that (2.16)–(2.18) hold, $T > 0$, and $\Omega \subset \mathbf{R}^N$ with $N \leq 3$. Let $(\theta_1, \chi_1, \xi_1, p_1, y_1)$ and $(\theta_2, \chi_2, \xi_2, p_2, y_2)$ be two solutions of (2.24)–(2.26), (2.28), and (2.7), combined with (1.20). Let one of the following conditions be satisfied:*

(A1) $N \leq 2$ and $(\theta_i, \chi_i, p_i, \xi_i, y_i)$, $i = 1, 2$, satisfy (2.19)–(2.23) and (2.30), (2.33)–(2.35);

(A2) $N = 3$ and $(\theta_i, \chi_i, p_i, \xi_i, y_i)$, $i = 1, 2$, satisfy (2.19)–(2.23), (2.30), (2.33)–(2.35), and, in addition,

$$p^{-1} \in L^4(0, T; L^6(\Omega)). \quad (2.36)$$

Then

$$\begin{aligned} \theta_1 &= \theta_2, & \chi_1 &= \chi_2, & p_1 &= p_2, \\ \xi_1 &= \xi_2, & y_1 &= y_2 & \text{a.e. in } Q. \end{aligned} \quad (2.37)$$

Remark 2.3. Theorems 2.2 and 2.3 provide a well-posedness result for the two-dimensional problem, at least when initial data are sufficiently regular. Indeed, by Theorem 2.2, assuming that the initial data satisfy (2.31)–(2.32), there exist solutions fulfilling in particular the regularity specified by (2.33)–(2.35). Thus, in this case, (A1) is satisfied, which leads to the uniqueness of the solution.

3. Proof of Theorem 2.1

This section is devoted to the proof of the existence result stated by Theorem 2.1. We apply the Schauder fixed-point theorem to a suitable operator we are going to introduce. To this aim, let

$$\begin{aligned} \mathcal{X} &= \{(\chi, \theta) \in L^2(0, T; V) \times L^2(0, T; H), \chi \in [0, 1] \text{ a.e. in } Q, \\ &\quad \|\chi\|_{L^2(0, T; V)} + \|\theta\|_{L^2(0, T; H)} \leq R\}, \end{aligned} \quad (3.1)$$

where $R > 0$ is chosen sufficiently large (also with respect to the initial data). Hence, take an arbitrary couple of functions $(\widehat{\chi}, \widehat{\theta}) \in \mathcal{X}$ and substitute $\widehat{\chi}$ in (2.27) instead of χ . By (3.1) it follows that

$$1 \leq 1 + \widehat{\chi} \leq 2 \quad \text{or analogously} \quad \frac{1}{2} \leq \frac{1}{1 + \widehat{\chi}} \leq 1. \quad (3.2)$$

Fairly standard results for parabolic evolution equations (see [2]) show that there exists a unique solution

$$p = \mathcal{T}_1(\widehat{\chi}) = \widetilde{p}_t,$$

solving the resulting equation. Indeed, (2.27) can be regarded as a parabolic equation with respect to the variable $\widetilde{p} = 1 * p$ (see also (2.11), (2.12), and (3.2)) with the right-hand side (a.e. positive) in $L^\infty(0, T; H)$ and initial condition $\widetilde{p}(0) = 0$. We now proceed by establishing some a priori estimates on the solution p of (2.27), where $\widehat{\chi}$ is considered in place of χ .

First estimate. Test (2.27) by p and integrate over $(0, t)$. Recalling that (3.2) holds, using Young's inequality, and integrating by parts in time, we write (see (2.3), (2.14), and (2.15))

$$\begin{aligned} &\frac{1}{2} \|p\|_{L^2(0, t; H)}^2 + \frac{1}{2} \|1 * \nabla p(t)\|_H^2 + \frac{1}{2} \|1 * p(t)\|_{L^2(\Gamma)}^2 \\ &\leq \int_0^t \int_\Omega \frac{p_0}{1 + \chi_0} p \leq \frac{1}{4} \|p\|_{L^2(0, t; H)}^2 + c \int_0^t \left\| \frac{p_0}{1 + \chi_0} \right\|_H^2 \\ &\leq c + \frac{1}{4} \|p\|_{L^2(0, t; H)}^2. \end{aligned} \quad (3.3)$$

Consequently, we get

$$\|p\|_{L^2(0, T; H)} + \|1 * p\|_{L^\infty(0, T; V)} \leq c. \quad (3.4)$$

Note that, from now on, for the sake of simplicity we denote by the same symbol c possibly different positive constants. Hence, by (3.2), (3.4), and (2.14)–(2.15) a comparison in (2.27) leads to

$$\|1 * \mathcal{B}p\|_{L^2(0,T;H)} \leq c,$$

and owing to (3.3)–(3.4)

$$\|1 * p\|_{L^2(0,T;H^2(\Omega))} \leq c. \quad (3.5)$$

Second estimate. To simplify notation we introduce the new variable

$$u := \frac{p}{1 + \chi}. \quad (3.6)$$

Hence, we rewrite (2.27) in terms of u ,

$$u + 1 * \mathcal{B}(u(1 + \chi)) = \frac{p_0}{1 + \chi_0}, \quad (3.7)$$

and take $\widehat{\chi}$ in place of χ in (3.6) and (3.7). Before proceeding, we point out that by (3.4), the definition of u , and the regularity of $\widehat{\chi}$ (see (3.1)) we have

$$\|u\|_{L^2(0,T;H)} + \|1 * u(1 + \widehat{\chi})\|_{L^\infty(0,T;V)} \leq c. \quad (3.8)$$

Now, we formally proceed differentiating (3.7) with respect to time:

$$u_t + \mathcal{B}u(1 + \widehat{\chi}) = 0. \quad (3.9)$$

Then we test (3.9) by

$$H(u) := 1 \quad \text{if } u \geq 0 \quad \text{and} \quad H(u) := 0 \quad \text{if } u < 0.$$

We recall that the Heaviside graph $H(\cdot) = \partial j(\cdot)$ can be seen as the subdifferential of the positive part function $j(\cdot) = (\cdot)^+$, which is defined by $(f)^+ := f$ if $f \geq 0$ and $(f)^+ := 0$ otherwise. The procedure we are exploiting is formal and the rigorous computation can be performed by use of the Moreau–Yosida approximation $H_\varepsilon = \partial j_\varepsilon$ of the graph H , j_ε being the corresponding regularization of the positive part function j . In particular, we let $H_\varepsilon(u) = 1$ if $u \geq \varepsilon$, $H_\varepsilon(u) = 0$ if $u < 0$, and $H_\varepsilon(u) = \varepsilon^{-1}u$ if $0 \leq u < \varepsilon$. Integrating by parts in time and recalling that $p_0(1 + \chi_0)^{-1} > 0$ a.e. in Ω , we infer that

$$\int_0^t \int_\Omega u_t H_\varepsilon(u) = \int_\Omega j_\varepsilon(u(t)) - \int_\Omega \frac{p_0}{1 + \chi_0}. \quad (3.10)$$

Then we have

$$\begin{aligned} \int_0^t \langle \mathcal{B}(u(1 + \widehat{\chi})), H_\varepsilon(u) \rangle &= \int_0^t \int_\Omega H'_\varepsilon(u) \nabla u(1 + \widehat{\chi}) \nabla u + \int_0^t \int_\Gamma u(1 + \widehat{\chi}) H_\varepsilon(u) \\ &= \int_0^t \int_{\Omega \cap \{0 \leq u \leq \varepsilon\}} \varepsilon^{-1} \nabla(u(1 + \widehat{\chi})) \nabla u \\ &\quad + \int_0^t \int_\Gamma u(1 + \widehat{\chi}) H_\varepsilon(u). \end{aligned} \quad (3.11)$$

