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Emmanuel Gobet, Stéphane Menozzi

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By Emmanuel Gobet and Stéphane Menozzi

ENSIMAG-INP Grenoble,Université Denis Diderot Paris 7

Abstract For a stopped diffusion process in a time dependent domain, we obtain the asymptotics of the triplet exit time/exit position/overshoot for the discretely stopped Euler scheme. Here, the overshoot means the distance to the boundary of the process when it exits the domain. As a first consequence of this result, we obtain an expansion for the weak error. From the expansion and the sensitivity of the underlying Dirichlet problem with respect to the domain, we finally derive a procedure to improve the convergence by suitably restraining the domain.

1. Introduction.

1.1. *Statement of the problem.* We consider a d-dimensional diffusion process whose dynamics is given by

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \tag{1.1}$$

where W is a standard d'-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. The mappings b and σ are Lipschitz continuous in space and locally bounded in time, so that (1.1) has a unique strong solution. For a fixed deterministic time $T > 0$ and a given bounded time dependent domain $\mathcal{D} \subset]0, T[\times\mathbb{R}^d$ assumed to be smooth we define $\tau := \inf\{t > 0 : (t, X_t) \notin \mathcal{D}\}$. Note that τ is bounded by T. For $x \in \mathcal{D}_0 := \{y \in \mathbb{R}^d : (0, y) \in \partial\mathcal{D}\backslash\overline{\partial\mathcal{D} \cap (]0, T[\times\mathbb{R}^d)}\}$, where $\partial\mathcal{D}$ is the boundary of $\mathcal{D}$, and given continuous functions $g, f, k : \bar{\mathcal{D}} \to \mathbb{R}$, we are interested in estimating the quantity

$$\mathbb{E}_x[g(\tau, X_\tau)Z_\tau + \int_0^\tau Z_s f(s, X_s)ds], \ Z_s = \exp(-\int_0^s k(r, X_r)dr), \tag{1.2}$$

where as usual $\mathbb{E}_x[.] := \mathbb{E}[.|X_0 = x]$ (resp. $\mathbb{P}_x[.] := \mathbb{P}[.|X_0 = x]$). The approximation of such quantities is a well known issue in finance, since it

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represents in this framework the price of a barrier option, see e.g. Andersen and Brotherton-Ratcliffe [ABR96]. Those quantities also arise through the Feynman-Kac representation of the solution of a parabolic PDE with Cauchy-Dirichlet boundary conditions, see Costantini *et al.* [CGK06]. They can therefore also be related to problems of heat diffusion in time dependent domains.

We then choose to approximate the expectation in (1.2) by Monte Carlo simulation. This approach is natural and especially appropriate if the dimension d is large. To this end we approximate the diffusion (1.1) by its Euler scheme with time step $\Delta > 0$. Define $\forall t \geq 0, \phi(t) := \sup\{t_i := i\Delta,\ i \in \mathbf{N} : t_i \leq t < t_{i+1}\}$ and introduce

$$X_t^\Delta = x + \int_0^t b(\phi(s), X_{\phi(s)}^\Delta)ds + \int_0^t \sigma(\phi(s), X_{\phi(s)}^\Delta)dW_s. \tag{1.3}$$

We now associate to (1.3) the discrete exit time $\tau^\Delta := \inf\{t_i,\ i \in \mathbf{N}^* : (t_i, X_{t_i}^\Delta) \notin \mathcal{D}\}$. Approximating the functional $V_\tau := g(\tau, X_\tau)Z_\tau + \int_0^\tau Z_s f(s, X_s)\, ds$ by

$$V_{\tau^\Delta}^\Delta := g(\tau^\Delta, X_{\tau^\Delta}^\Delta)Z_{\tau^\Delta}^\Delta + \int_0^{\tau^\Delta} Z_{\phi(s)}^\Delta f(\phi(s), X_{\phi(s)}^\Delta)ds$$
$$\text{with} \quad Z_t^\Delta = e^{-\int_0^t k(\phi(r), X_{\phi(r)}^\Delta)dr},$$

we introduce the quantity

$$\mathrm{Err}(T, \Delta, g, f, k, x) = \mathbb{E}_x[V_{\tau^\Delta}^\Delta - V_\tau] \tag{1.4}$$

that will be referred to as the weak error. Note that in $V_{\tau^\Delta}^\Delta$, g is a.s. not evaluated on the boundary (g must be understood as a function defined in a neighborhood of $\partial\mathcal{D}$). At first sight, this approximation can seem coarse. Anyhow, it does not affect the convergence rate and really reduces the computational cost with respect to the alternative that would consist in taking the projection on $\partial\mathcal{D}$.

Our main purpose is to expand the error (1.4) at the first order. Actually, the crucial tool is the asymptotics of the triplet exit time/exit position/overshoot for the Euler scheme. Here, the overshoot means the distance to the boundary of the process when it exits the domain. In addition, we improve the initial convergence rate by a boundary correction procedure.

We mention that we could also consider the diffusion process discretely stopped: results below would remain the same.

1.2. *Existing results and contribution of the paper.* Up to now, the behavior of (1.4) had mainly been analyzed for cylindrical domains, in the killed case, without source and potential terms (i.e. when the error writes $\mathrm{Err}(T,\Delta,g,0,0,x) = \mathbb{E}[g(X_T^{\Delta})\mathbf{1}_{\tau^{\Delta}>T}] - \mathbb{E}[g(X_T)\mathbf{1}_{\tau>T}] := \mathcal{E}$). Let us first mention the work of Broadie *et al.* [BGK97], who first derived the correction procedure of Section 2 in the one dimensional geometric Brownian motion setting (Black and Scholes model). In [Gob00] and [GM04], it had been shown that, under some (hypo)ellipticity conditions on the coefficients and some smoothness of the domain and the coefficients, $\mathcal{E}$ was lower and upper bounded at order 1/2 w.r.t. the time step Δ. Also, an expansion result for the killed Brownian motion in a cone as well as the associated correction procedure are available in [Men06].

All these works emphasize that the crucial quantity to control in order to obtain an expansion is the overshoot above the spatial boundary of the discrete process. In the Brownian one-dimensional framework such controls go back to Siegmund [Sie79] and Siegmund and Yuh [SY82]. We manage to extend their results to obtain the asymptotic distribution of the overshoot of the Euler scheme, see Sections 2 and 3. Concerning the asymptotics of the overshoot of stochastic processes, let us mention the works of Alsmeyer [Als94] or Fuh and Lai [FL01] for ergodic Markov chains and Doney and Kyprianou for Lévy processes [DK06]. These works are all based on renewal arguments.

Our results about the overshoot combined to sharp techniques of error analysis, allow us to derive an expansion of the form $\mathrm{Err}(T,\Delta,g,f,k,x) = C\sqrt{\Delta} + o(\Delta)$ in the very general framework of stopped processes in time dependent domains. Some sufficient assumptions can be formulated in terms of the uniform ellipticity of σ and some smoothness properties for $\mathcal{D}, b, \sigma, g, f, k$.

From a numerical viewpoint, error expansions are the first step for a procedure that aims to improve the convergence. A very popular one is the Romberg extrapolation, see [TT90]. Using the recent results of Costantini *et al.* [CGK06] concerning the sensitivity of the Dirichlet problem w.r.t. the boundary, we propose an alternative simulation procedure that converges as $o(\sqrt{\Delta})$. Namely, since the discrete stopping yields an overestimation of the exact stopping (see Boyle and Lau [BL94], Baldi [Bal95], [GM04]) we stop the discrete process when it leaves a suitable smaller domain. Furthermore, this technique does not require any refinement of the time step. Therefore, it does not increase the empirical variance for the associated Monte Carlo estimator as for the Romberg extrapolation.

Let us finally mention that under some quite usual assumptions the previous results about the error expansion and correction still hold in the station-

ary setting, see Section 4, which also seems to be new. Numerical applications are left to further works.

1.3. *Outline of the paper.* Notations and assumptions used throughout the paper are stated in Section 1.4. In Section 2 we give our main results concerning the asymptotics of the overshoot, the error expansion and the boundary correction. These results are proved in Section 3, which is the technical core of the paper. Eventually, Section 4 deals with the stationary extension of our results. We still manage to obtain an expansion and a correction for elliptic PDEs. Some technical results are postponed in Appendix.

1.4. *General notation and assumptions.*

1.4.1. *Miscellaneous.*

• *Differentiation.* For smooth functions $g(t,x)$, we denote by $\partial_x^\beta g(t,x)$ the derivative of g w.r.t. x according to the multi-index β, whereas the time derivative of g is denoted by $\partial_t g(t,x)$. The notation $\nabla g(t,x)$ stands for the usual gradient w.r.t. x (as a row vector) and the Hessian matrix of g (w.r.t. the space variable x) is denoted by $Hg(t,x)$.
The second order linear operator L below stands for the infinitesimal generator of the diffusion process X in (1.1):

$$Lg(t,x) = \nabla g(t,x)b(t,x) + \frac{1}{2}\mathrm{Tr}(Hg(t,x)[\sigma\sigma^*](t,x)). \tag{1.5}$$

• *Metric.* The Euclidean norm is denoted by $|\cdot|$.
We set $B_d(x,\epsilon)$ for the usual Euclidean d-dimensional open ball with center x and radius ϵ and $\mathbf{d}(x,C)$ for the Euclidean distance of a point x to a closed set C.

• *Functions.* For an open set $\mathcal{D}' \subset \mathbb{R}\times\mathbb{R}^d$ and $l \in \mathbf{N}$, $\mathcal{C}^{\lfloor \frac{l}{2}\rfloor,l}(\mathcal{D}')$ (resp. $\mathcal{C}^{\lfloor \frac{l}{2}\rfloor,l}(\overline{\mathcal{D}'})$) is the space of continuous functions f defined on $\mathcal{D}'$ with continuous derivatives $\partial_x^\beta\partial_t^j f$ for $|\beta|+2j \le l$ (resp. defined in a neighborhood of $\overline{\mathcal{D}'}$). Also, for $a = l+\theta,\ \theta\in]0,1],\ l\in\mathbf{N}$, we denote by $\mathbf{H}_a(\mathcal{D}')$ (resp. $\mathbf{H}_a(\bar{\mathcal{D}}')$) the Banach space of functions of class $\mathcal{C}^{\lfloor \frac{l}{2}\rfloor,l}(\mathcal{D}')$ (resp. $\mathcal{C}^{\lfloor \frac{l}{2}\rfloor,l}(\bar{\mathcal{D}}')$) having l^{th} space derivatives uniformly θ-Hölder continuous and $\lfloor l/2\rfloor$ time derivatives uniformly $(a/2-\lfloor l/2\rfloor)$-Hölder continuous, see Lieberman [Lie96], p. 46 for details. We may simply write $\mathcal{C}^{\lfloor \frac{l}{2}\rfloor,l}$ or $\mathbf{H}_a$ when $\mathcal{D}' = \mathbb{R}\times\mathbb{R}^d$.

• *Floating constants.* As usual, we use the same symbol C for all finite, non-negative constants which appear in our computations : they may depend on $\mathcal{D}, T, b, \sigma, g, f, k$ but they will not depend on Δ or x. We reserve the notation c for constants also independent of T, g, f and k. Other possible dependences

will be explicitly indicated.
In the following $O_{pol}(\Delta)$ (resp. $O(\Delta)$) stands for every quantity $R(\Delta)$ such that $\forall k \in \mathbf{N}$, for some $C > 0$, one has $|R(\Delta)| \leq C\Delta^k$ (resp. $|R(\Delta)| \leq C\Delta$) (uniformly in the starting point x).

1.4.2. *Time-space domains.* In the sequel $\mathcal{D}$ stands for a bounded time-space domain in $]0,T[\times\mathbb{R}^d$ (T is a fixed terminal time). Let

$$\mathcal{D}_0 = \left\{x : (0,x) \in \partial\mathcal{D}\backslash\overline{\partial\mathcal{D} \cap (]0,T[\times\mathbb{R}^d)}\right\},$$
$$\mathcal{D}_T = \left\{x : (T,x) \in \partial\mathcal{D}\backslash\overline{\partial\mathcal{D} \cap (]0,T[\times\mathbb{R}^d)}\right\}.$$

$\mathcal{D}_0$ and $\mathcal{D}_T$ are open sets and we assume that they are nonempty domains that coincide with the interior of their closure (cf. [Fri64], Section 3.2). We assume also (cf. again [Fri64], Section 3.2) that the time section of $\mathcal{D}$,

$$\mathcal{D}_t = \{x : (t,x) \in \mathcal{D}\}, \qquad t \in]0,T[, \tag{1.6}$$

is a domain that coincides with the interior of its closure, for every $t \in]0,T[$.

Regularity assumptions on the domain $\mathcal{D}$ will be formulated in terms of Hölder spaces with *time-space* variables (see [Lie96] p.46 and [Fri64] Section 3.2). Namely, we say that the domain $\mathcal{D}$ is of class $\mathbf{H}_a$, $a \geq 1$ if for every boundary point $(t_0,x_0) \in \overline{\partial\mathcal{D} \cap (]0,T[\times\mathbb{R}^d)}$, there exists a neighborhood $]t_0,t_0+\varepsilon_0^2[\times B_d(x_0,\varepsilon_0)$, an index $i \in [\![1,d]\!]$ and a function $\varphi_0 \in \mathbf{H}_a(]t_0,t_0+\varepsilon_0^2[\times B_{d-1}(x_0^1,...,x_0^{i-1},x_0^{i+1},...,x_0^d),\varepsilon_0)$ s.t.

$$\begin{aligned}\overline{\partial\mathcal{D} \cap (]0,T[\times\mathbb{R}^d)} \cap]t_0,t_0+\varepsilon_0^2[\times B_d(x_0,\varepsilon_0)\\ := \{(t,x) \in (]t_0,t_0+\varepsilon_0^2[\cap[0,T]) \times B_d(x_0,\varepsilon_0)\\ : x_i = \varphi_0(t,x_1,...,x_{i-1},x_{i+1},...,x_d)\}.\end{aligned}$$

In the following we freely use the notations of [Lie96].

