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Abstract

We derive Mok-Siu-Yeung type formulas for horizontal maps from compact contact locally sub-symmetric spaces into strictly pseudoconvex CR manifolds and we obtain some rigidity theorems for the horizontal pseudoharmonic maps.

Author Keywords: Contact locally sub-symmetric spaces, Tanaka-Webster connection, pseudoharmonic maps, CR maps, Rumin complex, Mok-Siu-Yeung type formulas, rigidity results.

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1 Introduction

The main ingredient in the harmonic maps approach of superrigidity for semi-simple Lie groups is the Mok-Siu-Yeung formula [23],[32] for harmonic maps defined on compact locally symmetric spaces of non-compact type (cf. also [19]). Actually, by means of this formula, it can be shown that a harmonic map from a compact locally symmetric space of non-compact type (with some exceptions) into a Riemannian manifold with nonpositive curvature is rigid in the sense of it is a totally geodesic isometric imbedding. The contact locally sub-symmetric spaces defined by Bieliavsky, Falbel and Gorodski [6],[17],[18] are the contact analogues of the riemannian locally symmetric spaces. These spaces can be characterized as contact metric manifolds for which the curvature and the torsion of the Tanaka-Webster connection are parallel in the direction of the contact distribution. Moreover, these spaces are strictly pseudoconvex CR manifolds. In an other hand, in the setting of contact metric manifolds, the analogue of harmonic maps seems to be the pseudoharmonic maps defined by Barletta, Dragomir and Urakawa [3],[5]. Also the main purpose of this article is to derive Mok-Siu-Yeung type formulas for horizontal maps (i.e. maps preserving the contact distributions) from compact contact locally sub-symmetric spaces into strictly pseudoconvex CR manifolds in order to obtain some rigidity theorems for horizontal pseudoharmonic maps under curvature assumptions. The plan of this article is the following. The section 2 begins to recall basic facts concerning the contact metric manifolds and the strictly pseudoconvex CR manifolds, next, we focus our attention on the pseudo-hermitian curvature tensor of a strictly pseudoconvex CR manifold, which plays a central part in the following. In section 3, we investigate the contact sub-symmetric spaces, the main result of this section (Theorem 3.3) which is related to the work of Cho [14], is an explicit formula for the pseudo-hermitian curvature tensor of a contact locally sub-symmetric space with non zero pseudo-hermitian torsion. The section 4 is devoted to derive Mok-Siu-Yeung type formulas for horizontal maps between strictly pseudoconvex CR manifolds (Proposition 4.2). In section 5, we extend the notion of pseudoharmonic maps defined in [3],[5] to the setting of horizontal maps between contact metric manifolds and we define the notion of CR-pluriharmonic maps for horizontal maps between strictly pseudoconvex CR manifolds. It is interesting to note that these two notions are strongly related to the Rumin complex [27]. In section 6, we obtain some rigidity theorems for the horizontal pseudoharmonic maps when the source manifold is a compact contact locally sub-symmetric space. The main result of this section (Theorem 6.1) asserts that any horizontal pseudoharmonic map ϕ from a compact contact locally sub-symmetric space of non-compact type, holonomy irreducible and torsionless, (with some exceptions) into a Sasakian manifold with nonpositive pseudo-Hermitian complex sectional curvature satisfies $\nabla d\phi = 0$ where $\nabla d\phi$ is the covariant derivative of $d\phi$ with respect to Tanaka-Webster connections. As application (Corollary 6.1) we deduce that ϕ preserves some special curves called parabolic geodesics [15] and therefore ϕ is, in some sense, totally geodesic. In section 7, we restrict our attention to CR maps from com-

pact contact locally sub-symmetric spaces into strictly pseudoconvex CR manifolds and we obtain the following rigidity result (Theorem 7.1): any horizontal pseudoharmonic CR map from a compact contact locally sub-symmetric space of non-compact type (with some exceptions) into a pseudo-Hermitian space form with negative pseudo-Hermitian scalar curvature is constant. This article is a first step in the study of horizontal pseudoharmonic maps from compact strictly pseudoconvex CR manifolds into strictly pseudoconvex CR manifolds with nonpositive pseudo-Hermitian sectional curvature. In particular, some existence results are missing for the moment (excepted if the target manifold is Tanaka-Webster flat).