To handle the right-hand side of (3.11) we proceed as follows:

$$\begin{aligned}
& \int_0^t \int_{\Omega \cap \{0 \leq u \leq \varepsilon\}} \varepsilon^{-1} \nabla(u(1 + \widehat{\chi})) \nabla u \\
&= \int_0^t \int_{\Omega \cap \{0 \leq u \leq \varepsilon\}} \varepsilon^{-1} (1 + \widehat{\chi})^{-1} |\nabla(u(1 + \widehat{\chi}))|^2 \\
&\quad - \int_0^t \int_{\Omega \cap \{0 \leq u \leq \varepsilon\}} \varepsilon^{-1} (1 + \widehat{\chi})^{-1} \nabla(u(1 + \widehat{\chi})) u \nabla \widehat{\chi}. \tag{3.12}
\end{aligned}$$

Thus, by (3.10)–(3.12), using Young’s inequality, and letting $\varepsilon < 1$, we have

$$\begin{aligned}
& \int_{\Omega} j_{\varepsilon}(u(t)) + \int_0^t \int_{\Omega \cap \{0 \leq u \leq \varepsilon\}} \varepsilon^{-1} (1 + \widehat{\chi})^{-1} |\nabla(u(1 + \widehat{\chi}))|^2 \\
&\quad + \int_0^t \int_{\Gamma} u(1 + \widehat{\chi}) H_{\varepsilon}(u) \\
&\leq \int_{\Omega} \frac{p_0}{1 + \chi_0} + \int_0^t \int_{\Omega \cap \{0 \leq u \leq \varepsilon\}} \varepsilon^{-1} (1 + \widehat{\chi})^{-1} \nabla(u(1 + \widehat{\chi})) u \nabla \widehat{\chi} \\
&\leq c + \frac{1}{2\varepsilon} \int_0^t \int_{\Omega \cap \{0 \leq u \leq \varepsilon\}} (1 + \widehat{\chi})^{-1} |\nabla(u(1 + \widehat{\chi}))|^2 \\
&\quad + c \int_0^t \int_{\Omega \cap \{0 \leq u \leq \varepsilon\}} (1 + \widehat{\chi})^{-1} |\nabla \widehat{\chi}|^2 \\
&\leq c(R) + \frac{1}{2\varepsilon} \int_0^t \int_{\Omega \cap \{0 \leq u \leq \varepsilon\}} (1 + \widehat{\chi})^{-1} |\nabla(u(1 + \widehat{\chi}))|^2. \tag{3.13}
\end{aligned}$$

Note that the last inequalities are justified as we are integrating over $\{0 \leq u \leq \varepsilon\}$ and recalling the definition of \mathcal{X} . Now, passing to the limit as $\varepsilon \searrow 0$, we deduce

$$\int_{\Omega} (u)^+(t) \leq c(R), \tag{3.14}$$

for a.e. t . Here and in what follows by $c(R)$ we denote possibly different positive constants depending on R .

Third estimate. We still deal with (3.9) and formally test by $-u^{-1}$. After integrating over $(0, t)$, we get

$$\begin{aligned}
- \int_0^t \int_{\Omega} u_t u^{-1} &= - \int_0^t \int_{\Omega} \frac{d}{dt} (\log u) \\
&= - \int_{\Omega} \log u(t) + \int_{\Omega} \log u(0) \\
&= \int_{\Omega} (\log u)^-(t) - \int_{\Omega} (\log u)^+(t) + \int_{\Omega} \log u(0), \tag{3.15}
\end{aligned}$$

where $(\log u)^-$ and $(\log u)^+$ denote the negative and positive parts of the function $\log u$, respectively. Observe that well-known properties of the logarithm function and (3.14) yield

$$\|(\log u)^+\|_{L^\infty(0,T;L^1(\Omega))} \leq c(R). \quad (3.16)$$

In addition, (2.14)–(2.15) imply (see (3.6)) $\log u(0) = \log p_0 - \log(1 + \chi_0) \in L^1(\Omega)$. Hence, we have

$$\begin{aligned} & \int_0^t \langle \mathcal{B}u(1 + \widehat{\chi}), -u^{-1} \rangle \\ &= \int_0^t \int_\Omega u^{-2} |\nabla u|^2 (1 + \widehat{\chi}) + \int_0^t \int_\Omega u^{-1} \nabla u \nabla \widehat{\chi} - \int_0^t \int_\Gamma u(1 + \widehat{\chi}) u^{-1} \\ &= \int_0^t \int_\Omega (1 + \widehat{\chi}) |\nabla \log u|^2 + \int_0^t \int_\Omega \nabla \log u \nabla \widehat{\chi} - \int_0^t \int_\Gamma (1 + \widehat{\chi}). \end{aligned} \quad (3.17)$$

Note that the last integral in (3.17) is bounded as $\widehat{\chi} \in \mathcal{X}$. Now, combining (3.15)–(3.17), using (3.2), and applying Young's inequality we have

$$\|(\log u)^-(t)\|_{L^1(\Omega)} + \frac{1}{2} \|\nabla \log u\|_{L^2(0,t;H)}^2 \leq c(1 + \|\widehat{\chi}\|_{L^2(0,t;V)}^2) \leq c(R). \quad (3.18)$$

Remark 3.1. We briefly detail the procedure to make the above estimate rigorous. Let $n \in \mathbb{N}$ and $\alpha_n(\cdot)$ be defined by

$$\begin{aligned} \alpha_n(x) &:= -x^{-1} & \text{if } x \geq \frac{1}{n}, \\ \alpha_n(x) &:= -n & \text{if } x \leq \frac{1}{n}. \end{aligned} \quad (3.19)$$

Then introduce a primitive function $\widehat{\alpha}_n$ defined by

$$\widehat{\alpha}_n(u) = \int_1^u \alpha_n(x) dx, \quad (3.20)$$

so that it results that

$$\begin{aligned} \widehat{\alpha}_n(u) &= -\log u & \text{if } u \geq \frac{1}{n}, \\ \widehat{\alpha}_n(u) &= 1 + \log n - nu & \text{if } u \leq \frac{1}{n}. \end{aligned} \quad (3.21)$$

Hence, we test (3.9) by $\alpha_n(u)$ and integrate over $(0, t)$. Integrating by parts in time we have

$$\begin{aligned} & \int_\Omega \widehat{\alpha}_n(u(t)) + \int_0^t \int_\Omega \nabla(u(1 + \widehat{\chi})) \cdot \nabla \alpha_n(u) \\ &+ \int_0^t \int_\Gamma u(1 + \widehat{\chi}) \alpha_n(u) - \int_\Omega \widehat{\alpha}_n(u_0) = 0. \end{aligned} \quad (3.22)$$

First, we observe that as $u_0 > 0$ a.e. in Ω (see (2.15) and (2.14)) we may deduce

$$\left| \int_{\Omega} \widehat{\alpha}_n(u_0) \right| \leq c. \quad (3.23)$$

Then we have

$$\int_0^t \int_{\Omega} \nabla(u(1 + \widehat{\chi})) \cdot \nabla \alpha_n(u) = \int_0^t \int_{\Omega \cap \{u \geq n^{-1}\}} \nabla(u(1 + \widehat{\chi})) \cdot \nabla(u^{-1}), \quad (3.24)$$

where the integral on the right-hand side of (3.24) can be treated as in (3.17). Hence, we have (see (3.22))

$$\begin{aligned} \int_0^t \int_{\Gamma} u(1 + \widehat{\chi}) \alpha_n(u) &= - \int_0^t \int_{\Omega \cap \{u \geq n^{-1}\}} (1 + \widehat{\chi}) \\ &\quad - \int_0^t \int_{\Omega \cap \{0 \leq u \leq n^{-1}\}} nu(1 + \widehat{\chi}) \\ &\quad - \int_0^t \int_{\Omega \cap \{u \leq 0\}} nu(1 + \widehat{\chi}). \end{aligned} \quad (3.25)$$