If $\mathcal{D}$ is of class $\mathbf{H}_2$, all domains $\mathcal{D}_t$, for $t \in [0,T]$, satisfy the uniform interior and exterior sphere condition with the same radius r_0. Moreover (see [Lie96], Section X.3), the signed spatial distance F, given by

$$F(t,x) = \begin{cases} -\mathbf{d}(x,\partial\mathcal{D}_t), & \text{for } x \in \mathcal{D}_t^c,\ \mathbf{d}(x,\partial\mathcal{D}_t) \leq r_0,\ 0 < t < T, \\ \mathbf{d}(x,\partial\mathcal{D}_t), & \text{for } x \in \mathcal{D}_t,\ \mathbf{d}(x,\partial\mathcal{D}_t) \leq r_0,\ 0 < t < T, \end{cases}$$

belongs to $\mathbf{H}_2\left(\{(t,x) : 0 < t < T,\ \mathbf{d}(x,\partial\mathcal{D}_t) < r_0\}\right)$ and $\nabla F(t,x)$ is the unit inward normal vector to $\mathcal{D}_t$ at $\pi_{\partial\mathcal{D}_t}(x)$ the nearest point to x in $\partial\mathcal{D}_t$. F can be extended as a $\mathbf{H}_2([0,T]\times\mathbb{R}^d)$ function, preserving the sign.

In the following we denote $\forall r \in \mathbb{R}^+$, by $V_{\partial\mathcal{D}}(r) := \{(t,x) \in [0,T]\times\mathbb{R}^d : \mathbf{d}(x,\partial\mathcal{D}_t) \leq r\}$ a neighborhood of size r of the so called *side*.

1.4.3. *Diffusion processes stopped at the boundary.* We specify the properties of the coefficients (b, σ) in (1.1) with assumption

$(\mathbf{A}_\theta)$ (with $\theta \in]0,1]$)

1. *Smoothness.* b and σ are functions of class $\mathbf{H}_{1+\theta}$.

2. *Uniform ellipticity.* For some $a_0 > 0$, it holds $\xi^*[\sigma\sigma^*](t,x)\xi \geq a_0|\xi|^2$ for any $(t,x,\xi) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$.

We also introduce assumption $(\mathbf{A}'_\theta)$ for which **2.** is replaced by the weaker assumption

2'. *Uniform non characteristic boundary.* For some $r_0 > 0$ there exists $a_0 > 0$ s.t. $\nabla F(t,x)[\sigma\sigma^*](t,x)\nabla F(t,x) \geq a_0$ for any $(t,x) \in V_{\partial\mathcal{D}}(r_0)$.

The asymptotic results concerning the overshoot hold true under $(\mathbf{A}'_\theta)$, see Section 2.1.

We mention that the additional smoothness of b and σ w.r.t. the time variable is required for the connection with PDEs. In the following we use the superscript t,x to indicate the usual Markovian dependence, i.e. $\forall s \geq t,\ X_s^{t,x} = x + \int_t^s b(u, X_u^{t,x})du + \int_t^s \sigma(u, X_u^{t,x})dW_u$. Now let

$$\tau^{t,x} := \inf\{s > t : X_s^{t,x} \notin \mathcal{D}_s\} = \inf\{s > t : (s, X_s^{t,x}) \notin \mathcal{D}\} \tag{1.7}$$

be the first exit time of $X_s^{t,x}$ from $\mathcal{D}_s$ or, equivalently, the first exit time of the time-space process $(s, X_s^{t,x})_{s\in[t,T]}$ from the domain $\mathcal{D}$. Note that $\tau^{t,x}$ is bounded by T. For functionals of the process X stopped at the exit from $\mathcal{D}$, of the form

(1.8)

$$u(t,x) = \mathbb{E}\big[g(\tau^{t,x}, X_{\tau^{t,x}}^{t,x})e^{-\int_t^{\tau^{t,x}} k(r,X_r^{t,x})dr} + \int_t^{\tau^{t,x}} e^{-\int_t^s k(r,X_r^{t,x})dr} f(s, X_s^{t,x})ds\big],$$

we now recall (see [CGK06]) that the Feynman-Kac representation holds in the time-space domain. Introduce the parabolic boundary $\mathcal{PD} = \partial\mathcal{D}\backslash\{0\} \times \mathcal{D}_0$.

Proposition 1.1 [Feynman-Kac's formula and a priori estimates on u]
Assume $(\mathbf{A}_\theta)$, $\mathcal{D} \in \mathbf{H}_1$, $k \in \mathbf{H}_\theta$, $f \in \mathbf{H}_\theta$ *and* $g \in \mathcal{C}^{0,0}$ *with* $\theta \in]0,1[$. *Then, there is a unique solution of class* $\mathcal{C}^{1,2}(\mathcal{D}) \cap \mathcal{C}^{0,0}(\overline{\mathcal{D}})$ *to*

$$\begin{cases} \partial_t u + Lu - ku + f = 0 & \text{in } \mathcal{D}, \\ u = g & \text{on } \mathcal{PD}, \end{cases} \tag{1.9}$$

and it is given by (1.8).
In addition, if for some $\theta \in]0,1[$, $\mathcal{D}$ *is of class* $\mathbf{H}_{1+\theta}$, $g \in \mathbf{H}_{1+\theta}$ *then* $u \in$

$\mathbf{H}_{1+\theta}$. *In particular* ∇u *exists and is* θ*-Hölder continuous up to the boundary. Eventually, for* $\mathcal{D} \in \mathbf{H}_{3+\theta}$, $k, f \in \mathbf{H}_{1+\theta}$, $g \in \mathbf{H}_{3+\theta}$ *satisfying the first order compatibility condition* $(\partial_t + L - k)g(T,x) + f(T,x)|_{x\in\partial\mathcal{D}_T} = 0$, *then the function* u *belongs to* $\mathbf{H}_{3+\theta}$.

Proof. The first two existence and uniqueness result for (1.9) are respectively implied by Theorems 5.9 and 5.10 and Theorem 6.45 in Lieberman, [Lie96]. The probabilistic representation is then a usual verification argument, see e.g. Appendix B.1 in [CGK06]. The additional smoothness can be derived from exercice 4.5 Chapter IV in [Lie96] or Theorem 12, Chapter 3 in [Fri64]. □

2. Main Results.

2.1. *Controls concerning the overshoot.* The overshoot is associated to the distance of the process to the boundary, when it exits the domain. To be precise, we use F the signed distance function and we consider the quantity $F(t_i, X^\Delta_{t_i})$. It remains positive for $t_i < \tau^\Delta$, and at time $t_i = \tau^\Delta$, it becomes non positive. The overshoot is thus defined by $F^-(\tau^\Delta, X^\Delta_{\tau^\Delta})$. Since F is Lipschitz continuous in time and space, it is easy to see that $F^-(\tau^\Delta, X^\Delta_{\tau^\Delta})$ is of order $\sqrt{\Delta}$ (in L_p-norm for instance). Thus, it is natural to study the asymptotics of the rescaled overshoot

$$\Delta^{-1/2} F^-(\tau^\Delta, X^\Delta_{\tau^\Delta}).$$

Adapting the proof of Proposition 6 in [GM04] to our time dependent context, see also the Proof of Proposition 4.2 for a simpler version, one has the following proposition.

Proposition 2.1 (Tightness of the overshoot) *Assume* $(\mathbf{A}'_\theta)$, *and that* $\mathcal{D}$ *is of class* $\mathbf{H}_2$. *Then, for some* $c > 0$ *one has*

$$\sup_{\Delta>0, s\in[0,T]} \mathbb{E}_x[\exp(c[\Delta^{-1/2} F^-(s\wedge\tau^\Delta, X^\Delta_{s\wedge\tau^\Delta})]^2)] < +\infty.$$

It is quite plain to prove by pathwise convergence of X^Δ towards X that $(\tau^\Delta, X^\Delta_{\tau^\Delta})$ converges in probability to (τ, X_τ). The next theorem also includes the rescaled overshoot.

Theorem 2.2 (Joint limit laws associated to the overshoot) *Assume* $(\mathbf{A}'_\theta)$, *and that* $\mathcal{D}$ *is of class* $\mathbf{H}_2$. *Let* φ *be a continuous function with compact support. For all* $t \in [0,T]$, $x \in \mathcal{D}_0$, $y \geq 0$,

$$\mathbb{E}_x[\mathbf{1}_{\tau^\Delta\leq t} Z^\Delta_{\tau^\Delta} \varphi(X^\Delta_{\tau^\Delta}) \mathbf{1}_{F^-(\tau^\Delta, X^\Delta_{\tau^\Delta})\geq y\sqrt{\Delta}}] \underset{\Delta\to 0}{\longrightarrow}$$
$$\mathbb{E}_x[\mathbf{1}_{\tau\leq t} Z_\tau \varphi(X_\tau)(1 - H(y/|\sigma^* \nabla F(\tau, X_\tau)|))]$$

with $H(y) := (\mathbb{E}_0[s_{\tau^+}])^{-1} \int_0^y dz \mathbb{P}_0[s_{\tau^+} > z]$ *and* $s_0 := 0, \forall n \geq 1, s_n := \sum_{i=1}^n G^i$, *the* G^i *being i.i.d. standard centered normal variables,* $\tau^+ := \inf\{n \geq 0 : s_n > 0\}$.

In other words, $(\tau^\Delta, X^\Delta_{\tau^\Delta}, \Delta^{-1/2} F^-(\tau^\Delta, X^\Delta_{\tau^\Delta}))$ weakly converges to $(\tau, X_\tau, |\sigma^* \nabla F(\tau, X_\tau)| Y)$ where Y is a random variable independent of (τ, X_τ), and which cumulative function is equal to H. Actually, Y has the asymptotic law of the renormalized Brownian overshoot. In the following analysis, the mean of the overshoot is an important quantity and it is worth noting that one has $\mathbb{E}(Y) = \frac{\mathbb{E}_0[s^2_{\tau^+}]}{2\mathbb{E}_0[s_{\tau^+}]} := c_0$. One knows from [Sie79] that

$$c_0 = -\frac{\zeta(1/2)}{\sqrt{2\pi}} = 0.5823... \tag{2.1}$$

The above theorem is the crucial tool in the derivation of our main results. The proof is given in Section 3.1.

2.2. *Error expansion and boundary correction.* For notational convenience introduce for $x \in \mathcal{D}_0$,

$$u(\mathcal{D}) = \mathbb{E}_x(g(\tau, X_\tau) Z_\tau + \int_0^\tau Z_s f(s, X_s) ds),$$
$$u^\Delta(\mathcal{D}) = \mathbb{E}_x(g(\tau^\Delta, X^\Delta_{\tau^\Delta}) Z^\Delta_{\tau^\Delta} + \int_0^{\tau^\Delta} Z^\Delta_{\phi(s)} f(\phi(s), X^\Delta_{\phi(s)}) ds).$$

Theorem 2.3 (First order expansion) *Under* $(\mathbf{A}_\theta)$, *for a domain of class* $\mathbf{H}_2$, $g \in \mathbf{H}_{1+\theta}$, $k, f \in \mathbf{H}_{1+\theta}$ *and for* Δ *small enough*

$$\mathrm{Err}(T, \Delta, g, f, k, x) = u^\Delta(\mathcal{D}) - u(\mathcal{D})$$
$$= c_0 \sqrt{\Delta} \mathbb{E}_x(\mathbf{1}_{\tau < T} Z_\tau (\nabla u - \nabla g)(\tau, X_\tau) \cdot \nabla F(\tau, X_\tau) |\sigma^* \nabla F(\tau, X_\tau)|) + o(\sqrt{\Delta}),$$

where c_0 *is defined in* (2.1).

Define now a smaller domain $\mathcal{D}^\Delta \subset \mathcal{D}$, which time section is given by $\mathcal{D}^\Delta_t = \{x \in \mathcal{D}_t : \mathbf{d}(x, \partial \mathcal{D}_t) > c_0 \sqrt{\Delta} |\sigma^* \nabla F(t, x)|\}$. Introduce the exit time of the Euler scheme from this smaller domain: $\hat{\tau}^\Delta = \inf\{t_i \geq 0 : (t_i, X^\Delta_{t_i}) \notin \mathcal{D}^\Delta\} \leq \tau^\Delta$. The boundary correction procedure consists in simulating

$$g(\hat{\tau}^\Delta, X^\Delta_{\hat{\tau}^\Delta}) Z^\Delta_{\hat{\tau}^\Delta} + \int_0^{\hat{\tau}^\Delta} Z^\Delta_{\phi(s)} f(\phi(s), X^\Delta_{\phi(s)}) ds. \tag{2.2}$$

As above, we do not compute any projection on the boundary. We denote the expectation of (2.2) by $u^\Delta(\mathcal{D}^\Delta)$. One has:

Theorem 2.4 (Boundary correction) *Under the assumptions of Theorem 2.3, if we additionally suppose $\nabla F(t,x)|\sigma^*\nabla F(t,x)|$ is of class $\mathcal{C}^{1,2}$, then one has:*

$$u^\Delta(\mathcal{D}^\Delta) - u(\mathcal{D}) = o(\sqrt{\Delta}).$$

The additional assumption is due to technical considerations in [CGK06]. It is automatically fulfilled for domains of class $\mathcal{C}^3$ and σ in $\mathcal{C}^{1,2}$.

