



**HAL**  
open science

# Retrieval of a Time-Dependent Source in an Acoustic Propagation Problem

Ginette Saracco, Philippe Tchamitchian

► **To cite this version:**

Ginette Saracco, Philippe Tchamitchian. Retrieval of a Time-Dependent Source in an Acoustic Propagation Problem. *Inverse Problems & Theoretical Image*, SPRINGER-VERLAG, 1990, pp.207-211. hal-00157751

**HAL Id: hal-00157751**

**<https://hal.science/hal-00157751>**

Submitted on 27 Jun 2007

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## Retrieval of a Time-Dependent Source in an Acoustic Propagation Problem

*G. Saracco*<sup>1</sup> and *Ph. Tchamitchian*<sup>2</sup>

<sup>1</sup>L.M.A., Equipe Ultra-Sons, CNRS 31, Chemin Joseph Aiguier,  
 F-13009 Marseille, France

<sup>2</sup>Centre de Physique Théorique, CNRS-Luminy, Case 907,  
 F-13288 Marseille Cédex 9, France

**Abstract :** Consider two média separated by a plane interface, a time-dependent source  $F(t)$  at a point  $S$  in the first medium (of the lower celerity), receivers in the second medium, located at a large radial distance from the source. Thanks to a diffusion-like behaviour of the transmitted wave, we are able to retrieve the term  $F(t)$ , under high frequencies hypothesis in an elementary way.

### 1. About homogeneity and cancellation

We begin with the Poisson semi-group of operators

$$P_y = e^{-y\Lambda},$$

where  $y > 0$  and  $\Lambda = \sqrt{-\Delta}$ , acting on  $L^2(\mathbb{R}^n)$ . Then, we define

$$Q_y = -y \frac{\partial}{\partial y} P_y.$$

Because of the fact that  $P_y$  is an approximation of the identity, which means that

$$I = \lim_{y \rightarrow 0} P_y \quad \text{and} \quad 0 = \lim_{y \rightarrow +\infty} P_y$$

in the strong topology of  $L^2(\mathbb{R}^n)$ , we deduce the following decomposition of the identity :

$$I = \int_0^{+\infty} Q_y \frac{dy}{y}. \quad (1)$$

This formula is known as the Littlewood-Paley decomposition, and has extensively been studied in harmonic analysis since the 30's. See for example the classical treatise by Zygmund [4]. For the amateurs, we mention that is in fact the ancestor of the so-called wavelet transform. But we do not want to go further into this matter.

What we would like to point out here is that analogous formulas to (1) can be obtained, and may be useful, in some physical situations which a priori have nothing to do with diffusion.

Before stating what we have in mind, let us have a closer look at (1), on the Fourier side.

(1) may be interpreted as a way of analysing signals with a continuous chain of filters associated to the operators  $Q_y$ . More precisely, if we denote these filters by  $Q_y(\omega)$ , then :

$$Q_y(\omega) = y |\omega| e^{-y|\omega|}$$

Now, it becomes apparent why (1) is true. It is because of the following two basic properties of  $Q_y(\omega)$ :

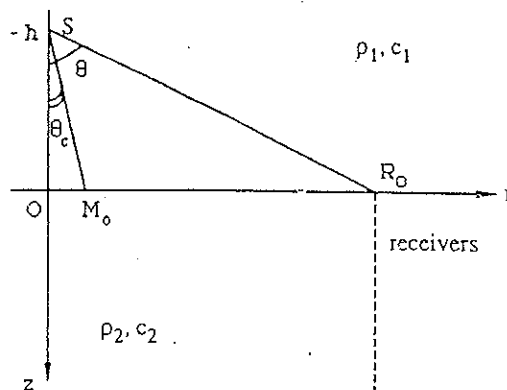
a. an homogeneity property :  $Q_y(\omega)$  depends effectively on the product  $y|\omega|$ . It is centered around the sphere in  $\mathbb{R}^n$  of radius  $\frac{1}{y}$ , with a bandwidth  $\Delta Q_y = \frac{c_n}{y}$ , where  $c_n$  is a constant (varying with your normalisation). Notice that  $\frac{dy}{y}$  is the dilation-invariant measure on  $\mathbb{R}^+$ .

b. a cancellation property :  $Q_y(\omega)$  cancels at the origin. Such a property is necessary in order to insure the convergence at  $y = 0$  of the integral in (1). A contrario, the same integral with  $P_y$  instead of  $Q_y$  would strongly diverge.