The last integral on the right-hand side is non-negative, while it is easy to check that

$$- \int_0^t \int_{\Omega \cap \{0 \leq u \leq n^{-1}\}} nu(1 + \widehat{\chi}) \geq - \int_0^t \int_{\Omega \cap \{0 \leq u \leq n^{-1}\}} (1 + \widehat{\chi}). \quad (3.26)$$

Moving $\int_0^t \int_{\Omega \cap \{u \geq 0\}} (1 + \widehat{\chi})$ on the right-hand side of (3.22) it turns out to be uniformly bounded. We point out that the last integral in (3.25) is non-negative. Eventually, from (3.22), we can obtain

$$\int_{\Omega} \widehat{\alpha}_n(u(t)) + \frac{1}{2} \int_0^t \int_{\Omega \cap \{u \geq n^{-1}\}} |\nabla \log u|^2 \leq c(R). \quad (3.27)$$

Hence, we point out that (3.14) implies that

$$- \int_{\Omega \cap \{u \geq 1\}} \widehat{\alpha}_n(u(t)) \leq c, \quad (3.28)$$

so that this term can be controlled if it is moved to the right-hand side. To pass to the limit as $n \rightarrow +\infty$ in (3.27), rewritten as

$$\int_{\Omega \cap \{u \leq 1\}} \widehat{\alpha}_n(u(t)) + \frac{1}{2} \int_0^t \int_{\Omega \cap \{u \geq n^{-1}\}} |\nabla \log u|^2 \leq c(R), \quad (3.29)$$

we apply the monotone convergence theorem and finally get

$$\int_{\Omega} \widehat{\alpha}(u(t)) + \frac{1}{2} \int_0^t \int_{\Omega \cap \{u \geq 0\}} |\nabla \log u|^2 \leq c(R), \quad (3.30)$$

where $\widehat{\alpha}(u) = (\log u)^-$ if $u > 0$ and $\widehat{\alpha}(u) = +\infty$ otherwise. In particular, (3.30) implies that (3.18) holds and $u > 0$ a.e.

As a consequence of (3.16), (3.18), and exploiting the Poincaré–Wirtinger inequality, we eventually deduce

$$\|\log u\|_{L^\infty(0,T;L^1(\Omega))\cap L^2(0,T;V)} \leq c(R). \quad (3.31)$$

Finally, owing to (3.6) and (3.1), by (3.31), standard arguments show

$$\|\log p\|_{L^\infty(0,T;L^1(\Omega))\cap L^2(0,T;V)} \leq c(R). \quad (3.32)$$

Now, in (2.25) fix $\widehat{\theta}$ instead of θ and let $y = \log p$ with $p = \mathcal{T}_1(\widehat{\chi})$. Then, by (2.17) and (3.32), the right-hand side of the resulting equation is in $L^\infty(Q) + L^2(0, T; V)$. Thus, the theory of evolution equations associated with the maximal monotone operators ensures that there exists a unique corresponding solution (see [8])

$$\chi = \mathcal{T}_2(\widehat{\theta}, p),$$

fulfilling the Cauchy condition specified by (1.20) (see (2.14)). We now proceed by exploiting some formal a priori estimates on χ .

Fourth a priori estimate. Test (2.25) by χ_t and integrate over $(0, t)$. After integrating by parts in time and exploiting the Young inequality, we get (see (3.32) and (2.14), (2.16), (2.17))

$$\begin{aligned} & \|\chi_t\|_{L^2(0,t;H)}^2 + \frac{1}{2} \|\nabla \chi(t)\|_{\mathbb{H}}^2 - \frac{1}{2} \|\nabla \chi_0\|_{\mathbb{H}}^2 \\ & \leq \frac{1}{2} \|\chi_t\|_{L^2(0,t;H)}^2 + c(\|h(\widehat{\theta})\|_{L^2(0,t;H)}^2 + \|y\|_{L^2(0,t;H)}^2). \end{aligned} \quad (3.33)$$

The above estimate is formal. Indeed, we should proceed by introducing the Moreau–Yosida approximation of $\partial I_{[0,1]}$, performing the rigorous estimates, and then passing to the limit. However, for the sake of simplicity, we omit the detail and refer, e.g., to [5] for rigorous computations. In particular, we have exploited the chain rule and the fact that $\xi \in \partial I_{[0,1]}(\chi)$ to (formally) get

$$\int_0^t \int_{\Omega} \xi \chi_t = \int_{\Omega} I_{[0,1]}(\chi(t)) - \int_{\Omega} I_{[0,1]}(\chi_0) = \int_{\Omega} I_{[0,1]}(\chi(t)) \geq 0.$$

Thus, combining (2.17) and (3.32) with (3.33), we finally get

$$\|\chi_t\|_{L^2(0,T;H)} + \|\chi\|_{L^\infty(0,T;V)} \leq c(R). \quad (3.34)$$

Fifth a priori estimate. We proceed by formally testing (2.25) (where $\widehat{\theta}$ and p are fixed) by $\mathcal{A}\chi$ and integrate over $(0, t)$. By monotonicity of the subdifferential operator $\partial I_{[0,1]}$ we have

$$\int_0^t \int_{\Omega} \xi \mathcal{A}\chi \geq 0.$$

Thus, integrating by parts in time and proceeding as in (3.33), applying Young’s inequality (see also (3.34)), we easily deduce

$$\|\mathcal{A}\chi\|_{L^2(0,T;H)} \leq c(R), \quad (3.35)$$

which leads to (see (3.34))

$$\|\chi\|_{L^2(0,T;W)} \leq c(R). \quad (3.36)$$

By a comparison in (2.25), (3.34) and (3.36) yield

$$x \|\xi\|_{L^2(0,T;H)} \leq c(R). \quad (3.37)$$

Finally, we deal with (2.24) where $\chi = \mathcal{T}_2(\widehat{\theta}, p)$ is fixed (with $p = \mathcal{T}_1(\widehat{\chi})$) and $\widehat{\theta}$ is considered in the nonlinear coefficients of θ_t and χ_t , i.e., we consider the equation

$$(1 + \widehat{\theta}h''(\widehat{\theta})\chi)\theta_t + \mathcal{A}\theta = -\widehat{\theta}h'(\widehat{\theta})\chi_t. \quad (3.38)$$

Then, owing to (2.18), by the theory of evolution parabolic equations (see, e.g., [2]) there exists a unique

$$\theta = \mathcal{T}_3(\chi, \widehat{\theta}),$$

solving (3.38) with the associated Cauchy condition expressed by (1.20).