2.3. *Proof of Theorems 2.3 and 2.4.*

2.3.1. *Error expansion.* By usual weak convergence arguments, Theorem 2.3 is a direct consequence of Proposition 2.1 (tightness), Theorem 2.2 (joint limit laws associated to the overshoot) and Theorem 2.5 below.

Theorem 2.5 (First order approximation) *Under the assumptions of Theorem 2.3*

$$u^\Delta(\mathcal{D}) - u(\mathcal{D}) = o(\sqrt{\Delta}) +$$
$$\mathbb{E}_x(\mathbf{1}_{\tau^\Delta \le T} Z^\Delta_{\tau^\Delta}(\nabla u - \nabla g)(\tau^\Delta, \pi_{\partial\mathcal{D}_{\tau^\Delta}}(X^\Delta_{\tau^\Delta})) \cdot \nabla F(\tau^\Delta, X^\Delta_{\tau^\Delta}) F^-(\tau^\Delta, X^\Delta_{\tau^\Delta})).$$

Remark 2.6 *In the above statement, we use projections on a non convex set, which needs a clarification. With the notation of Section 1.4.2, introduce $\tau^{r_0} := \inf\{s > 0 : (s, X^\Delta_s) \notin \mathcal{D} \cup V_{\partial\mathcal{D}}(r_0)\}$. For $s \in [0, \tau^{r_0}]$ the projection $\pi_{\bar{\mathcal{D}}_s}(X^\Delta_s)$ is uniquely defined by*

$$\pi_{\bar{\mathcal{D}}_s}(X^\Delta_s) = X^\Delta_s + F^-(s, X^\Delta_s)\nabla F(s, X^\Delta_s). \tag{2.3}$$

Large deviation arguments (see Lemma 3.1 below) also give $\mathbb{P}_x[\tau^{r_0} \le \tau^\Delta] = O_{pol}(\Delta)$. Thus, in the following, for $s \ge \tau^{r_0}$, $\pi_{\bar{\mathcal{D}}_s}(X^\Delta_s)$ and $\pi_{\partial\mathcal{D}_s}(X^\Delta_s)$ denote an arbitrary point on $\partial\mathcal{D}_s$. This choice yields an exponentially small contribution in our estimates.

Proof. Denote e^Δ the above error. Write now

$$\begin{aligned} e^\Delta &= \mathbb{E}_x[g(\tau^\Delta, X^\Delta_{\tau^\Delta}) Z^\Delta_{\tau^\Delta} - g(\tau^\Delta, \pi_{\bar{\mathcal{D}}_{\tau^\Delta}}(X^\Delta_{\tau^\Delta})) Z^\Delta_{\tau^\Delta}] \\ &\quad + \mathbb{E}_x[g(\tau^\Delta, \pi_{\bar{\mathcal{D}}_{\tau^\Delta}}(X^\Delta_{\tau^\Delta})) Z^\Delta_{\tau^\Delta} + \int_0^{\tau^\Delta} Z^\Delta_{\phi(s)} f(\phi(s), X^\Delta_{\phi(s)}) ds] - u(0, X^\Delta_0) \\ &:= e^\Delta_1 + e^\Delta_2. \end{aligned}$$

We introduce here the projection for the error analysis. From (2.3), a second order Taylor expansion and standard computations yield

$$\begin{aligned} e^\Delta_1 = &- \mathbb{E}_x[\mathbf{1}_{\tau^\Delta \le T} Z^\Delta_{\tau^\Delta} \nabla g(\tau^\Delta, \pi_{\partial\mathcal{D}_{\tau^\Delta}}(X^\Delta_{\tau^\Delta})) \cdot \nabla F(\tau^\Delta, X^\Delta_{\tau^\Delta}) F^-(\tau^\Delta, X^\Delta_{\tau^\Delta})] \\ &+ O(\Delta). \end{aligned} \tag{2.4}$$

For clarity we assume for the rest of the proof that $u \in \mathbf{H}_{3+\theta}$. Sufficient conditions to have such a smoothness are provided in Proposition 1.1. The proof under the assumptions of Theorem 2.3 is presented in Appendix.

Now, in order to isolate the overshoot F^- in e_2^Δ, we use a Taylor formula up to order 3. This is a more direct approach than the Itô-Tanaka expansion in [GM04] that also yields some local time terms.

In the following, we write $U \overset{\mathbb{E}}{=} V$ (resp $U \overset{\mathbb{E}}{\leq} V$) when the equality between U and V holds in mean up to a $O_{pol}(\Delta)$ (resp. $\mathbb{E}_x(U) \leq \mathbb{E}_x(V) + O_{pol}(\Delta)$). One has:

$$
\begin{aligned}
&e_2^\Delta \overset{\mathbb{E}}{=} \\
&\Big(\sum_{0\leq t_i<\tau^\Delta} u(t_{i+1}, \pi_{\bar{\mathcal{D}}_{t_{i+1}}}(X^\Delta_{t_{i+1}}))Z^\Delta_{t_{i+1}} - u(t_i, \pi_{\bar{\mathcal{D}}_{t_i}}(X^\Delta_{t_i}))Z^\Delta_{t_i} + Z^\Delta_{t_i} f(t_i, X^\Delta_{t_i})\Delta \Big) \\
&\mathbf{1}_{\tau^{r_0}>\tau^\Delta} \overset{\mathbb{E}}{=} \Big(\sum_{0\leq t_i<T} \mathbf{1}_{t_i<\tau^\Delta} \Big[u(t_{i+1}, \pi_{\bar{\mathcal{D}}_{t_{i+1}}}(X^\Delta_{t_{i+1}}))Z^\Delta_{t_{i+1}} - u(t_i, X^\Delta_{t_i})Z^\Delta_{t_i} \\
&+ Z^\Delta_{t_i} f(t_i, X^\Delta_{t_i})\Delta \Big] \Big) \mathbf{1}_{\tau^{r_0}>\tau^\Delta}
\end{aligned}
$$

since for $t_i < \tau^\Delta$, $X^\Delta_{t_i} \in \mathcal{D}_{t_i}$ and thus $\pi_{\bar{\mathcal{D}}_{t_i}}(X^\Delta_{t_i}) = X^\Delta_{t_i}$. Exploiting (2.3), writing $Z^\Delta_{t_{i+1}} = Z^\Delta_{t_i}(1 - k(t_i, X^\Delta_{t_i})\Delta) + O(\Delta^2)$ and performing a Taylor expansion of u at point $(t_i, X^\Delta_{t_i}) \in \mathcal{D}$ (where u is smooth), one gets

$$
\begin{aligned}
(2.5) \quad e_2^\Delta \overset{\mathbb{E}}{=} \Big(\sum_{0\leq t_i<T} \mathbf{1}_{t_i<\tau^\Delta} \Big[& \Delta \times Z^\Delta_{t_i} (\partial_t u + Lu - ku + f)(t_i, X^\Delta_{t_i}) \\
(2.6) \qquad & + Z^\Delta_{t_i} \nabla u(t_i, X^\Delta_{t_i}) \cdot \nabla F(t_{i+1}, X^\Delta_{t_{i+1}}) F^-(t_{i+1}, X^\Delta_{t_{i+1}}) \\
(2.7) \qquad & + O(|F^-(t_{i+1}, X^\Delta_{t_{i+1}})||X^\Delta_{t_{i+1}} - X^\Delta_{t_i}|) + O(|F^-(t_{i+1}, X^\Delta_{t_{i+1}})|^2) \Big] \\
(2.8) \qquad & + O(\Delta^{3/2+\theta/2}) + O(|W_{t_{i+1}} - W_{t_i}|^{3+\theta}) \Big) \mathbf{1}_{\tau^{r_0}>\tau^\Delta}.
\end{aligned}
$$

The contribution (2.5) equals 0 owing to the PDE. Both remainder terms in (2.8) are of order $O(\Delta^{3/2+\theta/2})$, and contribute to the sum as $O(\Delta^{1/2+\theta/2}) = o(\sqrt{\Delta})$. Finally, regarding (2.7), defining $\tau_{t_i} := \inf\{s \geq t_i : (s, X^\Delta_s) \notin \mathcal{D}\}$ one has

$$
\begin{aligned}
\sum_{0\leq t_i<T} \mathbf{1}_{t_i<\tau^\Delta} |F^-(t_{i+1}, X^\Delta_{t_{i+1}})|^2 &\overset{\mathbb{E}}{=} \sum_{0\leq t_i<T} \mathbf{1}_{t_i<\tau^\Delta} \mathbf{1}_{\tau_{t_i}<t_{i+1}} |F^-(t_{i+1}, X^\Delta_{t_{i+1}})|^2 \\
&\overset{\mathbb{E}}{\leq} C\Delta \sum_{0\leq t_i<T} \mathbf{1}_{t_i<\tau^\Delta} \mathbf{1}_{\tau_{t_i}\leq t_{i+1}}
\end{aligned}
$$

$$\stackrel{\mathbb{E}}{\leq} C\Delta \sum_{0\leq t_i<T} \mathbf{1}_{t_i<\tau^\Delta} \mathbf{1}_{X^\Delta_{t_{i+1}}\notin \mathcal{D}_{t_{i+1}}}$$
$$\stackrel{\mathbb{E}}{=} C\Delta 1_{\tau^\Delta\leq T} \stackrel{\mathbb{E}}{=} O(\Delta),$$

where the last but one inequality is obtained adapting the proof of Lemma 16 in [GM04] to time dependent domains.
In the same way, one has $\sum_{0\leq t_i<T} \mathbf{1}_{t_i<\tau^\Delta}|F^-(t_{i+1},X^\Delta_{t_{i+1}})||X^\Delta_{t_{i+1}}-X^\Delta_{t_i}| \stackrel{\mathbb{E}}{=} O(\Delta)$.
Recalling (2.4), to get the expected result, it remains in (2.6) to approximate $Z^\Delta_{t_i}\nabla u(t_i,X^\Delta_{t_i})$ by $Z^\Delta_{t_{i+1}}\nabla u(t_{i+1},\pi_{\partial\mathcal{D}_{t_{i+1}}}(X^\Delta_{t_{i+1}}))$, which is done as before, up to an extra additional error of order $O(\Delta)$. □

2.3.2. *Boundary Correction.* One has

$$u^\Delta(\mathcal{D}^\Delta)-u(\mathcal{D}) = [u^\Delta(\mathcal{D}^\Delta)-u(\mathcal{D}^\Delta)]+[u(\mathcal{D}^\Delta)-u(\mathcal{D})] \tag{2.9}$$

1. The first contribution in (2.9) has been previously analyzed, except that the domain $\mathcal{D}^\Delta$ depends on Δ. It is equal to $c_0\sqrt{\Delta}\mathbb{E}(\mathbf{1}_{\tau<T}Z_\tau(\nabla u-\nabla g)(\tau,X_\tau)\cdot\nabla F(\tau,X_\tau)|\sigma^*\nabla F(\tau,X_\tau)|)+o(\sqrt{\Delta})$.
2. Finally, the last term is related to the sensitivity of a Dirichlet problem with respect to the domain. By an application of Theorem 2.2 in [CGK06] with $\Theta(t,x)=-c_0\nabla F(t,x)|\sigma^*\nabla F(t,x)|$ (in $\mathcal{C}^{1,2}$), one gets that this contribution equals

$$-c_0\sqrt{\Delta}\mathbb{E}(\mathbf{1}_{\tau<T}Z_\tau(\nabla u-\nabla g)(\tau,X_\tau)\cdot\nabla F(\tau,X_\tau)|\sigma^*\nabla F(\tau,X_\tau)|)+o(\sqrt{\Delta}).$$

This proves that the new procedure has an error $o(\sqrt{\Delta})$. □

3. Technical results concerning the overshoot. This section is devoted to the proof of Theorem 2.2. We first state some useful auxiliary results.

Lemma 3.1 (Bernstein's inequality) *Assume* **(A$_\theta$-1)**. *Consider two stopping times S,S' upper bounded by T with $0\leq S'-S\leq\Theta\leq T$. Then for any $p\geq 1$, there are some constants $c>0$ and $C:=C(\textbf{(A}_\theta\textbf{-1)}\ ,\Theta)$, such that for any $\eta\geq 0$, one has a.s:*

$$\mathbb{P}[\sup_{t\in[S,S']}|X^\Delta_t-X^\Delta_S|\geq\eta\mid\mathcal{F}_S]\leq C\exp\left(-c\frac{\eta^2}{\Theta}\right),$$
$$\mathbb{E}[\sup_{t\in[S,S']}|X^\Delta_t-X^\Delta_S|^p\mid\mathcal{F}_S]\leq C\Theta^{p/2}.$$

For a proof of the first inequality we refer to Chapter 3, §3 in [RY99]. The last inequality easily follows from the first one or from the BDG inequalities.

Lemma 3.2 (Convergence of exit time) *Assume* $(\mathbf{A}'_\theta)$ *and that the domain is of class* $\mathbf{H}_2$. *The following convergences hold in probability:*

1. $\lim_{\Delta\to 0}\tau^\Delta = \tau$;
2. $\lim_{\Delta\to 0} X^\Delta_{\tau^\Delta} = X_\tau$;
3. $\lim_{\Delta\to 0}\sup_{t\le T}|X^\Delta_{\phi(t)} - X_t| = 0$.

The proof of the first two assertions in the case of space-time domain is analogous to the case of cylindrical domain (see [GM05]) and thus left to the reader. The last convergence is standard.

The following results are key tools to prove Theorem 2.2.