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2 Connection and curvature on contact metric manifolds

. Contact metric manifolds

A contact form on a smooth manifold M of dimension $m = 2d + 1$ is a 1-form θ satisfying $\theta \wedge (d\theta)^d \neq 0$ everywhere on M. If θ is a contact form on M, the hyperplan subbundle H of TM given by $H = Ker\,\theta$ is called a contact structure. The Reeb field associated to θ is the unique vector field ξ on M satisfying $\theta(\xi) = 1$ and $d\theta(\xi, .) = 0$. By a contact manifold (M, θ) we mean a manifold M endowed with a fixed contact form θ. If (M, θ) is a contact manifold then TM decomposes as $TM = H \oplus \mathbb{R}\xi$. Consequently any p-tensor t on M decomposes as $t = t_H + t_\xi$ with $t_H = t \circ \Pi_H$ and $t_\xi = t \circ \Pi_{\mathbb{R}\xi}$ (Π_H and $\Pi_{\mathbb{R}\xi}$ are the canonical projections on H and $\mathbb{R}\xi$). The tensors t_H and t_ξ are respectively called the horizontal part and the vertical part of t. Note that for an antisymmetric p-tensor γ we have $\gamma_\xi = \theta \wedge i(\xi)\gamma$. We denote by $\wedge^*_H(M)$ and $\wedge^*_\xi(M)$ the bundles of horizontal and vertical antisymmetric tensors and by $\Omega^*_H(M)$ and $\Omega^*_\xi(M)$ the horizontal and vertical forms associated to.

Let (M, θ) be a contact manifold, then there exists a riemannian metric g_θ and a $(1, 1)$-tensor field J on M such that:

$$g_\theta(\xi, X) = \theta(X), \quad J^2 = -Id + \theta \otimes \xi, \quad g_\theta(JX, Y) = d\theta(X, Y), \quad X, Y \in TM.$$

The metric g_θ (called the Webster metric) is said to be associated to θ. We call $(\theta, \xi, J, g_\theta)$ a contact metric structure and $(M, \theta, \xi, J, g_\theta)$ a contact metric manifold (cf. Blair[7]). In the following $\omega_\theta := d\theta$.

Let $(M,\theta,\xi,J,g_\theta)$ be a contact metric manifold. We define $L:\Omega^k(M)\to\Omega^{k+2}(M)$ by $L=\omega_\theta\wedge$. The restriction of L to $\Omega^*_H(M)$ will be denoted by L_H and the adjoint of L_H for the usual scalar product on $\Omega^*_H(M)$ by $\wedge_H$. Recall that, for any $\gamma_H\in\Omega^p_H(M)$,

$$(\wedge_H\gamma_H)(X_1,\ldots,X_{p-2})=\frac{1}{2}tr_H\gamma_H(\,.\,,J\,.\,,X_1,\ldots,X_{p-2}),$$

where tr_H is the trace calculated with respect to a g_θ-orthonormal frame of H.

Let $\Omega^*_{H_0}(M):=\{\gamma_H\in\Omega^*_H(M),\ \wedge_H\gamma_H=0\}$ and $\mathcal{F}^*_\xi(M):=\{\gamma_\xi\in\Omega^*_\xi(M),\ L\gamma_\xi=0\}$ be the bundle of primitive horizontal forms on M and the bundle of coprimitive vertical forms on M. We recall the Lefschetz decomposition

$$\Omega^*_H(M)=\Omega^*_{H_0}(M)\oplus L_H\Omega^*_{H_0}(M)\oplus\ldots\oplus L^d_H\Omega^*_{H_0}(M).$$

. connection and curvature

For the torsion and the curvature of a connection ∇ we adopt the conventions $T(X,Y)=[X,Y]-\nabla_XY+\nabla_YX$ and $R(X,Y)=[\nabla_Y,\nabla_X]-\nabla_{[Y,X]}$.
In the following, N is the TM-valued 2-form given by:

$$N(Y,Z)=J^2[Y,Z]+[JY,JZ]-J[Y,JZ]-J[JY,Z]+\omega_\theta(Y,Z)\xi.$$

Proposition 2.1 *(Generalized Tanaka-Webster connection cf. [29],[30],[31]). Let $(M,\theta,\xi,J,g_\theta)$ be contact metric manifold, then there exists a unique affine connection ∇ on TM with torsion T (called the generalized Tanaka-Webster connection) such that:*
(a) $\nabla\theta=0$, $\nabla\xi=0$.
(b) $\nabla g_\theta=0$.
(c) $T_H=-\omega_\theta\otimes\xi$ *and* $i(\xi)T=-\frac{1}{2}i(\xi)N$.
(d) $(\nabla_X\omega_\theta)(Y,Z)=g_\theta((\nabla_XJ)(Y),Z)=\frac{1}{2}\omega_\theta(X,N_H(Y,Z))$ *for any* $X,Y,Z\in TM$.