Sixth a priori estimate. Test (3.38) by θ_t and integrate over $(0, t)$. Owing to (2.18) and the fact that $\chi \in [0, 1]$, we get

$$c_s \|\theta_t\|_{L^2(0,t;H)}^2 \leq \int_0^t \int_{\Omega} (1 + h''(\widehat{\theta})\widehat{\theta}\chi)\theta_t^2. \quad (3.39)$$

Hence, (2.17) implies

$$\left| \int_0^t \int_{\Omega} h'(\widehat{\theta})\widehat{\theta}\chi_t\theta_t \right| \leq c_h \|\chi_t\|_{L^2(0,t;H)} \|\theta_t\|_{L^2(0,t;H)}. \quad (3.40)$$

Thus, applying Young's inequality and integrating by parts in time (see (2.13)), by (3.34) it follows that

$$\frac{c_s}{2} \|\theta_t\|_{L^2(0,t;H)}^2 + \frac{1}{2} \|\nabla\theta(t)\|_H^2 \leq c(R), \quad (3.41)$$

and consequently

$$\|\theta\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} \leq c(R). \quad (3.42)$$

Finally, by a comparison in the equation and due to (2.17), (3.34), and (3.42) it is a standard matter to infer that

$$\|\theta\|_{L^2(0,T;W)} \leq c(R). \quad (3.43)$$

Now, (3.34) and (3.42) lead to

$$\|\chi\|_{L^2(0,t;V)} + \|\theta\|_{L^2(0,t;H)} \leq t^{1/2} (\|\chi\|_{L^\infty(0,T;V)} + \|\theta\|_{L^\infty(0,T;H)}) \leq t^{1/2} \widetilde{c}(R),$$

where $\widetilde{c}(R)$ does not depend on $t \in (0, T)$. Thus, letting \widehat{t} be sufficiently small, we can infer that

$$\|\theta\|_{L^2(0,\widehat{t};H)} + \|\chi\|_{L^2(0,\widehat{t};V)} \leq R, \quad (3.44)$$

so that the operator

$$\begin{aligned} \mathcal{T}(\widehat{\chi}, \widehat{\theta}) &:= (\chi, \theta), \\ \text{for } \chi &= \mathcal{T}_2(\widehat{\theta}, p), \quad p = \mathcal{T}_1(\widehat{\chi}), \quad \text{and } \theta = \mathcal{T}_3(\chi, \widehat{\theta}), \end{aligned}$$

is well defined from \mathcal{X} into itself, at least in the interval $(0, \widehat{t})$. However, as the estimates we have performed on the solutions do not depend on \widehat{t} , the argument can be extended to the whole time interval. Thus, for the sake of simplicity, we directly refer to the interval $(0, T)$.

For the sake of clarity, we summarize the estimates we have previously obtained:

$$\|\theta\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} \leq c, \quad (3.45)$$

$$\|\chi\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \cap L^\infty(Q)} \leq c, \quad (3.46)$$

$$\|\xi\|_{L^2(0,T;H)} \leq c, \quad (3.47)$$

$$\|\log p\|_{L^\infty(0,T;L^1(\Omega)) \cap L^2(0,T;V)} \leq c, \quad (3.48)$$

$$\|p\|_{L^2(0,T;H)} + \|1 * p\|_{L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))} \leq c. \quad (3.49)$$

Remark 3.2. Observe that as $\log p$ is bounded in $L^2(0, T; H) \cap L^2(0, T; V)$ it follows that $p > 0$ a.e. in Q . Thus, by (3.14) and the fact that $\chi \in [0, 1]$ a.e., it follows in addition that

$$\|p\|_{L^\infty(0,T;L^1(\Omega))} \leq c. \quad (3.50)$$

In particular, (3.45) and (3.46) imply that \mathcal{T} turns out to be a compact operator in \mathcal{X} endowed with the natural norm induced by $L^2(0, T; V) \times L^2(0, T; H)$. Thus, in order to achieve the Schauder theorem we need to prove that \mathcal{T} is continuous from \mathcal{X} into itself. To this aim, we consider a sequence $(\widehat{\chi}_n, \widehat{\theta}_n)$ strongly converging in \mathcal{X} to some $(\widehat{\chi}, \widehat{\theta})$, i.e.,

$$\widehat{\chi}_n \rightarrow \widehat{\chi} \quad \text{in } L^2(0, T; V), \quad (3.51)$$

$$\widehat{\theta}_n \rightarrow \widehat{\theta} \quad \text{in } L^2(0, T; H). \quad (3.52)$$

Then denote by $p_n = \mathcal{T}_1(\widehat{\chi}_n)$, $\chi_n = \mathcal{T}_2(\widehat{\theta}_n, p_n)$, $y_n = \log p_n$, $\xi_n \in \partial I_{[0,1]}(\chi_n)$, and $\theta_n = \mathcal{T}_3(\chi_n, \widehat{\theta}_n)$ the corresponding solutions to (2.24)–(2.28), where $\widehat{\chi}_n$ and $\widehat{\theta}_n$ are fixed as in the above argument. Our aim is to pass to the limit as $n \rightarrow +\infty$ and eventually obtain

$$\chi_n \rightarrow \chi \quad \text{in } L^2(0, T; V), \quad (3.53)$$

$$\theta_n \rightarrow \theta \quad \text{in } L^2(0, T; H), \quad (3.54)$$

with

$$\chi = \mathcal{T}_2(\widehat{\theta}, p), \quad \theta = \mathcal{T}_3(\chi, \widehat{\theta}), \quad p = \mathcal{T}_1(\widehat{\chi}). \quad (3.55)$$

We first observe that (3.45)–(3.50) hold for the above sequences with constants c chosen independently of n . Thus, (3.45)–(3.50) written for $(\theta_n, \chi_n, p_n, \xi_n, y_n)$ and well-known weak and weak star convergence results imply that, at least for some suitable subsequences, there hold

$$\theta_n \overset{*}{\rightharpoonup} \theta \quad \text{in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \quad (3.56)$$

$$\chi_n \overset{*}{\rightharpoonup} \chi \quad \text{in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \cap L^\infty(Q), \quad (3.57)$$

$$\xi_n \rightharpoonup \xi \quad \text{in } L^2(0, T; H), \quad (3.58)$$

$$y_n \rightharpoonup y \quad \text{in } L^2(0, T; V), \quad (3.59)$$

$$p_n \overset{*}{\rightharpoonup} p \quad \text{in } L^2(0, T; H), \quad (3.60)$$

$$1 * p_n \overset{*}{\rightharpoonup} 1 * p \quad \text{in } L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)). \quad (3.61)$$

Hence, we aim to show that (3.53)–(3.54) and (3.55) hold for θ and χ introduced by (3.56) and (3.57). Strong compactness results give, at least for some subsequences,

$$\theta_n \rightarrow \theta \quad \text{in } L^\infty(0, T; H) \cap L^2(0, T; V), \quad (3.62)$$

$$\chi_n \rightarrow \chi \quad \text{in } L^\infty(0, T; H) \cap L^2(0, T; V), \quad (3.63)$$

$$1 * p_n \rightarrow 1 * p \quad \text{in } L^\infty(0, T; H) \cap L^2(0, T; V). \quad (3.64)$$

Owing to (3.56)–(3.64) we can pass to the limit in (2.24)–(2.28), now written for the index n . We first consider the limit of (2.24). Dealing with nonlinear coefficients

$$\varphi_{n,1} = h''(\widehat{\theta}_n)\widehat{\theta}_n\chi_n, \quad \varphi_{n,2} = h'(\widehat{\theta}_n)\widehat{\theta}_n,$$

we observe that due to (2.16), (3.52), and (3.63), $\varphi_{n,1}$ and $\varphi_{n,2}$ converge almost everywhere to $h''(\widehat{\theta})\widehat{\theta}\chi$ and $h'(\widehat{\theta})\widehat{\theta}$, respectively. Moreover, they are uniformly bounded (see (2.17) and (3.46)). Eventually, by the Lebesgue theorem, we have

$$h''(\widehat{\theta}_n)\widehat{\theta}_n\chi_n \rightarrow h''(\widehat{\theta})\widehat{\theta}\chi \quad \text{and} \quad h'(\widehat{\theta}_n)\widehat{\theta}_n \rightarrow h'(\widehat{\theta})\widehat{\theta}, \quad (3.65)$$

in $L^q(Q)$ for any $q < +\infty$. Now, owing to (3.65) and (3.56)–(3.57) we can pass to the limit weakly in H in the equation written for n . We briefly detail the passage to the limit on the right-hand side. By (2.17) and (3.46) we get that $h'(\widehat{\theta}_n)\widehat{\theta}_n\chi_{nt}$ is bounded in $L^2(0, T; H)$. Then, by (3.65) and (3.57), we may identify the limit as

$$\widehat{\theta}_n h'(\widehat{\theta}_n)\chi_{nt} \rightharpoonup \widehat{\theta} h'(\widehat{\theta})\chi_t \quad \text{in } L^2(0, T; H).$$

We analogously proceed to conclude that $h''(\widehat{\theta}_n)\widehat{\theta}_n\chi_n\theta_{nt}$ converges weakly in $L^2(0, T; H)$ to $h''(\widehat{\theta})\widehat{\theta}\chi\theta_t$ (see (3.65) and (3.56)). Thus, by uniqueness of the solution to the limit equation, once $\widehat{\theta}$ and χ are fixed, we can identify $\theta = \mathcal{T}_3(\chi, \widehat{\theta})$, so that (3.54) follows by (3.62) (see also (3.55)). In addition, we may infer that the above convergences actually hold for the whole sequence.