Lemma 3.3 (Asymptotic independence of the overshoot and the discrete exit time). *Let* W *be a standard one dimensional BM. Put* $x>0$ *and consider the domain* $\mathcal{D} :=]0,T[\times]-\infty,x[$. *With the notation of Section 2, for any* $\varepsilon>0$ *we have*

(3.1)
$$\lim_{\Delta\longrightarrow 0}\sup_{t\in[0,T],y\ge 0,x\ge\Delta^{1/2-\varepsilon}}\left|\mathbb{P}_0[\tau^\Delta\le t,(W_{\tau^\Delta}-x)\le y\sqrt{\Delta}]-\mathbb{P}_0[\tau\le t]H(y)\right| = 0.$$

Lemma 3.4 *Assume* $(\mathbf{A}'_\theta)$, *and that the domain is of class* $\mathbf{H}_2$. *Let* $0<\beta<\alpha<1/2$. *For all* $\eta>0$, *there exists* $C := C_\eta > 0$ *s.t. for* Δ *small enough,* $\forall (s,x)\in V_{\partial\mathcal{D}}(\Delta^\alpha)\cap\bar{\mathcal{D}}$ *(*$s\in\Delta\mathbb{N}$*),*

$$\mathbb{P}[\tau^\Delta\ge\Delta^{2\beta}|X^\Delta_s = x]\le C(\Delta^{\alpha-\beta-\eta}+\Delta^\beta),$$

where $\tau^\Delta := \inf\{t_i > s : (t_i, X^\Delta_{t_i})\notin\mathcal{D}\}$.

Lemma 3.5 *Assume* $(\mathbf{A}'_\theta)$, *and that the domain is of class* $\mathbf{H}_2$. *There exists* $C>0$, *such that* $\forall(s,x)\in\mathcal{D}$ *with* $s\in\Delta\mathbb{N}$, $\forall t\ge s$ *and* $\forall b\ge a\ge 0$,

$$\mathbb{P}[\tau^\Delta\le t,\Delta^{-1/2}F^-(\tau^\Delta,X^\Delta_{\tau^\Delta})\in[a,b]|X^\Delta_s = x]\le C((b-a)+\Delta^{1/4})$$

where τ^Δ *is shifted as in the previous lemma.*

The proof of these three lemmas is postponed to Section 3.2.

We mention that if $\sigma\sigma^*$ is uniformly elliptic, Lemma 3.5 is valid without the $\Delta^{1/4}$. See the proof for details.

3.1. *Proof of Theorem 2.2.* Consider first the case $\mathcal{D} =]0,T[\times D$ where D is a half space. The theorem in the case of BM is then a direct consequence of Lemma 3.3. Now to deal with the Euler scheme, we introduce a neighborhood whose distance to the boundary goes to 0 with Δ at a speed lower than $\Delta^{1/2}$. The characteristic exit time for a starting point in this neighborhood is short,

thus the diffusion coefficients are somehow constant and we are almost in the BM framework. Also, the localization w.r.t. to the hitting time of this neighborhood guarantees that up to a rescaling we are far enough from the boundary to apply the renewal arguments needed for the overshoot.
For a more general time-space domain of class $\mathbf{H}_2$ two additional tools are used: a time-space change of chart and a local half space approximation of the domain by some tangent hyperplane.
For notational convenience, we assume from now on that the time section domains $(\mathcal{D}_t)_{t\in[0,T]}$ are convex so that $\pi_{\partial\mathcal{D}_t}$ is always uniquely defined on $\mathcal{D}_t^c$. To handle the case of general $\mathbf{H}_2$ domains, an additional localization procedure similar to the one of Theorem 2.5 is needed.
For the sake of clarity, we also assume $k \equiv 0$ ($Z \equiv 1$). This is an easy simplification since owing to Lemma 3.2, $Z^{\Delta}_{\tau^{\Delta}}$ converges to Z_τ in L_1.

Step 1: localization. For $\alpha < 1/2$ specified later on, define $\tau_{\Delta^\alpha} := \inf\{t_i \geq 0 : F(t_i, X^{\Delta}_{t_i}) \leq \Delta^\alpha\} \leq \tau^{\Delta}$. We aim at studying the convergence of

$$\Psi_\Delta(t,x,y) := \mathbb{E}_x[\mathbf{1}_{\tau^\Delta\leq t, F(\tau^\Delta, X^\Delta_{\tau^\Delta})^-\geq y\sqrt{\Delta}}\varphi(X^\Delta_{\tau^\Delta})]$$

and for this, we define for all $0 \leq s \leq t \leq T$ ($s \in \Delta\mathbb{N}$), $(\tilde{x}, y) \in \mathbb{R}^d \times \mathbb{R}^+$

$$\begin{aligned}\Psi_\Delta(s,t,\tilde{x},y) :=& \mathbb{P}[\tau^\Delta \leq t, F(\tau^\Delta, X^\Delta_{\tau^\Delta})^- \geq y\sqrt{\Delta}|X^\Delta_s = \tilde{x}],\\ \forall \varepsilon \in]0,1/2[,\ A(t,\alpha,\varepsilon) :=& \{\tau_{\Delta^\alpha} < \tau^\Delta, \tau_{\Delta^\alpha} < t, F(\tau_{\Delta^\alpha}, X^\Delta_{\tau_{\Delta^\alpha}}) \geq \Delta^{1/2-\varepsilon}\}.\end{aligned}$$

In the definition of Ψ_Δ, τ^Δ has to be understood as the shifted exit time $\inf\{t_i > s : (t_i, X^\Delta_{t_i}) \notin \mathcal{D}\}$. By Lemma 3.1, $\mathbb{P}_x[\tau^\Delta = \tau_{\Delta^\alpha} \leq t] + \mathbb{P}_x[\tau_{\Delta^\alpha} < t, F(\tau_{\Delta^\alpha}, X^\Delta_{\tau_{\Delta^\alpha}}) < \Delta^{1/2-\varepsilon}] = O_{pol}(\Delta)$ for any $\varepsilon > 0$ s.t. $\alpha < 1/2 - \varepsilon$. Hence,

$$\begin{aligned}\Psi_\Delta(t,x,y) =& \mathbb{E}_x[\mathbf{1}_{A(t,\alpha,\varepsilon), F(\tau^\Delta, X^\Delta_{\tau^\Delta})^-\geq y\sqrt{\Delta}}\varphi(X^\Delta_{\tau^\Delta})] + O_{pol}(\Delta)\\ =& \mathbb{E}_x[\mathbf{1}_{A(t,\alpha,\varepsilon), F(\tau^\Delta, X^\Delta_{\tau^\Delta})^-\geq y\sqrt{\Delta}}(\varphi(X^\Delta_{\tau^\Delta}) - \varphi(X^\Delta_{\tau_{\Delta^\alpha}}))]\\ &+ \mathbb{E}_x[\mathbf{1}_{A(t,\alpha,\varepsilon)}\varphi(X^\Delta_{\tau_{\Delta^\alpha}})\Psi_\Delta(\tau_{\Delta^\alpha}, t, X^\Delta_{\tau_{\Delta^\alpha}}, y)] + O_{pol}(\Delta).\end{aligned}$$

The first term in the right hand side above converges to 0, using the convergence in probability of $|X^\Delta_{\tau^\Delta} - X^\Delta_{\tau_{\Delta^\alpha}}|$ to 0 (analogously to Lemma 3.2). This gives

$$\Psi_\Delta(t,x,y) = \mathbb{E}_x[\mathbf{1}_{A(t,\alpha,\varepsilon)}\varphi(X^\Delta_{\tau_{\Delta^\alpha}})\Psi_\Delta(\tau_{\Delta^\alpha}, t, X^\Delta_{\tau_{\Delta^\alpha}}, y)] + o(1). \tag{3.2}$$

It remains to study the convergence of $\Psi_\Delta(.)$.

Step 2: diffusion with frozen coefficients. Denote $\tau_{\Delta^\alpha} := \tilde{s}$, $X^\Delta_{\tau_{\Delta^\alpha}} := \tilde{x}$. Conditionally to $\mathcal{F}_{\tilde{s}}$, introduce now the one dimensional process $(Y_s)_{s\geq\tilde{s}}$,

$Y_s = F(\tilde{s},\tilde{x}) + (\nabla F \cdot \sigma)(\tilde{s},\tilde{x})(W_s - W_{\tilde{s}})$. Note that we do not take into account the drift part in the frozen process. From the next localization procedure, it yields a negligible term. Since Y has constant coefficients, we apply below Lemma 3.3 to handle the overshoot of Y w.r.t. $\mathbb{R}^{+*}$. Define $\tau^{\Delta,Y} := \inf\{t_i \geq \tilde{s} : Y_{t_i} \leq 0\}$ and rewrite

$$\begin{aligned} \Psi_\Delta(\tilde{s},t,\tilde{x},y) &:= \Psi^C_\Delta(\tilde{s},t,\tilde{x},y) + R_\Delta(\tilde{s},t,\tilde{x},y), \\ \Psi^C_\Delta(\tilde{s},t,\tilde{x},y) &:= \mathbb{P}_{\tilde{s},\tilde{x}}[\tau^{\Delta,Y} \leq t, (Y_{\tau^{\Delta,Y}})^- \geq y\sqrt{\Delta}]. \end{aligned} \tag{3.3}$$

From **($\mathbf{A}'_\theta$-2')** that guarantees that Y has a non degenerate variance and Lemma 3.3, one gets

$$\sup_{(\tilde{s},\tilde{x})\in\mathcal{A}^{\alpha,\varepsilon}} |\Psi^C_\Delta(\tilde{s},t,\tilde{x},y) - \mathbb{P}_{\tilde{s},\tilde{x}}[\tau^{\Delta,Y} \leq t](1 - H(y/|(\sigma^*\nabla F)(\tilde{s},\tilde{x})|))| \underset{\Delta\to 0}{\longrightarrow} 0,$$

where $\mathcal{A}^{\alpha,\varepsilon} := \big(V_{\partial\mathcal{D}}(\Delta^\alpha) \backslash V_{\partial\mathcal{D}}(\Delta^{1/2-\varepsilon})\big) \cap \mathcal{D}$. Plug now this identity in (3.3) to obtain with the same uniformity

$$\begin{aligned} \Psi_\Delta(\tilde{s},t,\tilde{x},y) =& \mathbb{P}_{\tilde{s},\tilde{x}}[\tau^{\Delta,Y} \leq t](1 - H(y/|(\sigma^*\nabla F)(\tilde{s},\tilde{x})|)) \\ &+ R_\Delta(\tilde{s},t,\tilde{x},y) + o(1). \end{aligned} \tag{3.4}$$

Step 3. Control of the rests. We now show that $R_\Delta(\tilde{s},t,\tilde{x},y) = o(1)$ where the rest is still uniform for $(\tilde{s},\tilde{x}) \in \mathcal{A}^{\alpha,\varepsilon}$. Write first:

$$\begin{aligned} |R_\Delta|(\tilde{s},t,\tilde{x},y) \leq\ & R^1_\Delta(\tilde{s},t,\tilde{x}) \\ &+ \mathbb{P}_{\tilde{s},\tilde{x}}[\tau^\Delta \leq t, F(\tau^\Delta, X^\Delta_{\tau^\Delta})^- \geq y\sqrt{\Delta}, (Y_{\tau^{\Delta,Y}})^- < y\sqrt{\Delta}, \tau^\Delta = \tau^{\Delta,Y}] \\ &+ \mathbb{P}_{\tilde{s},\tilde{x}}[\tau^\Delta \leq t, F(\tau^\Delta, X^\Delta_{\tau^\Delta})^- < y\sqrt{\Delta}, (Y_{\tau^{\Delta,Y}})^- \geq y\sqrt{\Delta}, \tau^\Delta = \tau^{\Delta,Y}] \end{aligned} \tag{3.5}$$

with $R^1_\Delta(\tilde{s},t,\tilde{x}) \leq \mathbb{P}_{\tilde{s},\tilde{x}}[\tau^\Delta \leq t, \tau^\Delta \neq \tau^{\Delta,Y}] + \mathbb{P}_{\tilde{s},\tilde{x}}[\tau^{\Delta,Y} \leq t, \tau^\Delta \neq \tau^{\Delta,Y}] := (R^{11}_\Delta + R^{12}_\Delta)(\tilde{s},t,\tilde{x})$. Let y_Δ be a given positive function of the time step s.t. $y_\Delta \underset{\Delta\to 0}{\to} 0$ specified later on.
On the event $\{\tau^\Delta = \tau^{\Delta,Y}, |Y_{\tau^{\Delta,Y}} - F(\tau^{\Delta,Y}, X^\Delta_{\tau^{\Delta,Y}})| \leq y_\Delta\sqrt{\Delta}\}$ the conditions $F(\tau^\Delta, X^\Delta_{\tau^\Delta})^- \geq y\sqrt{\Delta}$ and $(Y_{\tau^{\Delta,Y}})^- < y\sqrt{\Delta}$ imply $\Delta^{-1/2}(Y_{\tau^{\Delta,Y}})^- \in [y - y_\Delta, y)$ (resp. $(Y_{\tau^{\Delta,Y}})^- \geq y\sqrt{\Delta}$ and $F(\tau^\Delta, X^\Delta_{\tau^\Delta})^- < y\sqrt{\Delta}$ imply $\Delta^{-1/2}(Y_{\tau^{\Delta,Y}})^- \in [y, y + y_\Delta)$). Hence,

$$\begin{aligned} &|R_\Delta(\tilde{s},t,\tilde{x},y)| \leq (R^1_\Delta + R^2_\Delta)(\tilde{s},t,\tilde{x}) \\ &+ \mathbb{P}_{\tilde{s},\tilde{x}}[\tau^{\Delta,Y} \leq t, \Delta^{-1/2}(Y_{\tau^{\Delta,Y}})^- \in [y - y_\Delta, y + y_\Delta), \tau^\Delta = \tau^{\Delta,Y}] \\ &:= (R^1_\Delta + R^2_\Delta)(\tilde{s},t,\tilde{x}) + R^3_\Delta(\tilde{s},t,\tilde{x},y), \end{aligned}$$

with $R^2_\Delta(\tilde{s},t,\tilde{x}) := 2\mathbb{P}_{\tilde{s},\tilde{x}}[\tau^{\Delta,Y} \le t, \tau^\Delta = \tau^{\Delta,Y}, |Y_{\tau^{\Delta,Y}} - F(\tau^{\Delta,Y}, X^\Delta_{\tau^{\Delta,Y}})| > y_\Delta\sqrt{\Delta}]$.

