

# Does Church-Turing Thesis Apply Outside Computer Science?

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24-26/04/2006, Transgressive Computing, in honor of  
Jean Della Dora

# Context and objectives

## JDD, Pensées

The future of mathematics comes from informatics, the future of informatics comes from mathematics.

## My hidden objective

To try again to convince Jean that decidability is relevant to dynamics.

## Contributors

**Technical:** A. Bouajjani, P. Collins, O. Maler, G. Pace, A. Pnueli, G. Schneider, S. Yovine

**Ideological:** J. Della Dora, V. Blondel, O. Bournez, M. Fränzle, J.-F. Raskin

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The chaotic case :  
PCD in  $\mathbb{R}^3$

The awful case :  
Zero PCD

The noisy case

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# Notion of algorithm - history

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**Antiquity:** Euclid's algorithm etc

**9th century:** Al-Khwarizmi's book on arithmetic

**Since 19th century:** Many concrete algorithms. Some frameworks (Ada Byron)

**1930s-50s:** *Formal definitions of Algorithm*

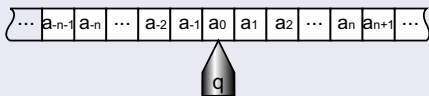
- Post machines
- Turing machines
- Partial recursive functions
- Markov normal algorithms

**since 1940s:** Computers and programming languages implementing these ideas

**since 1930s:** Theoretical computer science based on these ideas

# Important things about algorithms

## Sample model: Turing machines



$p, 0 : 1, L, q$

$p, 1 : 0, R, p$

$q, 0 : 0, R, p$

$q, 1 : \text{stop}$

## Common features

- unbounded discrete-time
- unbounded discrete-state memory,
- deterministic programs
- no-noise no-faults execution

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## Theorem

*All the definitions above are equivalent lead to the same class of **computable** functions.*

## Church-Turing thesis

- *Turing machines (and all the others) capture the general notion of algorithm.*
- *I.e. any reasonable (see above) computational model leads to the same (or smaller) class of computable functions as Turing machines*

What about unreasonable models???

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- A Turing machine  $M$  decides a set  $P \subset \text{Words}$  iff  
 $(w \in P) \Leftrightarrow M$  on  $w$  reaches a special state  $Y$   
 $(w \notin P) \Leftrightarrow M$  on  $w$  reaches a special state  $N$ .  
The set  $P$  is called recursive decidable.
- A Turing machine  $M$  recognizes a set  $P \subset \text{Words}$  iff  
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The set  $P$  is called recursively enumerable. It may be undecidable.
- A hierarchy of sets:
  - $\Delta_1$  - decidable sets.
  - $\Sigma_1$  - recursively enumerable sets.
  - $\Pi_1$  - complements of r.e. sets
  - $\Delta_2$  - decidable using an oracle for halting.
  - ...
  - $\Sigma_\omega$  - expressible in arithmetics

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# Extended CTT : a statement and some speculations

## Church-Turing thesis (Physical - PCTT)

*Feasible, realizable, reproducible physical computing devices have the same (or smaller) computational power as Turing machines.*

- If Everything is finite and bounded  $\Rightarrow$  PCTT **true** and trivial.
- Without reproducibility: measure cosmic rays. Probably uncomputable by a Turing machine  $\Rightarrow$  PCTT **false**
- For reproducible systems: measure the ratio of frequencies of some spectral lines of  $H$  and  $He$ . Are its digits a computable sequence?
- - It is a *philosophical* problem.
  - Maybe it makes sense in *physics*.
  - Let us speak about *mathematics*.

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## Classes of systems to consider

- Discrete :  $x := f(x)$  or continuous  $\dot{x} = f(x)$ .
- Deterministic:  $x := f(x)$  or contingent:  $x := F(x)$ .  
Maybe even probabilistic.
- I prefer piecewise trivial systems.

## Reachability problem

$$\mathcal{R}(P, Q) \Leftrightarrow \exists \text{ trajectory with } x_0 \in P, x_m \in Q$$

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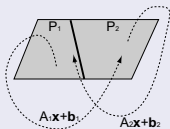
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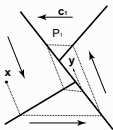
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## Piecewise-affine maps



$$x := A_j x + b_j \text{ for } x \in P_j$$

## My favorite class: Piecewise-constant derivatives (PCD)



$$\dot{x} = c_j \text{ for } x \in P_j$$

## And also

Linear systems, linear hybrid automata, differential inclusions, Turing machines ...