We now turn our attention to a seemingly very different problem : the propagation through a plane interface.

## 2. An asymptotic formula for retrieving a time-dependent source

The physical situation is the classical one : two infinite homogeneous and isotropic media are separated by a plane interface. We think to air and water for numerical and experimental purposes [3]. An acoustical point source is emitting in the first medium (air), at height  $h$ , and receivers are located in the second one (water), at a large radial distance  $r$  from the source, and on a vertical straight line starting from the interface at the point  $R_0$ :



$$\frac{\rho_2}{\rho_1} = n \sim 800 \quad (\text{densities})$$

$$\frac{c_1}{c_2} = n \sim 0.23 \quad (\text{celerities})$$

$$\theta_c = \arcsin n \sim 13^\circ \quad (\text{critical angle})$$

Fig. 1

Here, "large radial distance" means that  $\theta = \arctg \frac{r}{h}$  is greater than the critical angle  $\theta_c$  (that is to say,  $\frac{r}{\sqrt{r^2+h^2}} > n$ ).

If  $F(t)$  is the acoustic potential of the source, and if  $\psi_2(r,z,t)$  is the transmitted potential at a point  $(r,z)$ ,  $z > 0$ , and at time  $t$ , the associated pressure  $P_2(r,z,t)$  is

$$P_2(r, z, t) = \rho_2 \frac{\partial}{\partial y} \psi_2(r, z, t).$$

We recall that (see Landau and Lifshitz [2]) :

$$\psi_2(r, z, t) = \int_R e^{-i\omega x} \hat{F}(\omega) i \int_0^{+\infty} e^{i(k_1 h + k_2 z)} J_0(kr) \frac{1}{m k_1 + k_2} k dk d\omega,$$

$$\text{where } k_j = \sqrt{\left(\frac{\omega}{c_j}\right)^2 - k^2}.$$

We suppose that the source is of a high frequency nature :  $\hat{F}(\omega) \neq 0$  only if  $|\omega| \geq \omega_{\min}$ , and  $\frac{r \omega_{\min}}{c_1} \gg 1$ .

Then, we have the following asymptotic formula :

$$\int_0^{+\infty} P_2(r, z, t) dz \sim C_1(r, h) F(t-t_1) + C_2(r, h) F(t-t_2) + C_3(r, h) \mathcal{H}F(t-t_2) \quad (2)$$

where

$\mathcal{H}$  is the Hilbert transform ,

$$t_1 = \frac{h \sqrt{1-n^2} + nr}{c_1},$$

$$t_2 = \frac{\sqrt{r^2+h^2}}{c_1},$$

$C_j(r, h)$  are constants depending only on the positions of the source and the receivers.

$t_1$  is the time for a way to travel over the path  $S M_0 R_0$  with velocity  $c_1$  on  $S M_0$  and  $c_2$  on  $M_0 R_0$ , and  $t_2$  corresponds to the path  $S R_0$  with velocity  $c_1$  (see fig. 1).

The presence of the Hilbert transform is due to the condition  $\frac{r}{\sqrt{r^2+h^2}} > n$ . It disappears in the opposite case, where a similar (and much less interesting, at least for an air-water interface) formula holds :

$$\int_0^{+\infty} P_2(r, z, t) dt \sim C(r, h) F(t-t_2). \quad (2')$$

If we take into consideration the fact that  $m = \frac{\rho_2}{\rho_1} \approx 800$ , we have for the constants in (2) the following simplified expressions :

$$C_1(r, h) = \frac{\rho_1}{(1-n^2)^{1/4} [r(r\sqrt{1-n^2} - hn)]^{1/2}} + O\left(\frac{1}{m}\right), \quad C_2(r, h) = O\left(\frac{1}{m}\right),$$

$$C_3(r, h) = \frac{\rho_1}{\left[\frac{r^2}{r^2+h^2} - n^2\right]^{1/2} r} + O\left(\frac{1}{m}\right).$$

We are not going to enter in the details of the proof of the formula (2), which is straightforward and typical of asymptotic arguments (one could ask whether such a kind of argument is necessary or not, the answer is yes, for the result is simply false for arbitrary sources).

We prefer to conclude with a comparison between formulas (1) and (2).

Let us remark however that, once (2) is valid and the position of the source is known, it is easy to retrieve the source term  $F(t)$ . Maybe it is also possible to find the coordinates of the point source from (2), but this is still under investigations.