Term $R^3_\Delta(\tilde{s},t,\tilde{x},y)$. From Lemma 3.5 applied to the process with frozen coefficients, one gets

$$R^3_\Delta(\tilde{s},t,\tilde{x},y) \le C(y_\Delta + \Delta^{1/4}). \tag{3.6}$$

Term $R^2_\Delta(\tilde{s},t,\tilde{x})$. Introduce for $0 < \beta < \alpha < 1/2$, $\tau_{\Delta^\beta} := \inf\{s \ge \tilde{s} : |X^\Delta_s - \tilde{x}| \ge \Delta^\beta\} \wedge (\tilde{s} + \Delta^\delta)$, $\delta := 2\beta + \gamma$, $\gamma > 0$. One has

$$\begin{aligned}|R^2_\Delta(\tilde{s},t,\tilde{x})| \le 2\Big(&\mathbb{P}_{\tilde{s},\tilde{x}}[\tau^{\Delta,Y} \le t, \tau^\Delta = \tau^{\Delta,Y}, \tau^\Delta < \tau_{\Delta^\beta},\\ &|Y_{\tau^{\Delta,Y}} - F(\tau^{\Delta,Y}, X^\Delta_{\tau^{\Delta,Y}})| > y_\Delta\sqrt{\Delta}] + \mathbb{P}_{\tilde{s},\tilde{x}}[\tau^\Delta \ge \tau_{\Delta^\beta}, \tau^\Delta \le t]\Big)\\ &:= (R^{21}_\Delta + R^{22}_\Delta)(\tilde{s},t,\tilde{x}).\end{aligned}$$

Let us first deal with $R^{21}_\Delta(\tilde{s},t,\tilde{x})$. One has

$$\begin{aligned}&R^{21}_\Delta(\tilde{s},t,\tilde{x}) \le\\ &2\Delta^{-1}y_\Delta^{-2}\mathbb{E}_{\tilde{s},\tilde{x}}[\mathbf{1}_{\tau^\Delta < \tau_{\Delta^\beta}, \tau^{\Delta,Y} \le t, \tau^\Delta = \tau^{\Delta,Y}} |Y_{\tau^{\Delta,Y}} - F(\tau^{\Delta,Y}, X^\Delta_{\tau^{\Delta,Y}})|^2].\end{aligned} \tag{3.7}$$

Note that since $\mathcal{D}$ is of class $\mathbf{H}_2$, F has the same regularity, i.e. it is uniformly Lipschitz continuous in time and its first space derivatives are uniformly Lipschitz continuous in space. Thus, assuming up to a regularization procedure that $F \in C^{1,2}([0,T]\times\mathbb{R}^d)$, Itô's formula yields for all $t \ge \tilde{s}$,

$$\begin{aligned}F(t,X^\Delta_t) &= F(\tilde{s},\tilde{x}) + \int_{\tilde{s}}^t \nabla F(s,X^\Delta_s)\cdot dX^\Delta_s\\ &\quad + \int_{\tilde{s}}^t (\partial_s F(s,X^\Delta_s) + \frac{1}{2}\mathrm{tr}(H_F(s,X^\Delta_s)\sigma\sigma^*(\phi(s),X^\Delta_{\phi(s)})))ds\\ &:= F(\tilde{s},\tilde{x}) + \int_{\tilde{s}}^t \nabla F(s,X^\Delta_s)\cdot\sigma(\phi(s),X^\Delta_{\phi(s)})dW_s + R^\Delta_F(\tilde{s},t,\tilde{x}) \qquad (3.8)\\ &= Y_t + R^\Delta_F(\tilde{s},t,\tilde{x}) + \int_{\tilde{s}}^t (\sigma^*(\phi(s),X^\Delta_{\phi(s)})\nabla F(s,X^\Delta_s) - \sigma^*\nabla F(\tilde{s},\tilde{x}))\cdot dW_s.\end{aligned}$$

From $(\mathbf{A}'_\theta\text{-}\mathbf{1})$ and the assumptions on $\mathcal{D}$ one derives $|R^\Delta_F|(\tilde{s},t,\tilde{x}) \le C(t-\tilde{s})$. Thus, for a given stopping time $\tilde{s} \le U \le \tau_{\Delta^\beta}$, the working assumptions (i.e. smoothness of σ, F), standard computations and the BDG inequalities yield

$$\mathbb{E}[|F(U,X^\Delta_U) - Y_U|^2] \le C(\Delta^{2\beta+\delta} + \Delta^{\delta(1+\theta)}).$$

From (3.7) and the above control with $U = \tau^{\Delta,Y} \wedge \tau_{\Delta^\beta}$, one obtains

$$R_\Delta^{21}(\tilde{s},t,\tilde{x}) \leq C y_\Delta^{-2}\Delta^{-1}(\Delta^{2\beta+\delta} + \Delta^{\delta(1+\theta)}). \tag{3.9}$$

Let us now control $R_\Delta^{22}(\tilde{s},t,\tilde{x})$. From Lemmas 3.1 and 3.4, for any $\eta > 0$ we write

$$\begin{aligned} R_\Delta^{22}(\tilde{s},t,\tilde{x}) &\leq \mathbb{P}_{\tilde{s},\tilde{x}}[\tau_{\Delta^\beta} < \tilde{s} + \Delta^\delta] + \mathbb{P}_{\tilde{s},\tilde{x}}[\tau^\Delta \geq \tilde{s} + \Delta^\delta]\mathbf{1}_{\tilde{s}+\Delta^\delta \leq t} \\ &\leq C\big(\exp\left(-c\Delta^{2\beta-\delta}\right) + \Delta^{\alpha-\eta-\delta/2} + \Delta^{\delta/2}\big). \end{aligned} \tag{3.10}$$

Take now $\alpha = \frac{1+\frac{\theta}{2}}{2(1+\theta)} < 1/2,\ \eta = \frac{\theta}{16(\theta+1)},\ \gamma = \frac{1}{8(1+\theta)},\ y_\Delta = \Delta^{\theta/16}$. Check that for $\delta = 2\beta+\gamma = 2\alpha - 4\eta$, one has $\delta = \frac{1+\theta/4}{1+\theta},\ \beta = \frac{7/8+\theta/4}{2(1+\theta)} < \alpha,\ 3\eta < \alpha$. Thus, $R_\Delta^{22}(\tilde{s},t,\tilde{x}) = O(\Delta^\eta)$. In addition, $y_\Delta^{-2}\Delta^{\delta(1+\theta)-1} = \Delta^{\theta/8}, y_\Delta^{-2}\Delta^{2\beta+\delta-1} = O(\Delta^{1/(8(1+\theta))})$. Hence, from (3.9) and (3.10)

$$R_\Delta^{2}(\tilde{s},t,\tilde{x}) \leq C\big(\Delta^{1/(8(1+\theta))} + \Delta^{\theta/8} + \Delta^{\theta/(16(\theta+1))}\big) \leq C\Delta^{\theta/32}. \tag{3.11}$$

Term $R_\Delta^1(\tilde{s},t,\tilde{x})$. We give an upper bound for $R_\Delta^{11}(\tilde{s},t,\tilde{x})$. The term $R_\Delta^{12}(\tilde{s},t,\tilde{x})$ can be handled in the same way. From the previous control on $R_\Delta^{22}(\tilde{s},t,\tilde{x})$ and for the previous parameters, one gets

$$\begin{aligned} R_\Delta^{11}(\tilde{s},t,\tilde{x}) &= \mathbb{P}_{\tilde{s},\tilde{x}}[\tau^\Delta \leq t, \tau^\Delta \neq \tau^{\Delta,Y}, \tau^\Delta < \tau_{\Delta^\beta}] + O(\Delta^\eta) \\ &= \mathbb{P}_{\tilde{s},\tilde{x}}[\tau^\Delta \leq t, \tau^\Delta > \tau^{\Delta,Y}, \tau^\Delta < \tau_{\Delta^\beta}] \\ &\quad + \mathbb{P}_{\tilde{s},\tilde{x}}[\tau^\Delta \leq t, \tau^\Delta < \tau^{\Delta,Y}, \tau^\Delta < \tau_{\Delta^\beta}] + O(\Delta^\eta). \end{aligned}$$

Note that,

$$\begin{aligned} &R_\Delta^{11}(\tilde{s},t,\tilde{x}) \leq \mathbb{P}_{\tilde{s},\tilde{x}}[\tau^{\Delta,Y} \leq t, \Delta^{-1/2}(Y_{\tau^{\Delta,Y}})^- \leq y_\Delta] + \\ &\mathbb{P}_{\tilde{s},\tilde{x}}[\tau^\Delta \leq t, \tau^{\Delta,Y} < \tau^\Delta, \tau^\Delta < \tau_{\Delta^\beta}, \Delta^{-1/2}|Y_{\tau^{\Delta,Y}} - F(\tau^{\Delta,Y}, X^\Delta_{\tau^{\Delta,Y}})| \geq y_\Delta] \\ &+ \mathbb{P}_{\tilde{s},\tilde{x}}[\tau^\Delta \leq t, \tau^{\Delta,Y} > \tau^\Delta, \tau^\Delta < \tau_{\Delta^\beta}, \Delta^{-1/2}|Y_{\tau^\Delta} - F(\tau^\Delta, X^\Delta_{\tau^\Delta})| \geq y_\Delta] \\ &+ \mathbb{P}_{\tilde{s},\tilde{x}}[\tau^\Delta \leq t, \Delta^{-1/2}F(\tau^\Delta, X^\Delta_{\tau^\Delta})^- \leq y_\Delta] + C\Delta^\eta, \end{aligned}$$

for the previous function $(y_\Delta)_{\Delta>0}$. Since we could obtain the same type of bound for $R_\Delta^{12}(\tilde{s},t,\tilde{x})$, from Lemma 3.5 and following the computations that gave (3.9) we derive for the previous set of parameters

$$R_\Delta^{1}(\tilde{s},t,\tilde{x}) \leq C(y_\Delta^{-2}\Delta^{-1}(\Delta^{2\beta+\delta} + \Delta^{\delta(1+\theta)}) + \Delta^\eta + y_\Delta + \Delta^{1/4}) \leq C\Delta^{\theta/32}. \tag{3.12}$$

From (3.12), (3.11), (3.6) we finally obtain $R_\Delta(\tilde{s},t,\tilde{x},y) = O(\Delta^{\theta/32}) = o(1)$. The rest is uniform w.r.t. $(\tilde{s},\tilde{x},y) \in \mathcal{A}^{\alpha,\varepsilon} \times \mathbb{R}^+$.

Step 4. Final step. Plug the previous results in (3.4). We derive from (3.2)

$$\Psi_\Delta(t,x,y) = \mathbb{E}_x[\mathbf{1}_{A(t,\alpha,\varepsilon)}\varphi(X^\Delta_{\tau_{\Delta^\alpha}}) \times \mathbb{P}_{\tau_{\Delta^\alpha},X^\Delta_{\tau_{\Delta^\alpha}}}[\tau^{\Delta,Y} \le t](1 - H(y/|\sigma^*\nabla F(\tau_{\Delta^\alpha}, X^\Delta_{\tau_{\Delta^\alpha}})|))] + o(1).$$

The previous controls on $R^1_\Delta(\tilde{s},t,\tilde{x})$, $R^{22}_\Delta(\tilde{s},t,\tilde{x})$ give

$$\Psi_\Delta(t,x,y) = \mathbb{E}_x[\mathbf{1}_{\tau_{\Delta^\alpha}<t}\mathbb{P}_{\tau_{\Delta^\alpha},X^\Delta_{\tau_{\Delta^\alpha}}}[\tau^\Delta \le t, \tau_{\Delta^\beta} > \tau^\Delta] \varphi(X^\Delta_{\tau_{\Delta^\alpha}})(1 - H(y/|\sigma^*\nabla F(\tau_{\Delta^\alpha}, X^\Delta_{\tau_{\Delta^\alpha}})|))] + o(1).$$

Under $(\mathbf{A}'_\theta)$, by continuity arguments and Lemma 3.1 we eventually get

$$\Psi_\Delta(t,x,y) = \mathbb{E}_x[\mathbf{1}_{\tau^\Delta \le t}\varphi(X^\Delta_{\tau^\Delta})(1 - H(y/|\sigma^*\nabla F(\tau^\Delta, X^\Delta_{\tau^\Delta})|))] + o(1).$$

Now, Lemma 3.2 gives

$$\Psi_\Delta(t,x,y) \underset{\Delta\to 0}{\to} \mathbb{E}_x[\mathbf{1}_{\tau\le t}\varphi(X_\tau)(1 - H(y/|\sigma^*\nabla F(\tau, X_\tau)|))].$$

□

3.2. *Proof of Lemmas 3.3, 3.4 and 3.5.*

Proof of Lemma 3.3. We shall insist on the dependence of the exit times with respect to x, by setting $\tau^\Delta := \inf\{s \in \Delta\mathbf{N}^* : W_s \ge x\} := \tau^\Delta_x$ and analogously for $\tau = \tau_x$.