The endomorphism $\tau:=i(\xi)T$ is called the generalized Tanaka-Webster torsion or sub-torsion. Note that τ is g_θ-symmetric with trace-free and satisfies $\tau\circ J=-J\circ\tau$.

A contact metric manifold $(M,\theta,\xi,J,g_\theta)$ for which J is integrable (i.e. $N_H=0$ or equivalently $\nabla J=0$) is called a strictly pseudoconvex CR manifold. A strictly pseudoconvex CR manifold for which the Tanaka-Webster torsion vanishes is called a Sasakian manifold.

The curvature R of the generalized Tanaka-Webster connection ∇ satisfies the following Bianchi identities (cf. [17],[29]):
(First Bianchi identity)

$$\begin{aligned}R_H(X,Y)Z+R_H(Z,X)Y+R_H(Y,Z)X &= \omega_\theta(X,Y)\tau(Z)+\omega_\theta(Z,X)\tau(Y)\\ &\quad+\omega_\theta(Y,Z)\tau(X)\end{aligned}\qquad(1)$$

$$R(X,\xi)Z + R(\xi,Z)X = (\nabla_X\tau)(Z) - (\nabla_Z\tau)(X) \tag{2}$$

with $(\nabla_X\tau)(Z) = \nabla_X\tau(Z) - \tau(\nabla_X Z)$.

Remember that any horizontal 2-tensor t_H on M decomposes into $t_H = t_H^+ + t_H^-$, where $t_H^\pm := \frac{1}{2}(t_H \pm J^*t_H)$ are respectively the J-invariant part and the J-anti-invariant part of t_H.

If M is a strictly pseudoconvex CR manifold, then we have the decomposition:

$$R_H = R_H^+ + R_H^-,$$

with

$$R_H^-(X,Y) = -\frac{1}{2}\Big((\tau(X))^*\wedge(JY)^* - (\tau(Y))^*\wedge(JX)^* - (J\tau(X))^*\wedge Y^* + (J\tau(Y))^*\wedge X^*\Big). \tag{3}$$

Also we define the pseudo-Hermitian curvature tensor $R_H^{\mathcal{W}}$ of a strictly pseudoconvex CR manifold by:

$$R_H^{\mathcal{W}}(X,Y,Z,W) = g_\theta(R_H^+(X,Y)Z,W), \quad X,Y,Z,W \in H.$$

In order to give the algebraic properties of $R_H^{\mathcal{W}}$, we recall some definitions related to the curvature algebra (cf. [11]).

Let (V,q) be an euclidean space and $Q \in \bigotimes^4 V^*$. We define the Bianchi map $b(Q)$ by:

$$b(Q)(X,Y,Z,W) = Q(X,Y,Z,W) + Q(Z,X,Y,W) + Q(Y,Z,X,W), \quad X,Y,Z,W \in V.$$

Recall that $b(S^2(\wedge^2 V^*)) = \wedge^4 V^*$ and that we have the decomposition:

$$S^2(\wedge^2 V^*) = Ker\, b \oplus \wedge^4 V^*.$$

Let $h,k \in \bigotimes^2 V^*$. We define the symmetric product $h\odot k \in S^2(\bigotimes^2 V^*)$ and the Kulkarni product $h \owedge k \in \bigotimes^2(\wedge^2 V^*)$ respectively by:

$$(h\odot k)(X,Y,Z,W) = h(X,Y)k(Z,W) + h(Z,W)k(X,Y),$$

and

$$(h \owedge k)(X,Y,Z,W) = (h\odot k)(X,Z,Y,W) - (h\odot k)(X,W,Y,Z).$$

Note that if $h,k \in S^2V^*$ then $h \owedge k \in S^2(\wedge^2 V^*)$ and $b(h \owedge k) = 0$.
If $h,k \in \wedge^2 V^*$ then $h \owedge k \in S^2(\wedge^2 V^*)$ and $b(h \owedge k) = -2b(h\odot k)$.
If $h \in S^2V^*$ and $k \in \wedge^2 V^*$ then $h \owedge k \in \wedge^2(\wedge^2 V^*)$ and $b(h \owedge k) = -2b(k\otimes h)$.
We define the Ricci contraction $c(Q)$ by:

$$c(Q)(X,Y) = tr\, Q(.,X,.,Y),$$

with the trace is taken with respect to a q-orthonormal basis of V.