Now, we consider (2.27) written for n , with $\widehat{\chi}_n$ fixed, and pass to the limit. Arguing as above, it is now a standard matter to prove that $(1 + \widehat{\chi}_n)^{-1}$ converges to $(1 + \widehat{\chi})^{-1}$ strongly in $L^q(Q)$ for any $q < +\infty$ (see (3.51)). Then, as $p_n(1 + \widehat{\chi}_n)^{-1}$ is bounded in

$L^2(0, T; H)$ (see (3.49)) and due to (3.60), we may identify its weak limit as

$$\frac{p_n}{1 + \widehat{\chi}_n} \rightharpoonup \frac{p}{1 + \widehat{\chi}} \quad \text{in } L^2(0, T; H).$$

Finally, (3.61) allows us to pass to the limit in the convolution product. We observe that, as in the previous case, by uniqueness of the solution for the limit equation, once $\widehat{\chi}$ is fixed, we eventually entail $p = \mathcal{T}_1(\widehat{\chi})$ (see (3.55)) and deduce that the above convergences actually hold for the whole sequence.

By (3.52), (2.16), and (3.57)–(3.59) we can easily pass to the limit in (2.25). It remains to verify that (2.26) and (2.28) hold. Exploiting the monotonicity arguments, we can identify $\xi \in \partial I_{[0,1]}(\chi)$ by simply combining (3.58) with the strong convergence (3.63). Concerning the limit of $y_n = \log p_n$, we recall that

$$\log u_n = y_n - \log(1 + \widehat{\chi}_n) =: \zeta_n.$$

By (3.51) and the strong convergence of $(1 + \widehat{\chi}_n)^{-1}$ to $(1 + \widehat{\chi})^{-1}$ in $L^q(Q)$, for any $q < +\infty$, we may infer that

$$\log(1 + \widehat{\chi}_n) \rightharpoonup \log(1 + \widehat{\chi}) \quad \text{in } L^2(0, T; V). \quad (3.66)$$

Then (3.66) and (3.59) lead to

$$\zeta_n \rightharpoonup \zeta := y - \log(1 + \widehat{\chi}) \quad \text{in } L^2(0, T; V). \quad (3.67)$$

Now we proceed using semicontinuity arguments, as $\log(\cdot)$ is a monotone graph. We aim to show that

$$\limsup_{n \rightarrow +\infty} \int_0^t \int_{\Omega} u_n \zeta_n \leq \int_0^t \int_{\Omega} u \zeta,$$

to identify $\zeta = \log u$ (see [4]), from which it would follow that $y = \log p$, as (see (3.67))

$$y - \log(1 + \widehat{\chi}) = \zeta = \log u = \log p - \log(1 + \widehat{\chi}).$$

Using (3.7) and exploiting (3.64) (from which we recover a strong convergence for $1 * \mathcal{B}p_n$ in $L^2(0, T; V')$) and (3.67), we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_0^t \int_{\Omega} u_n \zeta_n &= \lim_{n \rightarrow +\infty} \int_0^t \int_{\Omega} \frac{p_0}{1 + \chi_0} \zeta_n - \int_0^t \langle 1 * \mathcal{B}p_n, \zeta_n \rangle \\ &= \int_0^t \int_{\Omega} \frac{p_0}{1 + \chi_0} \zeta - \int_0^t \langle 1 * \mathcal{B}p, \zeta \rangle = \int_0^t \int_{\Omega} u \zeta. \end{aligned} \quad (3.68)$$

Thus, the above arguments imply that χ can be identified with $\mathcal{T}_2(\widehat{\theta}, p)$. Then (3.53) and (3.55) follow from (3.63), which concludes the proof of the Theorem 2.1.

4. Proof of Theorem 2.2

In this section we present some further regularity results on the solutions to our problem, which hold in the two-dimensional case. Thus, to prove Theorem 2.2 we perform some

further a priori estimates on the solutions of (2.24)–(2.28). Before proceeding, we recall the following Gagliardo–Nirenberg inequalities (surely holding in the two-dimensional case, see [17]):

$$\|v\|_{L^4(\Omega)}^2 \leq c \|v\|_{\mathbb{H}} \|\nabla v\|_{\mathbb{H}}, \quad \forall v \in V, \quad (4.1)$$

$$\|\nabla v\|_{L^4(\Omega)}^2 \leq c \|v\|_{H^2(\Omega)} \|v\|_{L^\infty(\Omega)}, \quad \forall v \in H^2(\Omega). \quad (4.2)$$

Seventh a priori estimate. Let us deal with (3.9) and test it by u . After integrating over $(0, t)$ and using the Hölder inequality, we get (see (3.2), now holding for χ)

$$\begin{aligned} & \frac{1}{2} \|u(t)\|_{\mathbb{H}}^2 + \|\nabla u\|_{L^2(0,t;H)}^2 + \|u\|_{L^2(0,T;L^2(\Gamma))}^2 \\ & \leq \frac{1}{2} \|u(0)\|_{\mathbb{H}}^2 + \int_0^t \int_{\Omega} (1 + \chi) |\nabla u|^2 + \int_0^t \int_{\Gamma} u^2 (1 + \chi) \\ & \leq \frac{1}{2} \|u(0)\|_{\mathbb{H}}^2 + \int_0^t \int_{\Omega} |u| |\nabla \chi| |\nabla u| \\ & \leq \frac{1}{2} \|u(0)\|_{\mathbb{H}}^2 + \int_0^t \|u\|_{L^4(\Omega)} \|\nabla \chi\|_{L^4(\Omega)} \|\nabla u\|_{\mathbb{H}}. \end{aligned} \quad (4.3)$$

To handle the right-hand side of (4.3), we use the Young inequality and the Gagliardo–Nirenberg inequalities (4.1)–(4.2). We have

$$\begin{aligned} & \int_0^t \|u\|_{L^4(\Omega)} \|\nabla \chi\|_{L^4(\Omega)} \|\nabla u\|_{\mathbb{H}} \\ & \leq \frac{1}{4} \|\nabla u\|_{L^2(0,t;H)}^2 + c \int_0^t \|u\|_{L^4(\Omega)}^2 \|\nabla \chi\|_{L^4(\Omega)}^2 \\ & \leq \frac{1}{4} \|\nabla u\|_{L^2(0,t;H)}^2 + c \int_0^t \|u\|_{\mathbb{H}} \|\nabla u\|_{\mathbb{H}} \|\chi\|_W \|\chi\|_{L^\infty(\Omega)} \\ & \leq \frac{1}{2} \|\nabla u\|_{L^2(0,t;H)}^2 + c \int_0^t \|\chi\|_W^2 \|u\|_{\mathbb{H}}^2, \end{aligned} \quad (4.4)$$

where the last constant c depends in particular on $\|\chi\|_{L^\infty(\Omega)} \leq 1$. Hence, after observing that $\|\chi\|_W^2$ is bounded in $L^1(0, T)$ (see (3.46)), we combine (4.4) with (4.3) and make use of the Gronwall lemma to conclude (see [8])