Our proof relies on the following convergence (see equation (19) in Siegmund [Sie79]): if we set (for any $y, z \ge 0$)

$$D(z,y) = \mathbb{P}_0[W_{\tau^\Delta_z} - z \le y\sqrt{\Delta}] - H(y),$$

then

$$\lim_{z\Delta^{-1/2}\to+\infty} |D(z,y)| = 0.$$

Using the monotonicity and the uniform continuity of $H(y)$, Dini's Theorem yields that the above limit is actually uniform with respect to $y \ge 0$. It follows

$$\sup_{y\ge 0, z\in[\Delta^{1/2-\varepsilon/3},\infty)} |D(z,y)| \underset{\Delta\to 0}{\to} 0. \tag{3.13}$$

By similar monotonicity arguments,

$$\sup_{x\geq 0, t\geq 0} |\mathbb{P}_0(\tau_x^\Delta > t) - \mathbb{P}_0(\tau_x > t)| \underset{\Delta\to 0}{\to} 0. \tag{3.14}$$

We can now proceed to the proof. First, note that if $x/\sqrt{t} \geq \Delta^{-\varepsilon/3} \to +\infty$ as $\Delta \to 0$, $\mathbb{P}_0(\tau_x^\Delta > t)$ and $\mathbb{P}_0(\tau_x > t)$ are both $O_{pol}(\Delta)$. Thus, the difference in Lemma 3.3 converges to 0 as $\Delta \to 0$.

Suppose now that $x/\sqrt{t} \leq \Delta^{-\varepsilon/3}$, hence $\sqrt{t} \geq x\Delta^{\varepsilon/3} \geq \Delta^{1/2-2\varepsilon/3}$, and write for $t \in \Delta\mathbf{N}^*$

$$\begin{aligned} P &:= \mathbb{P}_0[\tau_x^\Delta > t, W_{\tau_x^\Delta} - x \leq y\sqrt{\Delta}] \\ &= \int_0^{+\infty} dz q_t^{x,\Delta}(0, x-z)\mathbb{P}_0[W_{\tau_z^\Delta} - z \leq y\sqrt{\Delta}] \end{aligned}$$

where $q_t^{x,\Delta}$ denotes the transition density of the discretely killed Brownian motion. Introduce the partition $\mathbb{R}^+ = \{z \in [0, \Delta^{1/2-\varepsilon/3})\} \cup \{z \in [\Delta^{1/2-\varepsilon/3}, +\infty)\}$. Then,

$$P = R + \int_{\Delta^{1/2-\varepsilon/3}}^{+\infty} q_t^{x,\Delta}(0, x-z)D(z,y)dz + \mathbb{P}_0[\tau_x^\Delta > t]H(y)$$

where $|R| \leq 2\mathbb{P}_0[W_t \in [x - \Delta^{1/2-\varepsilon/3}, x]] \leq \frac{2}{\sqrt{2\pi t}}\Delta^{1/2-\varepsilon/3} \leq \frac{2}{\sqrt{2\pi}}\Delta^{\varepsilon/3}$. Finally, taking advantage of the estimates (3.13) and (3.14) readily completes our proof. □

Proof of Lemma 3.4. We take $s = 0$ for notational simplicity. Introduce $\tau_{\Delta^\beta} := \inf\{t \geq 0 : (t, X_t^\Delta) \notin V_{\partial\mathcal{D}}(\Delta^\beta)\}$ and for $\gamma > 0$ write from Lemma 3.1 and the notation of (3.8) (up to the same regularization procedure concerning F)

$$\begin{aligned} \mathbb{P}_x[\tau^\Delta \geq \Delta^{2\beta}] =& \mathbb{P}_x[\inf_{i\in[\![0,\Delta^{2\beta-1}]\!]} F(0,x) + \int_0^{t_i} \nabla F(s, X_s^\Delta)\cdot\sigma(\phi(s), X_{\phi(s)}^\Delta)dW_s \\ &+ R_F^\Delta(0, t_i, x) \geq 0, \tau_{\Delta^\beta} \geq \Delta^{2\beta+\gamma}] + O_{pol}(\Delta) := Q, \end{aligned}$$

where under the assumptions of the Lemma, $|R_F^\Delta(0, t_i, x)| \leq Ct_i$. For a given $r > 0$, consider the event $\mathcal{A}_r = \{\exists s \leq T : |X_s^\Delta - X_{\phi(s)}^\Delta| \geq r\}$ where the increments of X^Δ between two close times are large: by Lemma 3.1, it has an exponentially small probability. Hence, if we set

$$M_u := \int_0^u \nabla F(s, X_s^\Delta)\cdot\sigma(\phi(s), X_{\phi(s)}^\Delta)dW_s := B_{<M>_u},\ \tilde{t}_i = \langle M\rangle_{t_i},$$

B is a standard Brownian motion (on a possibly enlarged probability space) owing to the Dambis, Dubbins-Schwarz Theorem, cf. Theorem V.1.7 in [RY99].

In addition, the above time change is strictly increasing on the set $\mathcal{A}_r^c$ and $\langle M\rangle_t - \langle M\rangle_s \geq (t-s)a_0/2$ $(t \geq s)$ up to taking r small enough, because **$(\mathbf{A}_\alpha'$-2)** is in force. It readily follows that

$$\begin{aligned}
Q \leq & \mathbb{P}_x[\inf_{i\in[\![0,\Delta^{2\beta+\gamma-1}]\!]} M_{t_i} + Ct_i \geq -\Delta^\alpha, \tau_{\Delta^\beta} \geq \Delta^{2\beta+\gamma}] + O_{pol}(\Delta) \\
\leq & \mathbb{P}_x[\inf_{i\in[\![0,\Delta^{2\beta+\gamma-1}]\!]} B_{\tilde{t}_i} + 2Ca_0^{-1}\tilde{t}_i \geq -\Delta^\alpha, \tau_{\Delta^\beta} \geq \Delta^{2\beta+\gamma}, \mathcal{A}_r^c] + O_{pol}(\Delta) \\
\leq & \mathbb{P}_x[\inf_{i\in[\![0,\Delta^{2\beta+\gamma-1}]\!]} B_{\tilde{t}_i} + 2Ca_0^{-1}\tilde{t}_i \geq -\Delta^\alpha, \tau_{\Delta^\beta} \geq \Delta^{2\beta+\gamma}, \\
& \qquad \inf_{s\in[0,\langle M\rangle_{\Delta^{2\beta+\gamma}}]} B_s + 2Ca_0^{-1}s \leq -\Delta^{\alpha-\zeta}, \mathcal{A}_r^c] + O_{pol}(\Delta) \\
& + \mathbb{P}_x[\tau_{\Delta^\beta} \geq \Delta^{2\beta+\gamma}, \inf_{s\in[0,\langle M\rangle_{\Delta^{2\beta+\gamma}}]} B_s + 2Ca_0^{-1}s \geq -\Delta^{\alpha-\zeta}, \mathcal{A}_r^c],
\end{aligned}$$

for $\zeta > 0$. Thus, from Lemma 3.1 and standard controls

$$\begin{aligned}
Q \leq & \mathbb{P}_x[\exists i \in [\![0,\Delta^{2\beta+\gamma-1}]\!], \sup_{s\in[\tilde{t}_i,\tilde{t}_{i+1}]} |B_s - B_{\tilde{t}_i} + 2Ca_0^{-1}(s-\tilde{t}_i)| \geq \Delta^{\alpha-\zeta} - \Delta^\alpha, \\
& \tau_{\Delta^\beta} \geq \Delta^{2\beta+\gamma}] + \mathbb{P}_x[\inf_{s\in[0,a_0\Delta^{2\beta+\gamma}/2]} B_s \geq -\Delta^{\alpha-\zeta} - C\Delta^{2\beta+\gamma}] + O_{pol}(\Delta) \\
& \leq O_{pol}(\Delta) + C(\Delta^{\alpha-\zeta-\beta-\gamma/2} + \Delta^{\beta+\gamma/2}).
\end{aligned}$$

Choose now γ, ζ s.t. $(\zeta + \frac{\gamma}{2}) = \eta > 0$. The proof is complete. □

Proof of Lemma 3.5. Taking also $s = 0$ for notational convenience, we write

$$\begin{aligned}
P := & \mathbb{P}_x[\tau^\Delta \leq t, \Delta^{-1/2}F^-(\tau^\Delta, X_{\tau^\Delta}^\Delta) \in [a,b]] \leq O_{pol}(\Delta) \\
& + \sum_{i=1}^{\lfloor t/\Delta\rfloor} \mathbb{E}_x[\mathbf{1}_{\tau^\Delta > t_{i-1}}, \mathbf{1}_{(t_{i-1},X_{t_{i-1}}^\Delta)\in V_{\partial\mathcal{D}}(r_0)} \mathbb{E}_{\mathcal{F}_{t_{i-1}}}[\mathbf{1}_{\Delta^{-1/2}F(t_i,X_{t_i}^\Delta)^-\in[a,b]}]] \qquad (3.15)
\end{aligned}$$

using Lemma 3.1 for the last identity.

A Taylor formula gives: $F(t_i, X_{t_i}^\Delta) = F(t_{i-1}, X_{t_{i-1}}^\Delta) + \Sigma_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}) + R_{t_{i-1},t_i}^\Delta := \mathcal{N}_{t_{i-1}} + R_{t_{i-1},t_i}^\Delta$ where $\Sigma_{t_{i-1}} = \sigma^*\nabla F(t_{i-1}, X_{t_{i-1}}^\Delta)$, $\mathbb{E}_{\mathcal{F}_{t_{i-1}}}[|R_{t_{i-1},t_i}^\Delta|^2] \leq C\Delta^2$. Conditionally to $\mathcal{F}_{t_{i-1}}$, $\mathcal{N}_{t_{i-1}}$ has a Gaussian distribution $\mathcal{N}(F(t_{i-1}, X_{t_{i-1}}^\Delta), \|\Sigma_{t_{i-1}}\|^2\Delta)$.

Also, on the event $(t_{i-1}, X_{t_{i-1}}^\Delta) \in V_{\partial\mathcal{D}(r_0)}$, $\|\Sigma_{t_{i-1}}\|^2\Delta \geq a_0\Delta$. Set $Q_{i-1} :=$

$\mathbb{P}_{\mathcal{F}_{t_{i-1}}}[F(t_i, X^\Delta_{t_i})^- \in [a\Delta^{1/2}, b\Delta^{1/2}]]$. We obtain

$$\begin{aligned}
Q_{i-1} &= \mathbb{P}_{\mathcal{F}_{t_{i-1}}}[(\mathcal{N}_{t_{i-1}} + R^\Delta_{t_{i-1},t_i})^- \in [a\Delta^{1/2}, b\Delta^{1/2}]] \\
&\le \mathbb{P}_{\mathcal{F}_{t_{i-1}}}[\mathcal{N}_{t_{i-1}} \in [-b\Delta^{1/2} - \Delta^{3/4}, -a\Delta^{1/2} + \Delta^{3/4}]] \\
&\quad + \mathbb{P}_{\mathcal{F}_{t_{i-1}}}[|R^\Delta_{t_{i-1},t_i}| \ge \Delta^{3/4}, X^\Delta_{t_i} \notin \mathcal{D}_{t_i}] \\
&\le \mathbb{P}_{\mathcal{F}_{t_{i-1}}}[\mathcal{N}_{t_{i-1}} \in [-\Delta^{1/2}(b + \Delta^{1/4}), -\Delta^{1/2}(a - \Delta^{1/4})]] \\
&\quad + C\Delta^{1/4} \exp\left(-c\frac{d(X^\Delta_{t_{i-1}}, \partial\mathcal{D}_{t_{i-1}})^2}{\Delta}\right)
\end{aligned}$$

using the Cauchy-Schwarz inequality and Lemma 3.1 for the last inequality. Hence, we derive from (3.15)

$$\begin{aligned}
P \le \sum_{i=1}^{\lfloor t/\Delta \rfloor} \mathbb{E}_x[\mathbf{1}_{\tau^\Delta > t_{i-1}, (t_{i-1}, X^\Delta_{t_{i-1}}) \in V_{\partial\mathcal{D}}(r_0)} \Bigg(C\Delta^{1/4} \exp\left(-c\frac{d(X^\Delta_{t_{i-1}}, \partial\mathcal{D}_{t_{i-1}})^2}{\Delta}\right) \\
+ \int_{-\Delta^{1/2}(b+\Delta^{1/4})}^{-\Delta^{1/2}(a-\Delta^{1/4})} \exp\left(-\frac{(y - F(t_{i-1}, X^\Delta_{t_{i-1}}))^2}{2\|\Sigma_{t_{i-1}}\|^2\Delta}\right) \frac{dy}{(2\pi\Delta)^{1/2}\|\Sigma_{t_{i-1}}\|}\Bigg)] \\
+ O_{pol}(\Delta) \le C(b - a + \Delta^{1/4})\Delta^{-1} \int_0^t ds \mathbb{E}_x[\mathbf{1}_{\tau^\Delta > \phi(s), (\phi(s), X^\Delta_{\phi(s)}) \in V_{\partial\mathcal{D}(r_0)}} \\
\times \exp\left(-c\frac{d(X^\Delta_{\phi(s)}, \partial\mathcal{D}_{\phi(s)})^2}{\Delta}\right)] + O_{pol}(\Delta).
\end{aligned}$$

Following the proof of Lemma 10 in [GM04], mainly based on the application of the occupation time formula for the distance process $F(s, X^\Delta_s)$, one can show that the above integral is bounded by $C\Delta$. This completes the proof. □

Remark 3.6 *Finally, we mention that if $\sigma\sigma^*$ is uniformly elliptic, the rest $R^\Delta_{t_{i-1},t_i}$ can be avoided and the result can be stated without the contribution $\Delta^{1/4}$. Indeed, we can directly exploit that the Euler scheme has conditionally a non degenerate Gaussian distribution and usual changes of chart associated to a parametrization of the boundary (see e.g. [Gob00]) give the expected result.*

4. Extension to the stationary case.

4.1. *Framework.* In this section we assume that the coefficients in (1.1) are time independent and that the mappings b, σ are uniformly Lipschitz

continuous, i.e. $(X_t)_{t\geq 0}$ is the unique strong solution of

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, t \geq 0,\ x \in \mathbb{R}^d.$$

For a bounded domain $D \subset \mathbb{R}^d$, and given functions $f, g, k : \bar{D} \to \mathbb{R}$, we are interested in estimating

$$u(x) := \mathbb{E}_x[g(X_\tau)Z_\tau + \int_0^\tau f(X_s)Z_s ds],\ Z_s = \exp(-\int_0^s k(X_r)dr), \tag{4.1}$$

where $\tau := \inf\{t \geq 0 : X_t \notin D\}$.