### 3. Link between the formulas (1) and (2)

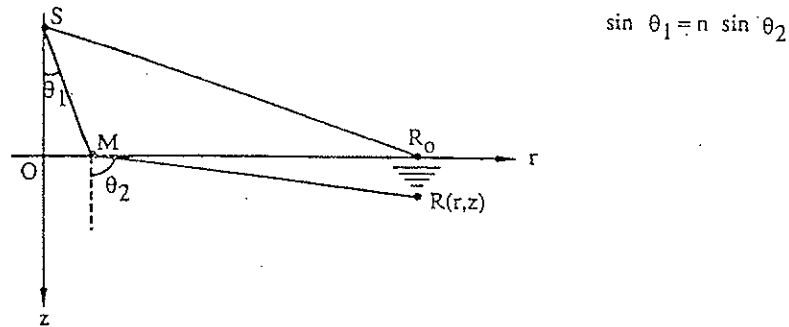


Fig. 2

Looking at  $P_2(r, z, t)$  as the result of a filtering process applied on  $F(t)$ , we denote by  $\Gamma(r, z, \omega)$  the associated impulse response.

$\Gamma(r, z, \omega)$  is nothing but the transmitted pressure at the point  $R(r, z)$  when the source is harmonic, with frequency  $\omega$ . It is usual, following Brekhovskikh [1], to decompose it in two contributions, under asymptotic conditions :

$$\Gamma(r, z, \omega) = \Gamma_{\text{geo}}(r, z, \omega) + \Gamma_{\text{lat}}(r, z, \omega).$$

$\Gamma_{\text{geo}}$  is given by the approximation of the geometrical acoustics (see fig. 2) :

$$\Gamma_{\text{geo}}(r, z, \omega) = \gamma_{\text{geo}}(r, z, h) i \omega e^{i \omega \left( \frac{SM}{c_1} + \frac{MR}{c_2} \right)}, \quad (3)$$

while  $\Gamma_{\text{lat}}$  is specific of the case when  $n = \frac{c_1}{c_2} < 1$  :

$$\Gamma_{\text{lat}}(r, z, \omega) = \gamma_{\text{lat}}(r, h) e^{i \omega \frac{\sqrt{r^2 + h^2}}{c_1}} i \omega e^{-\alpha |\omega| \frac{z}{c_2}}, \quad (4)$$

$$\alpha = \sqrt{\frac{r^2}{r^2 + h^2} \cdot n^2}.$$

Looking at these expressions, we may understand why (2) is valid.

Consider at first  $\Gamma_{lat}$ , which is the main term, at least near the interface (see Saracco [3]). It is usually said that  $\Gamma_{lat}$  corresponds to a wave propagating in the air along the path  $S R_0$  and then diffusing in the water (fig. 2). The term  $e^{-\alpha |\omega| \frac{z}{c_1}}$  in (4) is typical of a diffusion process, while  $e^{i\omega \frac{\sqrt{r^2+h^2}}{c_1}}$  is only a delay term.  $\Gamma_{lat}(r,z,\omega)$  appears to be very much like  $Q_y(\omega)$  in §1, with  $\alpha \frac{z}{c_1}$  playing the role of  $y$ .

What about  $\Gamma_{geo}$ ? It is different in nature from  $\Gamma_{lat}$ , and associated to a wave propagating in air and in water along the path SMR (fig. 2). But it still has the properties of homogeneity and cancellation we described in § 1, at least again near the interface. In fact, if  $\frac{z}{r}$  is small, we have

$$\Gamma_{geo}(r,z,\omega) \sim \gamma_{geo}'(r,h) e^{i\omega \frac{h\sqrt{1-n^2+m}}{c_1}} i\omega z e^{i \frac{\omega z^2}{c_1 \ell}}, \quad (5)$$

where  $\ell = \frac{r}{n} - \frac{h}{\sqrt{1-n^2}}$ . The cancellation property is already clear in (3), and the homogeneity is apparent in (5).

These remarks explain why the formulas (1) and (2) are analogous.

#### REFERENCES

- [1] L.N. Brekhovskikh, *Waves in Layered Media*, 1960 Wiley, New-York.
- [2] L.D. Landau, E.M. Lifshitz, *Fluid mechanics*, Pergamon Press 1959.
- [3] G. Saracco, *Etude de la transmission acoustique à travers le dioptre air-eau*, Note-Laboratoire de Mécanique et d'Acoustique n° 101, CNRS: Marseille, ISSN 0750-7356, 1986.
- [4] A. Zygmund, *Trigonometric series*, Cambridge University Press 1968.