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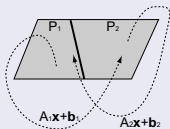
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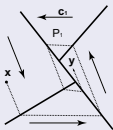
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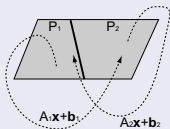
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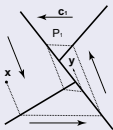
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Dyn. syst.  $H$  recognizes a set  $P \subset \text{Words}$ :

- There is an encoding  $e : \text{Words} \rightarrow \text{States}$
- an acceptor  $Y \subset \text{States}$
- and an equivalence  $(w \in P) \Leftrightarrow \mathcal{R}(e(w), Y)$

$H$  decides a set  $P \subset \text{Words}$ :

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# Another extension of Church-Turing thesis

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### Conclusions

## Church-Turing thesis (Mathematical - MCTT)

*Any reasonable class of Dynamical systems has the same (or smaller) computational power as Turing machines*

- *It can recognize at most  $\Sigma_1$  (recursively enumerable) sets.*
- *It can decide at most  $\Delta_1$  (recursive) sets.*

## Some evident restrictions

- The system  $H$ , the encoding  $e$ , the acceptor/rejector  $Y/N$  should admit finite description
- We prefer all the parameter to be rational

# Another extension of Church-Turing thesis

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E. Asarin

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PCD in  $\mathbb{R}^2$

The chaotic case :  
PCD in  $\mathbb{R}^3$

The awful case :  
Zero PCD

The noisy case

Conclusions

## Church-Turing thesis (Mathematical - MCTT)

*Any reasonable class of Dynamical systems has the same (or smaller) computational power as Turing machines*

- *It can recognize at most  $\Sigma_1$  (recursively enumerable) sets.*
- *It can decide at most  $\Delta_1$  (recursive) sets.*

## Some evident restrictions

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- DS much weaker than TM  $\Rightarrow$  decidability of  $\mathcal{R}$ , verification algorithms.
- DS as strong as TM (or stronger)  $\Rightarrow$  undecidability of  $\mathcal{R}$ .
- DS as strong as TM  $\Rightarrow$  a way to build computers
- DS stronger than TM  $\Rightarrow$  a way to build super-computers
- This gives an insight into PhysicalCTT
- A computational power is tightly related to complexity of behaviour.
- Informatics has an excellent scale to measure computational power.
- This scale is similar to but finer and simpler than descriptive set theory
- It is a fun.



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# Planar PCD

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Conclusions

- Trajectories of ODE on the plane are simple: no chaos, spirals at worst. (Poincaré-Bendixson, Maler-Pnueli).
- MP94: an algorithm to decide reachability for planar PCD systems. Based on enumeration of all possible spirals.
- Planar PCD are weaker than TM. They can only recognize some  $\Delta_1$  sets, while TM can recognize all  $\Sigma_1$
- Not enough space to compute on the plane!

# Extension to inclusions on the plane

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## Our result

Everything fine for piecewise constant inclusions on the plane - no chaos, only spirals, reachability decidable, automatic analysis possible. Weaker than TM.

## Our tool SPEEDI

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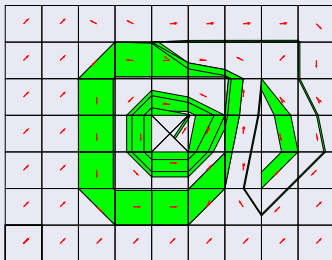
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Conclusions

- PCD in 3d and more are as strong as TM. Proof: analog programming of digital computing.
- Corollary: reachability undecidable, automatic analysis impossible in 3d.
- **Remark:** Our construction is chaotic.
- **Theorem:** A TM is chaotic  $\Rightarrow$  any class of DS capable to simulate TMs admits chaos.
- A typical situation, PCD as strong as TM, Math CTT holds.
- If you want, you can build analog computers (but they will be fragile).

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# The horror

## Zeno phenomenon

PCD is discontinuous. A trajectory is a broken line, it can have a transfinite number of angles. Let us allow this.

## Its consequences

- We : PCD (using Zeno) can decide any arithmetic predicate ( $\Sigma_\omega$ ). Bournez: Dimension 5 suffices.
- MathCTT fails here. This corresponds to something worse than chaos, but depends on the definition of a trajectory. A mathematical phenomenon to explore.
- Should we build supercomputers? Black hole computers? Probably impossible to realize. Related to PhysCTT

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# A folk conjecture

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## Optimistic practitioners say:

Reachability is decidable for realistic, unprecise, noisy, “fuzzy”, “robust” systems. They are hence weaker than TM and analyzable.

## Arguments

- The only known proof method uses unbounded precision (or unbounded state space)
- Noise could regularize...
- This world is nice and bad things never happen...
- Engineers design systems and never deal with undecidability.



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- **Henzinger, Raskin: a finite contingent noise without lower bound does not help. It remains  $\Sigma_1$  and non-analyzable.**
- We: a contingent noise tending to zero  $\Rightarrow \Pi_1$ . Strange. Meaning: sometimes it is easier to find obstacles than trajectories.
- We: a stochastic noise tending to zero  $\Rightarrow \Delta_2$ . Meaning: limit of large deviations is weird.
- **Challenge:** Maybe a finite contingent noise with lower bound can help?

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Conclusions

- Some mathematical systems are “super-Turing”  $\Rightarrow$  Extended CTC fails in maths.
- This is unlikely to be useful for super-computers.
- Computational power is a rough but adequate complexity characteristic of dynamical systems.
- It provides some funny mathematics and funny informatics.

# Open problems

Contact me for details

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Conclusions

- Decidability of very simple questions for very simple systems.
- Regularization by noise
- A more general theory.
- Something sensible about physical CTC.