Adapting freely the previous notations for Hölder spaces to the elliptic setting, introduce for $\theta \in]0, 1[$:

$(\mathbf{A}_\theta)$ **1.** *Smoothness of the coefficients.* $b, \sigma \in \mathbf{H}_{1+\theta}$.

2. *Uniform ellipticity.* For some $a_0 > 0$, $\forall (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$, $\xi^* \sigma\sigma^*(x)\xi \geq a_0|\xi|^2$.

$(\mathbf{D}_\theta)$ *Smoothness of the domain.* The bounded domain D is of class $\mathbf{H}_2$.

$(\mathbf{C}_\theta)$ *Other coefficients.* The boundary data $g \in \mathbf{H}_{1+\theta}$, $f, k \in \mathbf{H}_{1+\theta}$ and $k \leq 0$.

Note that under $(\mathbf{A}_\theta)$ and since D is bounded, Lemma 3.1 Chapter III of [Fre85] yields $\sup_{x\in\bar{D}} \mathbb{E}_x[\tau] < \infty$. Thus, (4.1) is well defined under our current assumptions.

From Theorem 6.13, the final notes of Chapter 6 in [GT98] and Theorem 2.1 Chapter II in Freidlin [Fre85], the Feynman-Kac representation in our elliptic setting writes

Proposition 4.1 (Elliptic Feynman-Kac's formula and estimates) *Assume* $(\mathbf{A}_\theta)$, $(\mathbf{D}_\theta)$, $(\mathbf{C}_\theta)$ *are in force. Then, there is a unique solution of class* $\mathbf{H}_{1+\theta} \cap \mathcal{C}^2(D)$ *to*

$$\begin{cases} Lu - ku + f = 0, \text{ in } D, \\ u|_{\partial D} = g \end{cases} \tag{4.2}$$

and it is given by (4.1).

In the following we denote by $F(x)$ the signed spatial distance to the boundary ∂D. Under $(\mathbf{D}_\theta)$, D satisfies the exterior and interior uniform sphere condition with radius $r_0 > 0$ and $F \in \mathbf{H}_2(V_{\partial D}(r_0))$ where $V_{\partial D}(r_0) := \{x \in \mathbb{R}^d : d(x, \partial D) \leq r_0\}$. Also, F can be extended to a $\mathbf{H}_2$ function preserving the sign. For more details on the distance function, we refer to Appendix 14.6 in [GT98].

4.2. *Tools and results.* Below, we keep the previous notations concerning the Euler scheme. We also use the symbol C for nonnegative constants that may depend on D, b, σ, g, f, k but not on Δ or x. We reserve the notation c for constants also independent of D, g, f, k. Let us now state the main results of Section 2 in our current framework.

Proposition 4.2 (Tightness of the overshoot) *Assume* **($\mathbf{A}_\theta$-2)**, *and that D is of class* $\mathbf{H}_2$. *Then, for some $c > 0$,*

$$\sup_{\Delta>0} \mathbb{E}_x[\exp(c[\Delta^{-1/2} F^-(X^\Delta_{\tau^\Delta})]^2)] < +\infty.$$

From the proof of Theorem 2.2 and Theorem 4.2 in Gobet and Maire [GM05] we derive:

Theorem 4.3 (Joint limit laws associated to the overshoot) *Assume* **($\mathbf{A}_\theta$)**, *and that D is of class* $\mathbf{H}_2$. *Let φ be a continuous function with compact support. With the notation of Theorem 2.2, for all $x \in D$, $y \geq 0$,*

$$\mathbb{E}_x[Z^\Delta_{\tau^\Delta} \varphi(X^\Delta_{\tau^\Delta}) \mathbf{1}_{F^-(X^\Delta_{\tau^\Delta}) \geq y\sqrt{\Delta}}] \underset{\Delta\to 0}{\longrightarrow} \mathbb{E}_x\big[Z_\tau \varphi(X_\tau)(1 - H(y/|\sigma^* \nabla F(X_\tau)|))\big].$$

4.3. *Error expansion and boundary correction.* For notational convenience introduce for $x \in D$,

$$u(D) = \mathbb{E}_x(g(X_\tau) Z_\tau + \int_0^\tau Z_s f(X_s) ds),$$
$$u^\Delta(D) = \mathbb{E}_x(g(X^\Delta_{\tau^\Delta}) Z^\Delta_{\tau^\Delta} + \int_0^{\tau^\Delta} Z^\Delta_{\phi(s)} f(X^\Delta_{\phi(s)}) ds).$$

The second quantity is well defined according to Theorem 4.2 in Gobet and Maire [GM05] that states

$$\forall p \geq 1, \ \limsup_{\Delta\to 0} \sup_{x\in\bar{D}} \mathbb{E}_x[(\tau^\Delta)^p] < \infty. \tag{4.3}$$

Theorem 4.4 (First order expansion) *Under* **($\mathbf{A}_\theta$)**, **($\mathbf{D}_\theta$)**, **($\mathbf{C}_\theta$)**, *for Δ small enough and with the notation of Theorem 2.3*

$$\begin{aligned}&\mathrm{Err}(\Delta, g, f, k, x) = u^\Delta(D) - u(D)\\ &= c_0 \sqrt{\Delta} \mathbb{E}_x(Z_\tau (\nabla u - \nabla g)(X_\tau) \cdot \nabla F(X_\tau) |\sigma^* \nabla F(X_\tau)|) + o(\sqrt{\Delta}).\end{aligned}$$

Define now $D^\Delta = \{x \in D : \mathbf{d}(x, \partial D) > c_0 \sqrt{\Delta} |\sigma^* \nabla F(x)|\}$. Introduce $\hat{\tau}^\Delta = \inf\{t_i \geq 0 : X^\Delta_{t_i} \in D^\Delta\}$. Set

$$u^\Delta(D^\Delta) = \mathbb{E}_x[g(X^\Delta_{\hat{\tau}^\Delta}) Z^\Delta_{\hat{\tau}^\Delta} + \int_0^{\hat{\tau}^\Delta} Z^\Delta_{\phi(s)} f(X^\Delta_{\phi(s)}) ds].$$

One has:

Theorem 4.5 (Boundary correction) *Under* $(\mathbf{A}_\theta)$, $(\mathbf{D}_\theta)$, $(\mathbf{C}_\theta)$ *and assuming additionally* $\nabla F(x)|\sigma^*\nabla F(x)|$ *is of class* $\mathcal{C}^2$, *then for* Δ *small enough one has*

$$u^\Delta(D^\Delta) - u(D) = o(\sqrt{\Delta}).$$

4.4. *Proofs.* Note carefully that all the constants appearing in the error analysis for the parabolic case have at most linear growth w.r.t the fixed final time T. Equation (4.3) allows to control uniformly the integrability of these constants in our current framework. Thus, since the arguments remain the same, we only give below sketches of the proofs.

Proof of Proposition 4.2. It is sufficient to prove that there exist constants $\tilde{c} > 0$ and C s.t. $\forall A \geq 0$, $\sup_{\Delta>0} \mathbb{P}_x[F^-(X^\Delta_{\tau^\Delta}) \geq A\Delta^{1/2}] \leq C\exp(-\tilde{c}A^2)$. Then any choice of $c < \tilde{c}$ is valid. For $x \in D$, we write

$$\begin{aligned} P &:= \mathbb{P}_x[F^-(X^\Delta_{\tau^\Delta}) \geq A\Delta^{1/2}] \\ &= \sum_{i\in\mathbf{N}^*} \mathbb{E}[\mathbf{1}_{\tau^\Delta > t_{i-1}} \mathbf{1}_{\tau^\Delta_{t_{i-1}} < t_i} \mathbb{P}[F^-(X^\Delta_{t_i}) \geq A\Delta^{1/2} | \mathcal{F}_{\tau^\Delta_{t_{i-1}}}]] \end{aligned}$$

where $\tau^\Delta_{t_{i-1}} := \inf\{s \geq t_{i-1} : X^\Delta_s \notin D\}$. From Lemma 3.1, we get

$$P \leq C\exp(-\tilde{c}A^2) \sum_{i\in\mathbf{N}^*} \mathbb{P}[\tau^\Delta > t_{i-1}, \tau^\Delta_{t_{i-1}} < t_i].$$

Lemma 16 from [GM04] remains valid under our current assumptions and yields $P \leq C\exp(-\tilde{c}A^2)\sum_{i\in\mathbf{N}^*}\mathbb{E}[\mathbf{1}_{\tau^\Delta>t_{i-1}}(\mathbb{P}[X^\Delta_{t_i} \notin D] + O_{pol}(\Delta))]$. From (4.3), $\mathbb{P}_x[\tau^\Delta < \infty] = 1$. Since $\sum_{i\in\mathbf{N}^*}\mathbb{P}_x[\tau^\Delta > t_{i-1}] = \Delta^{-1}\mathbb{E}_x[\tau^\Delta]$, we also derive from (4.3) and the previous upper bound on P that $P \leq C\exp(-\tilde{c}A^2)$ which concludes the proof. □

Proof of Theorem 4.4. Similarly to the proof of Theorem 2.5 we suppose first that $u \in \mathbf{H}_{3+\theta}$. The general case can be deduced as in the parabolic case using suitable Schauder estimates, given in the final notes of Chapter 6 in [GT98], see also our Appendix.

In this simplified setting, we obtain

$$\begin{aligned} \mathrm{Err}(\Delta, g, f, k, x) \overset{\mathbb{E}}{=} &- Z^\Delta_{\tau^\Delta} \nabla g(\pi_{\partial D}(X^\Delta_{\tau^\Delta}))\nabla F(X^\Delta_{\tau^\Delta})F^-(X^\Delta_{\tau^\Delta}) + \\ &(\sum_{i\in\mathbf{N}} \mathbf{1}_{t_i<\tau^\Delta} \Big[\Delta \times Z^\Delta_{t_i}(Lu - ku + f)(X^\Delta_{t_i}) \qquad (4.4) \\ &+ Z^\Delta_{t_i}\nabla u(X^\Delta_{t_i}) \cdot \nabla F(X^\Delta_{t_{i+1}})F^-(X^\Delta_{t_{i+1}}) \qquad (4.5) \\ &+ O(|F^-(X^\Delta_{t_{i+1}})||X^\Delta_{t_{i+1}} - X^\Delta_{t_i}|) + O(|F^-(X^\Delta_{t_{i+1}})|^2)\Big] \qquad (4.6) \\ &+ O(\Delta^{3/2+\theta/2}) + O(|W_{t_{i+1}} - W_{t_i}|^{3+\theta}))\mathbf{1}_{\tau^{r_0}>\tau^\Delta}. \qquad (4.7) \end{aligned}$$

The contribution (4.4) cancels owing to the PDE (4.2). The global contribution associated to the remainders (4.7) can be bounded by $C\Delta^{3/2+\theta/2}(\Delta^{-1}\mathbb{E}_x[\tau^\Delta])$. From (4.3), this quantity is a $O(\Delta^{1/2+\theta/2}) = o(\sqrt{\Delta})$. For (4.6) write

$$
\begin{aligned}
&\Big(\sum_{i\in\mathbf{N}} \mathbf{1}_{t_i<\tau^\Delta}(|F^-(X^\Delta_{t_{i+1}})||X^\Delta_{t_{i+1}} - X^\Delta_{t_i}|) + |F^-(X^\Delta_{t_{i+1}})|^2\Big) \\
&\overset{\mathbb{E}}{\leq} C\Delta \sum_{i\in\mathbf{N}} \mathbf{1}_{t_i<\tau^\Delta}\mathbf{1}_{\tau^\Delta_{t_{i-1}}\leq t_{i+1}} \overset{\mathbb{E}}{\leq} C\Delta \sum_{i\in\mathbf{N}} \mathbf{1}_{t_i<\tau^\Delta}\mathbf{1}_{X^\Delta_{t_{i+1}}\notin D} \overset{\mathbb{E}}{=} C\Delta \mathbf{1}_{\tau^\Delta<\infty}
\end{aligned}
$$

where we used Lemma 16 from [GM04] for the last inequality. Thus the global contribution of this term is a $O(\Delta)$. We eventually derive the result as in Section 2. □

Theorem 4.5 can be proved as Theorem 2.4, using a sensitivity result analogous to Theorem 2.2 in [CGK06] for elliptic problems, see e.g. Simon [Sim80].

5. Conclusion. In the parabolic setting, the error expansion, Theorem 2.3, and the associated correction, Theorem 2.4, have been obtained under "usual" assumptions from a PDE viewpoint, see $(\mathbf{A}_\theta)$. A natural question concerns the possible extension of these results to a hypoelliptic framework for a stopped diffusion with time dependent coefficients. The main tool needed is the smoothness of the law of the diffusion. For coefficients that are C^1 in time, this point is discussed in Cattiaux and Mesnager [CM02] for a cylindrical domain. Up to an extension of their results to time dependent domains, our main results should in some sense remain valid. For the elliptic case, the extension of Theorems 4.4, 4.5 to a hypoelliptic framework is open. Indeed, we strongly exploited some controls on the Euler scheme in large time, that are far from being easy to establish when the coefficients degenerate. This will concern further research. For PDE results in this framework, see [Bon69] for instance.