$$\|u\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leq c. \quad (4.5)$$

Now we discuss the regularity of $p = u(1 + \chi)$, which can be deduce by use of (4.5) and (3.46). Owing to (4.1)–(4.2) and the Hölder inequality we have

$$\begin{aligned} & \int_0^t \|\nabla(u(1 + \chi))\|_{\mathbb{H}}^2 \\ & \leq \int_0^t \|\nabla u\|_{\mathbb{H}}^2 \|1 + \chi\|_{L^\infty(\Omega)}^2 + \int_0^t \|u\|_{L^4(\Omega)}^2 \|\nabla \chi\|_{L^4(\Omega)}^2 \end{aligned}$$

$$\begin{aligned}
&\leq c \|\nabla u\|_{L^2(0,t;H)}^2 + c \|\chi\|_{L^\infty(Q)} \int_0^t \|u\|_H \|\nabla u\|_H \|\chi\|_W \\
&\leq c(\|\nabla u\|_{L^2(0,T;H)}^2 + \|u\|_{L^\infty(0,T;H)} \|\nabla u\|_{L^2(0,T;H)} \|\chi\|_{L^2(0,T;W)}) \leq c. \quad (4.6)
\end{aligned}$$

Thus, we can easily infer that

$$\|p\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leq c. \quad (4.7)$$

We point out that by (4.7) we can now solve (2.7) in the duality between V' and V (see Remark 3.2), after obtaining by comparison in (3.9) (see (3.6) and (4.7))

$$\|u_t\|_{L^2(0,T;V')} = \left\| \left(\frac{p}{1+\chi} \right)_t \right\|_{L^2(0,T;V')} \leq c, \quad (4.8)$$

which concludes the first part of the Theorem 2.2.

Now, we proceed to prove (2.34)–(2.35) by assuming (2.31) and (2.32) as further regularities on the initial data.

Eighth a priori estimate. Test (3.9) by $-u^{-3}$ and integrate over $(0, t)$ (this is a formal estimate, for a rigorous justification proceed as in Remark 3.1). Observe that by (2.31) and (2.14) we deduce $u^{-1}(0) \in H$. We first have

$$-\int_0^t \int_\Omega u_t u^{-3} = \frac{1}{2} \int_0^t \int_\Omega \frac{d}{dt} (u^{-2}) = \frac{1}{2} \|u^{-1}(t)\|_H^2 - \frac{1}{2} \|u^{-1}(0)\|_H^2. \quad (4.9)$$

Then, by definition of \mathcal{B} , we write

$$\begin{aligned}
&\int_0^t \langle \mathcal{B}(u(1+\chi)), u^{-3} \rangle \\
&= 3 \int_0^t \int_\Omega (1+\chi) |\nabla u|^2 u^{-4} - \int_0^t \int_\Gamma u^{-2} (1+\chi) \\
&\quad + 3 \int_0^t \int_\Omega u^{-3} \nabla u \nabla \chi. \quad (4.10)
\end{aligned}$$

Then the third integral on the right-hand side of (4.10) is estimated as follows (see (4.1)–(4.2))

$$\begin{aligned}
&3 \int_0^t \int_\Omega |u^{-3} \nabla u \nabla \chi| \\
&\leq c \int_0^t \|\nabla \chi\|_{L^4(\Omega)} \|\nabla(u^{-1})\|_H \|u^{-1}\|_{L^4(\Omega)} \\
&\leq \frac{3}{4} \|\nabla(u^{-1})\|_{L^2(0,t;H)}^2 + c \int_0^t \|\chi\|_W \|u^{-1}\|_H \|\nabla(u^{-1})\|_H \\
&\leq \frac{3}{2} \|\nabla(u^{-1})\|_{L^2(0,t;H)}^2 + c \int_0^t \|\chi\|_W^2 \|u^{-1}\|_H^2. \quad (4.11)
\end{aligned}$$

As the trace operator is compact $V \rightarrow L^2(\Gamma)$, we may deduce that, for any $\sigma > 0$, there exists $C_\sigma > 0$ such that

$$\|v|_\Gamma\|_{L^2(\Gamma)}^2 \leq \sigma \|v\|_V^2 + C_\sigma \|v\|_H^2, \quad v \in V.$$

Thus, we can control the boundary integral in (4.10) as follows:

$$\begin{aligned} \left| \int_0^t \int_\Gamma u^{-2}(1 + \chi) \right| &\leq 2 \int_0^t \|u^{-1}\|_{L^2(\Gamma)}^2 \\ &\leq \sigma \|\nabla u^{-1}\|_{L^2(0,t;H)}^2 + C_\sigma \|u^{-1}\|_{L^2(0,t;H)}^2. \end{aligned} \quad (4.12)$$

Combining (4.9)–(4.12), for a sufficiently small σ , we have

$$\|u^{-1}(t)\|_H^2 + \|\nabla(u^{-1})\|_{L^2(0,t;H)}^2 \leq c \left(1 + \int_0^t (1 + \|\chi\|_W^2) \|u^{-1}\|_H^2 \right). \quad (4.13)$$

Due to (3.46) we have that $\|\chi\|_W^2$ is bounded in $L^1(0, T)$. Thus, the Gronwall lemma applied to (4.13) yields

$$\|u^{-1}\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leq c. \quad (4.14)$$

Ninth a priori estimate. We explicitly write (2.7) as follows:

$$\frac{p_t}{1 + \chi} - \frac{p\chi_t}{(1 + \chi)^2} + \mathcal{B}p = 0, \quad (4.15)$$

and formally test by p_t . After integrating over $(0, t)$, we can write, by Sobolev's embeddings,

$$\begin{aligned} &\frac{1}{2} \|p_t\|_{L^2(0,t;H)}^2 + \frac{1}{2} \|\nabla p(t)\|_H^2 + \frac{1}{2} \|p(t)\|_{L^2(\Gamma)}^2 \\ &\leq c \|p_0\|_V^2 + c \int_0^t \|p\|_{L^4(\Omega)} \|\chi_t\|_{L^4(\Omega)} \|p_t\|_H \\ &\leq c + \frac{1}{8} \|p_t\|_{L^2(0,t;H)}^2 + c \int_0^t \|p\|_H \|\nabla p\|_H \|\chi_t\|_H \|\nabla \chi_t\|_H \\ &\leq c + \frac{1}{8} \|p_t\|_{L^2(0,t;H)}^2 + \frac{1}{4} \|\nabla \chi_t\|_{L^2(0,t;H)}^2 + c \int_0^t \|p\|_H^2 \|\nabla p\|_H^2 \|\chi_t\|_H^2, \end{aligned} \quad (4.16)$$

where $\|p\|_H^2 \|\nabla p\|_H^2$ is bounded in $L^1(0, T)$ since (4.7) holds.

Tenth a priori estimate. Let us differentiate (2.25) with respect to time, and then test it by χ_t . Note that by a comparison, (2.31) and (2.32) imply $\chi_t(0) \in H$. After integrating over $(0, t)$ and exploiting the monotonicity of the subdifferential operator, we get

$$\begin{aligned} &\frac{1}{2} \|\chi_t(t)\|_H^2 + \|\nabla \chi_t\|_{L^2(0,t;H)}^2 \\ &\leq \frac{1}{2} \|\chi_t(0)\|_H^2 + c_h \int_0^t \|\theta_t\|_H \|\chi_t\|_H + \int_0^t \|p_t\|_H \|p^{-1}\|_{L^4(\Omega)} \|\chi_t\|_{L^4(\Omega)} \end{aligned}$$

$$\begin{aligned}
&\leq c(1 + \|\theta_t\|_{L^2(0,t;H)}^2 + \|\chi_t\|_{L^2(0,t;H)}^2) + \frac{1}{8} \|p_t\|_{L^2(0,t;H)}^2 \\
&\quad + c \int_0^t \|u^{-1}\|_{L^4(\Omega)}^2 \|(1 + \chi)^{-1}\|_{L^\infty(\Omega)}^2 \|\chi_t\|_H \|\nabla \chi_t\|_H \\
&\leq c + \frac{1}{8} \|p_t\|_{L^2(0,t;H)}^2 + \frac{1}{2} \|\nabla \chi_t\|_{L^2(0,t;H)}^2 \\
&\quad + c \int_0^t \|u^{-1}\|_H^2 \|\nabla u^{-1}\|_H^2 \|\chi_t\|_H^2, \tag{4.17}
\end{aligned}$$

where by (4.14) we have $\|u^{-1}\|_H^2 \|\nabla u^{-1}\|_H^2 \in L^1(0, T)$. Thus, combining (4.17) with (4.16), the Gronwall lemma ensures that

$$\|p\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} + \|\chi_t\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leq c, \tag{4.18}$$

which concludes the proof of Theorem 2.2.

5. Proof of Theorem 2.3

In this section we prove the uniqueness result stated by Theorem 2.3 which holds for sufficiently regular solutions (see (A1) and (A2)). In particular, by (A1), we get the uniqueness of the solution for the two-dimensional problem, whose existence is established by Theorem 2.2, at least once (2.31)–(2.32) are assumed. For the sake of simplicity, when this is possible, we prefer to treat by the same estimates the two different situations, corresponding to the sets of assumptions (A1) or (A2). We detail some estimates which need to be performed using different tools depending on whether (A1) or (A2) is assumed.

We first introduce

$$\begin{aligned}
[\theta] &= \theta_1 - \theta_2, & [\chi] &= \chi_1 - \chi_2, & [\xi] &= \xi_1 - \xi_2 \\
[u] &= u_1 - u_2, & [p] &= p_1 - p_2,
\end{aligned} \tag{5.1}$$

where $(\theta_i, \chi_i, \xi_i, p_i, y_i)$ are solutions of our problem fulfilling the hypotheses of Theorem 2.3. Then we make use of the two trivial identities

$$a_1 b_1 - a_2 b_2 = [ab] = [a]b_2 + a_1[b] = [a]b_1 + a_2[b], \tag{5.2}$$

so that, without loss of generality, in what follows, we rewrite (5.2) and subsequent computations omitting the subscripts, i.e.,

$$[ab] = a[b] + [a]b. \tag{5.3}$$

Then, for $N \leq 3$, we recall the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and the Gagliardo–Nirenberg inequality, holding, at least, for $N = 3$,

$$\|v\|_{L^3(\Omega)}^2 \leq c \|v\|_H \|\nabla v\|_H. \tag{5.4}$$

We first consider (2.7) written for (p_1, χ_1) and (p_2, χ_2) , take the difference, and integrate with respect to time. Then test the resulting equation by $[p]$ and integrate once more over $(0, t)$. We get

$$\begin{aligned} & \frac{1}{2} \|[p]\|_{L^2(0,t;H)}^2 + \frac{1}{2} \|1 * \nabla [p](t)\|_H^2 + \frac{1}{2} \|1 * [p](t)\|_{L^2(\Gamma)}^2 \\ & \leq - \int_0^t \int_{\Omega} p \left[\frac{1}{1 + \chi} \right] [p]. \end{aligned} \quad (5.5)$$

We recall that by (A1) (or (A2)), the norm $\|p\|_{L^\infty(0,T;V)}$ is bounded by the data of the problem. Thus, due to (5.4) and using the mean value theorem, we handle the right-hand side of (5.5) as follows:

$$\begin{aligned} & \left| \int_0^t \int_{\Omega} p \left[\frac{1}{1 + \chi} \right] [p] \right| \\ & \leq c \int_0^t \|p\|_{L^6(\Omega)} \|\chi\|_{L^3(\Omega)} \|[p]\|_H \left\| \frac{1}{(1 + \chi)^2} \right\|_{L^\infty(\Omega)} \\ & \leq \frac{1}{4} \|[p]\|_{L^2(0,t;H)}^2 + c \int_0^t \|p\|_V^2 \|\chi\|_H \|\nabla \chi\|_H \\ & \leq \frac{1}{4} \|[p]\|_{L^2(0,t;H)}^2 + \frac{1}{4} \|\nabla \chi\|_{L^2(0,t;H)}^2 + c \int_0^t \|\chi\|_H^2. \end{aligned} \quad (5.6)$$

Now we take the difference of (2.25) written for $(\theta_1, \chi_1, \xi_1, y_1)$ and $(\theta_2, \chi_2, \xi_2, y_2)$, test it by $[\chi]$, and integrate over $(0, t)$. We first observe that, by the monotonicity of the subdifferential operator, we have

$$\int_0^t \int_{\Omega} [\xi][\chi] \geq 0. \quad (5.7)$$

Thus, it results that

$$\begin{aligned} & \frac{1}{2} \|\chi(t)\|_H^2 + \|\nabla \chi\|_{L^2(0,t;H)}^2 \\ & \leq \int_0^t \int_{\Omega} |[h(\theta)]| |\chi| + \int_0^t \int_{\Omega} |[\log p]| |\chi|. \end{aligned} \quad (5.8)$$

We handle the first integral on the right-hand side of (5.8) by using the Young inequality and exploiting (2.16)–(2.17),

$$\int_0^t \int_{\Omega} |[h(\theta)]| |\chi| \leq \sigma \|\theta\|_{L^2(0,t;H)}^2 + c \|\chi\|_{L^2(0,t;H)}^2, \quad (5.9)$$

and the possible different constants σ will be fixed later. Hence, we treat the second integral using different tools for $N \leq 2$ or $N = 3$, i.e., in the case when we assume (A1) or (A2). We first consider the two-dimensional case, i.e., we let (A1) hold. Applying the Gagliardo–Nirenberg inequalities, in \mathbf{R}^2 , and exploiting the mean value theorem

yields

$$\begin{aligned}
& \int_0^t \int_{\Omega} |[\log p]| |[\chi]| \\
& \leq c \int_0^t \|p^{-1}\|_{L^4} \| [p] \|_{\mathbb{H}} \| [\chi] \|_{L^4} \\
& \leq \sigma \| [p] \|_{L^2(0,t;H)}^2 + c \int_0^t \|u^{-1}\|_{L^4(\Omega)}^2 \| [\chi] \|_{\mathbb{H}} \| \nabla [\chi] \|_{\mathbb{H}} \\
& \leq \sigma \| [p] \|_{L^2(0,t;H)}^2 + \sigma \| \nabla [\chi] \|_{L^2(0,t;H)}^2 \\
& \quad + c \int_0^t \int_{\Omega} \|u^{-1}\|_{\mathbb{H}}^2 \| \nabla u^{-1} \|_{\mathbb{H}}^2 \| [\chi] \|_{\mathbb{H}}^2, \tag{5.10}
\end{aligned}$$

where $\|u^{-1}\|_{\mathbb{H}}^2 \| \nabla u^{-1} \|_{\mathbb{H}}^2$ is bounded in $L^1(0, T)$. In the case $N = 3$, we have to assume some further regularity on the solution, i.e., let (A2) (and in particular (2.36)) hold. Then, exploiting (5.4), we have

$$\begin{aligned}
& \int_0^t \int_{\Omega} |[\log p]| |[\chi]| \leq \int_0^t \|p^{-1}\|_{L^6} \| [p] \|_{\mathbb{H}} \| [\chi] \|_{L^3} \\
& \leq \sigma \| [p] \|_{L^2(0,t;H)}^2 + c \int_0^t \|p^{-1}\|_{L^6(\Omega)}^2 \| [\chi] \|_{\mathbb{H}} \| \nabla [\chi] \|_{\mathbb{H}} \\
& \leq \sigma \| [p] \|_{L^2(0,t;H)}^2 + \sigma \| \nabla [\chi] \|_{L^2(0,t;H)}^2 \\
& \quad + c \int_0^t \int_{\Omega} \|p^{-1}\|_{L^6(\Omega)}^4 \| [\chi] \|_{\mathbb{H}}^2, \tag{5.11}
\end{aligned}$$

and now (2.36) yields $\|p^{-1}\|_{L^6(\Omega)}^4$ bounded in $L^1(0, T)$. Finally, we have to deal with the difference of θ 's. We consider (2.24), rewritten for the sake of simplicity as follows:

$$(\theta + \chi(\theta h'(\theta) - h(\theta)))_t + \mathcal{A}\theta + h(\theta)\chi_t = 0. \tag{5.12}$$