To conclude, we note that the boundary correction procedure could be at least formally extended to general Itô processes of the form $dX_t = b_t dt + \sigma_t dW_t$. In that case, the smaller domain would be defined ω by ω replacing $\sigma^*(t,x)\nabla F(t,x)$ by $\sigma_t^*\nabla F(t,X_t)$. Even if our current proof relies on Markovian properties, we conjecture that the correction should once again give a $o(\sqrt{\Delta})$ independently of the Markovian structure. We mention that it is already known [GM06] that the error associated to the discrete sampling of an exit time for a general Itô process yields an error of order $\sqrt{\Delta}$.

APPENDIX A: PROOF OF THEOREM 2.5 IN THE GENERAL SETTING

In this section, we detail how the proof of Section 2 has to be modified under the assumptions of Theorem 2.3, i.e. for $g \in \mathbf{H}_{1+\theta}$ and without compatibility condition so that $u \in \mathbf{H}_{1+\theta}$.

A.1. Preliminary notation and controls. Introduce the parabolic distance $\mathbf{pd}$: for $(s,x),(t,y) \in \bar{\mathcal{D}}$, $\mathbf{pd}((s,x),(t,y)) = \max(|s-t|^{1/2}, |x-y|)$. We also denote for a closed set $\mathcal{A} \in \bar{\mathcal{D}}$ and $(s,x) \in \mathcal{D}$, $\mathbf{pd}((s,x),\mathcal{A})$ the parabolic distance of (s,x) to $\mathcal{A}$.

Under our current assumptions, $\exists C > 0,\ \forall (s,x) \in \mathcal{D}$,

$$
\begin{aligned}
|Hu(s,x)| + \sup_{\alpha,\ |\alpha|=3} |\partial_x^\alpha u(s,x)| &\leq C\mathbf{pd}((s,x), \mathcal{PD} \cap \{v \geq s\})^{-2}, \\
&\sup_{\alpha,|\alpha|=3,\ (t,y)\in\mathcal{D},\ (t,y)\neq(s,x)} \frac{|\partial_x^\alpha u(s,x) - \partial_x^\alpha u(t,y)|}{\mathbf{pd}((s,x),(t,y))^\theta} \\
&\leq C(\mathbf{pd}((s,x), \mathcal{PD} \cap \{v \geq s\}) \wedge \mathbf{pd}((t,y), \mathcal{PD} \cap \{v \geq t\})^{-2-\theta}, \\
&\sup_{(t,x)\in\mathcal{D},\ t\neq s} \frac{|\partial_t u(s,x) - \partial_t u(t,x)|}{|t-s|^{(1+\theta)/2}} \\
&\leq C(\mathbf{pd}((s,x), \mathcal{PD} \cap \{v \geq s\}) \wedge \mathbf{pd}((t,x), \mathcal{PD} \cap \{v \geq t\})^{-2-\theta}. \qquad \text{(A.1)}
\end{aligned}
$$

These inequalities are obtained with the interior Schauder estimates for the PDEs satisfied by the partial derivatives $(\partial_{x_i} u)_{i\in[\![1,d]\!]}$, see Theorem 4.9 in [Lie96].

We first state an important proposition for the error analysis with possibly explosive controls as in (A.1) for the derivatives. Namely, under our current regularity assumptions, in order to perform a Taylor expansion we have to work with interior points s.t. the whole segment between these points belongs to the time space domain, with the distance to the boundary uniformly lower bounded along the segment. The proposition states that this is the case if the points are "far enough" from the side of $\mathcal{D}$.

Proposition A.1 *Assume $\mathcal{D} \in \mathbf{H}_2$ and take $\varepsilon \in]0,1[$. For all $(t,x) \in \bar{\mathcal{D}} \cap V_{\partial\mathcal{D}}(r_0/2) \setminus V_{\partial\mathcal{D}}(2\Delta^{1/2(1-\varepsilon)})$, where r_0 is defined in Section 1.4.2, $t \leq T - \Delta$ and $\forall y \in B(x, \Delta^{1/2(1-\varepsilon)}) \cap \bar{\mathcal{D}}_{t+\Delta}$, one has*

$$
\begin{aligned}
&I_\varepsilon(t,x,y,\Delta) := [(t,x),(t+\Delta,y)] \in \bar{\mathcal{D}}, \\
&\forall (s,z) \in I_\varepsilon(t,x,y,\Delta),\ F(s,z) \geq \frac{1}{4} F(t,x)
\end{aligned}
$$

for Δ small enough.

The proof is postponed to the end of the Section.

A.2. Error analysis. Recall from the previous proof of Theorem 2.3 that the main term to analyze is

$$e_2^\Delta \stackrel{\mathbb{E}}{=} \Big(\sum_{0\le t_i<T-4\Delta^{1-\varepsilon}} \mathbf{1}_{t_i<\tau^\Delta} \Big[u(t_{i+1},\pi_{\bar{\mathcal{D}}_{t_{i+1}}}(X^\Delta_{t_{i+1}}))Z^\Delta_{t_{i+1}} - u(t_i,X^\Delta_{t_i})Z^\Delta_{t_i} + Z^\Delta_{t_i} f(t_i,X^\Delta_{t_i})\Delta \Big] \big(\mathbf{1}_{(t_i,X^\Delta_{t_i})\notin V_{\partial\mathcal{D}}(2\Delta^{1/2(1-\varepsilon)})} + \mathbf{1}_{(t_i,X^\Delta_{t_i})\in V_{\partial\mathcal{D}}(2\Delta^{1/2(1-\varepsilon)})}\big)\big)\mathbf{1}_{\tau^{r_0}>\tau^\Delta} + e^\Delta_{23} := e^\Delta_{21} + e^\Delta_{22} + e^\Delta_{23},$$

where $\stackrel{\mathbb{E}}{=}$ denotes an equality for the expectation up to a $O_{pol}(\Delta)$. The terms $e^\Delta_{21}, e^\Delta_{22}, e^\Delta_{23}$ are respectively associated to the events for which the $(t_i, X^\Delta_{t_i})_i$ are far from the parabolic boundary of $\mathcal{D}$, close to its side and close to its top.

Control of e^Δ_{21}. From Proposition A.1, Lemma 3.1 and (A.1), the pointwise θ-Hölder controls for the third spatial derivatives are bounded by $C \times F(t_i, X^\Delta_{t_i})^{-2-\theta}$, as well the $(1+\theta)/2$-Hölder controls for the time derivative. Hence, adapting the previous analysis of Section 2, we get

$$|e^\Delta_{21}| \stackrel{\mathbb{E}}{\le} C \sum_{0\le t_i<T} \mathbf{1}_{t_i<\tau^\Delta} \Big(r_0^{-2-\theta}(|W_{t_{i+1}} - W_{t_i}|^{3+\theta} + \Delta^{(3+\theta)/2}) + \mathbf{1}_{2\Delta^{1/2(1-\varepsilon)}\le F(t_i,X^\Delta_{t_i})\le r_0/2} F(t_i,X^\Delta_{t_i})^{-2-\theta}(|W_{t_{i+1}} - W_{t_i}|^{3+\theta} + \Delta^{(3+\theta)/2}) \Big).$$

The terms involving $r_0^{-2-\theta}$ readily give a $O(\Delta^{1/2+\theta/2}) = o(\Delta^{1/2})$. For the other terms, the key tool is the occupation times formula associated to some sharp controls from [GM04] for the expectation of the local time $\big(L^y_s(F(.,X^\Delta_.))\big)_s$ at level y of the continuous semi-martingale distance process $(F(s,X^\Delta_s))_s$. Indeed, an easy adaptation of the proof of Lemma 17 [GM04] to our time dependent domain framework gives

$$\mathbb{E}[L^y_{T\wedge\tau^\Delta}(F(.,X^\Delta_.))] \le C(|y| + \sqrt{\Delta}). \tag{A.2}$$

Thus, one has

$$\begin{aligned} |e^\Delta_{21}| &\stackrel{\mathbb{E}}{\le} C\Delta^{1/2+\theta/2}\Big(\int_0^{T\wedge\tau^\Delta} \mathbf{1}_{F(\phi(t),X^\Delta_{\phi(t)})\in[2\Delta^{1/2(1-\varepsilon)},r_0/2]} F(\phi(t),X^\Delta_{\phi(t)})^{-2-\theta} dt + 1 \Big) \\ &\stackrel{\mathbb{E}}{\le} C\Delta^{1/2+\theta/2}\Big(\int_0^{T\wedge\tau^\Delta} \mathbf{1}_{F(t,X^\Delta_t)\in[\Delta^{1/2(1-\varepsilon)},3r_0/4]} F(t,X^\Delta_t)^{-2-\theta} dt + 1 \Big) \\ &\stackrel{\mathbb{E}}{\le} C\Delta^{1/2+\theta/2}\Big(\int_{\Delta^{1/2(1-\varepsilon)}}^{3r_0/4} y^{-2-\theta} L^y_{T\wedge\tau^\Delta}(F(.,X^\Delta_.))dy + 1 \Big), \end{aligned}$$

using Lemma 3.1 for the last but one inequality, and the occupation time formula for $F(t, X_t^\Delta)$ for the last one (recall that σ is uniformly elliptic). Finally using (A.2), one gets

$$|e_{21}^\Delta| \le C\Delta^{1/2+\theta/2}\left(\int_{\Delta^{1/2(1-\varepsilon)}}^{3r_0/4} y^{-2-\theta}(y+\Delta^{1/2})dy + 1\right) \le C\Delta^{1/2+\theta\varepsilon/2}.$$

Thus, e_{21}^Δ is a $o(\Delta^{1/2})$. This technique will also be used for e_{22}^Δ.

Control of e_{22}^Δ. A Taylor formula gives:

$$e_{22}^\Delta \stackrel{\mathbb{E}}{=} \sum_{0\le t_i<T-4\Delta^{1-\varepsilon}} \mathbf{1}_{t_i<\tau^\Delta, F(t_i,X_{t_i}^\Delta)\in]0,2\Delta^{1/2(1-\varepsilon)}]}\Big\{\nabla u(t_i, X_{t_i}^\Delta)(F^-\nabla F)(t_{i+1}, X_{t_{i+1}}^\Delta) + (Z_{t_i}^\Delta f(t_i, X_{t_i}^\Delta)\Delta + O(\Delta^{1/2+\theta/2}))\Big\} \times \mathbf{1}_{\tau^{r_0}>\tau^\Delta} := e_{221}^\Delta + e_{222}^\Delta.$$

The term e_{221}^Δ corresponds to the overshoot. The term e_{222}^Δ can be controlled with techniques similar to the ones used for e_{21}^Δ. Namely,

$$\begin{aligned}
|e_{222}^\Delta| &\stackrel{\mathbb{E}}{\le} C\Delta^{1/2+\theta/2}(\Delta^{-1}\int_0^{T\wedge\tau^\Delta} \mathbf{1}_{(\phi(t),X_{\phi(t)})\in V_{\partial\mathcal{D}}(2\Delta^{1/2(1-\varepsilon)})}dt)\\
&\stackrel{\mathbb{E}}{\le} C\Delta^{1/2+\theta/2}(\Delta^{-1}\int_0^{T\wedge\tau^\Delta} \mathbf{1}_{(t,X_t^\Delta)\in V_{\partial\mathcal{D}}(3\Delta^{1/2(1-\varepsilon)})}dt)\\
&\stackrel{\mathbb{E}}{\le} C\Delta^{1/2+\theta/2}\Delta^{-1}\int_{-3\Delta^{1/2(1-\varepsilon)}}^{3\Delta^{1/2(1-\varepsilon)}} L_{T\wedge\tau^\Delta}^y(F(.,X_.^\Delta))dy\\
&\stackrel{\mathbb{E}}{\le} C\Delta^{1/2+\theta/2}\Delta^{-1}\int_{-3\Delta^{1/2(1-\varepsilon)}}^{3\Delta^{1/2(1-\varepsilon)}} (|y|+\Delta^{1/2})dy \le C\Delta^{1/2+\theta/2-\varepsilon}
\end{aligned}$$

which for ε small enough gives a $o(\Delta^{1/2})$.

Control of e_{23}^Δ. A Taylor formula gives the overshoot component for the time steps between $T-4\Delta^{1-\varepsilon}$ and T, and a $O(\Delta^{1/2+\theta/2-\varepsilon}) = o(\Delta^{1/2})$ for the other terms. This completes the proof. □

Proof of Proposition A.1. Fix $t \in [0, T-\Delta]$. For all $\lambda \in [0,1]$, let

$$s := t+\lambda\Delta, z := x+\lambda(y-x).$$

Since $F \in \mathbf{H}_2$,

$$F(s,z) \ge F(t,x) - C\Delta + \lambda\langle\nabla F(t,x), y-x\rangle - C\Delta^{1-\varepsilon}. \tag{A.3}$$

Now $\langle\nabla F(t,x), y-x\rangle = F(t+\Delta, y) - F(t,x) + O(\Delta^{1-\varepsilon})$ which plugged into (A.3) yields for Δ small enough

$$F(s,z) \ge F(t,x)(1-\lambda) + \lambda F(t+\Delta, y) - C\Delta^{1-\varepsilon}.$$

Also, since $(t,x) \notin V_{\partial\mathcal{D}}(2\Delta^{1/2(1-\varepsilon)})$, then, for Δ small enough, $F(t+\Delta, y) \ge F(t,x)/3$. The proof is complete. □

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E-mail: emmanuel.gobet@imag.fr.
Laboratoire Jean Kuntzmann
UMR 5224, LJK,
BP 53, F 38041 Grenoble Cedex 9,
FRANCE.

E-mail: menozzi@math.jussieu.fr.
Laboratoire de Probabilités et
Modèles Aléatoires,
175, rue de Chevaleret 75013 Paris
FRANCE.