Hence, we take the difference of (5.12), written for the two solutions (θ_1, χ_1) and (θ_2, χ_2) , and integrate with respect to time:

$$[\theta] + [\chi(\theta h'(\theta) - h(\theta))] + 1 * \mathcal{A}[\theta] + 1 * [h(\theta)\chi_t] = 0.$$

After some integration by parts in time and by the use of (5.3), we finally get

$$\begin{aligned}
& [\theta] + [(\theta h'(\theta) - h(\theta))\chi] + 1 * \mathcal{A}[\theta] \\
& = -(\theta h'(\theta) - h(\theta))[\chi] - h(\theta)[\chi] + 1 * h'(\theta)\theta_t[\chi] - 1 * [h(\theta)]\chi_t \\
& = -\theta h'(\theta)[\chi] + 1 * h'(\theta)\theta_t[\chi] - 1 * [h(\theta)]\chi_t. \tag{5.13}
\end{aligned}$$

Then we test (5.13) by $[\theta]$ and integrate over $(0, t)$. Note that, in our notation, we can write

$$(\theta h'(\theta) - h(\theta))' = \theta h''(\theta) \quad \text{and} \quad [(\theta h'(\theta) - h(\theta))\chi] = \theta h''(\theta)[\theta]\chi.$$

Thus, we can infer, after integrating by parts in time,

$$c_s \|\theta\|_{L^2(0,t;H)}^2 + \frac{1}{2} \|\nabla[\theta](t)\|_H^2 \leq \sum_{j=1}^3 I_j(t), \quad (5.14)$$

where the integrals I_j will be handled in a moment. We first consider (see (2.17))

$$\begin{aligned} |I_1(t)| &\leq \int_0^t \int_{\Omega} |\theta h'(\theta)[\chi][\theta]| \\ &\leq c \int_0^t \|\chi\|_H \|\theta\|_H \leq \sigma \|\theta\|_{L^2(0,t;H)}^2 + c_{\sigma} \|\chi\|_{L^2(0,t;H)}^2, \end{aligned} \quad (5.15)$$

where σ will be fixed later. Then we estimate the second integral as follows:

$$\begin{aligned} |I_2(t)| &\leq \int_{\Omega} |1 * h'(\theta)\theta_t[\chi]| |1 * [\theta]| + \int_0^t \int_{\Omega} |h'(\theta)\theta_t[\chi]| |1 * [\theta]| \\ &\leq \|1 * [\theta]\|_{L^4(\Omega)} \int_0^t \|h'(\theta)\theta_t\|_H \|\chi\|_{L^4(\Omega)} \\ &\quad + \int_0^t \|1 * [\theta]\|_{L^4(\Omega)} \|h'(\theta)\theta_t\|_H \|\chi\|_{L^4(\Omega)} \\ &\leq \sigma \|1 * [\theta](t)\|_V^2 + c \left(\int_0^t \|\theta_t\|_H^2 \right) \|\chi\|_{L^2(0,t;V)}^2. \end{aligned} \quad (5.16)$$

Finally, we have

$$\begin{aligned} |I_3(t)| &\leq \int_{\Omega} |1 * [\theta]| |1 * [h(\theta)]\chi_t| + \int_0^t \int_{\Omega} |1 * [\theta]| |[h(\theta)]\chi_t| \\ &\leq \sigma \|1 * [\theta]\|_{L^\infty(0,t;V)}^2 + c \left(\int_0^t \|\chi_t\|_V^2 \right) \|\theta\|_{L^2(0,t;H)}^2. \end{aligned} \quad (5.17)$$

Now, we combine (5.5), (5.6) and (5.8)–(5.10), with sufficient small σ 's. After observing that $\|\theta_t\|_H^2$ and $\|\chi_t\|_V^2$ belong to $L^1(0, T)$ (see (2.19) and (2.35)), we can find \widehat{t} such that

$$\max \left\{ c_1 \int_{\tau}^{\widehat{t}+\tau} \|\theta_t\|_H^2, c_2 \int_{\tau}^{\widehat{t}+\tau} \|\chi_t\|_V^2 \right\} \leq \min \left\{ \frac{c_s}{2}, \frac{1}{8} \right\},$$

for any $\tau \in [0, T - \widehat{t}]$. Thus, we finally get

$$\begin{aligned} &\frac{c_s}{2} \|\theta\|_{L^2(0,t;H)}^2 + \frac{1}{2} \|1 * [\theta](t)\|_V^2 \\ &\quad + \frac{1}{4} \|[p]\|_{L^2(0,t;H)}^2 + \frac{1}{2} \|1 * \nabla[p]\|_H^2 + \frac{1}{2} \|\chi\|_H^2 + \frac{1}{8} \|\nabla[\chi]\|_{L^2(0,t;H)}^2 \\ &\leq c \int_0^t \left(1 + \|p\|_V^4 + \|u^{-1}\|_H^2 \|\nabla u^{-1}\|_H^2 \right) \|\chi\|_H^2 \\ &\quad + \frac{1}{4} \|1 * [\theta]\|_{L^\infty(0,t;V)}^2, \end{aligned} \quad (5.18)$$

if (A1) holds, or

$$\begin{aligned}
& \frac{c_s}{2} \|\theta\|_{L^2(0,t;H)}^2 + \frac{1}{8} \|1 * [\theta](t)\|_V^2 \\
& \quad + \frac{1}{4} \|p\|_{L^2(0,t;H)}^2 + \frac{1}{2} \|1 * \nabla[p]\|_H^2 + \frac{1}{2} \|\chi(t)\|_H^2 + \frac{1}{8} \|\nabla[\chi]\|_{L^2(0,t;H)}^2 \\
& \leq c \int_0^t \left(1 + \|p\|_V^4 + \|p^{-1}\|_{L^6(\Omega)}^4\right) \|\chi\|_H^2 + \frac{1}{4} \|1 * [\theta]\|_{L^\infty(0,t;V)}^2 \quad (5.19)
\end{aligned}$$

if (A2) is verified. Then, in both cases, a generalized version of the Gronwall lemma (see [2]) ensures that

$$\begin{aligned}
& \|\theta\|_{L^2(0,t;H)} + \|1 * [\theta]\|_{L^\infty(0,t;V)} + \|1 * [p]\|_{L^\infty(0,t;V)} \\
& \quad + \|\chi\|_{L^\infty(0,t;H) \cap L^2(0,t;V)} + \|p\|_{L^2(0,T;H)} \leq 0, \quad (5.20)
\end{aligned}$$

for any $t \in (0, \tau)$. Finally, we can repeat the same estimates in the interval $(\tau, \tau + \widehat{t})$ and so on. Thus, iterating the above procedure (see [7]) we are allowed to extend (5.20) on the whole interval $(0, T)$, which concludes the proof of the uniqueness result stated by Theorem 2.3.

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