Let $(M, \theta, \xi, J, g_\theta)$ be a contact metric manifold and let $Q_H \in \bigotimes^2(\wedge^2_H(M))$. The endomorphism $\widehat{Q_H}$ of $\wedge^2_H(M)$ associated to Q_H is defined by:

$$(\widehat{Q_H}\gamma_H)(X,Y) = \frac{1}{2}\sum_{i,j} Q_H(\epsilon_i, \epsilon_j, X, Y)\gamma_H(\epsilon_i, \epsilon_j),$$

where $\gamma_H \in \wedge^2_H(M)$, $\{\epsilon_i\}$ is a local g_θ-orthonormal frame of H and $X, Y \in H$.

For $X, Y, Z, W \in H$, we have $\langle \widehat{Q_H}(X \wedge Y), Z \wedge W\rangle = Q_H(X, Y, Z, W)$.

Let $\wedge^2_{H_0}(M)$ be the bundle of primitif horizontal antisymmetric 2-tensors. For $Q_H \in S^2(\wedge^2_H(M))$, we define $Q_{H_0} \in S^2(\wedge^2_{H_0}(M))$ by:

$$Q_{H_0} = Q_H - \frac{1}{d}(\widehat{Q_H}\omega_\theta) \odot \omega_\theta + \frac{1}{2d^2}(\wedge_H(\widehat{Q_H}\omega_\theta))\omega_\theta \odot \omega_\theta.$$

Note that, for Q_H viewed as horizontal $\wedge^2_H(M)$-valued 2-form, we have $\widehat{Q_H}\omega_\theta = \wedge_H Q_H$.

Let $\wedge^{2,\pm}_H(M)$ be the bundle of J-invariant (J-anti-invariant) horizontal antisymmetric 2-tensors. For $Q_H \in S^2(\wedge^2_H(M))$, we define $Q^\pm_H \in S^2(\wedge^{2,\pm}_H(M))$ by:

$$\begin{aligned} Q^\pm_H(X,Y,Z,W) &= \frac{1}{4}(Q_H(X,Y,Z,W) \pm Q_H(JX,JY,Z,W) \pm Q_H(X,Y,JZ,JW) \\ &+ Q_H(JX,JY,JZ,JW)). \end{aligned}$$

Note that

$$(g_{\theta_H} \owedge g_{\theta_H})_0 = g_{\theta_H} \owedge g_{\theta_H} - \frac{1}{d}\omega_\theta \odot \omega_\theta$$

and

$$(g_{\theta_H} \owedge g_{\theta_H})^\pm = \frac{1}{2}(g_{\theta_H} \owedge g_{\theta_H} \pm \omega_\theta \owedge \omega_\theta),$$

with $g_{\theta_H} = g_{\theta/H}$.
For any $P_H, Q_H \in S^2(\wedge^2_H(M))$, the scalar product $\langle P_H, Q_H\rangle$ is defined by:

$$\langle P_H, Q_H\rangle = \frac{1}{2} tr_H \widehat{P_H} \circ \widehat{Q_H}.$$

We have

$$\langle (g_{\theta_H} \owedge g_{\theta_H})^\pm, Q_H\rangle = tr_H \widehat{Q^\pm_H}$$

and for $Q_H \in S^2(\wedge^2_H(M)) \cap Ker\, b$,

$$\langle \omega_\theta \odot \omega_\theta, Q_H\rangle = tr_H \widehat{Q^+_H} - tr_H \widehat{Q^-_H}.$$

If M is CR then $R_H^{\mathcal{W}}$ can be written by (3) as:

$$R_H^{\mathcal{W}} = R_H + \frac{1}{2}(\omega_\theta \owedge A_\theta - g_{\theta_H} \owedge B_\theta), \tag{4}$$

with $R_H(X,Y,Z,W) = g_\theta(R_H(X,Y)Z,W)$, $A_\theta(X,Y) = g_\theta(\tau(X),Y)$ and $B_\theta(X,Y) = \omega_\theta(\tau(X),Y)$.
It follows from (1) and (4) that $R_H^{\mathcal{W}}$ satisfies the following the algebraic properties:

$R_H^{\mathcal{W}} \in S^2(\wedge_H^2(M)) \cap Ker\, b$ and $R_H^{\mathcal{W}} \in S^2(\wedge_H^{2,+}(M))$.

The pseudo-Hermitian Ricci tensor $Ric_H^{\mathcal{W}} \in S_H^2(M)$, the pseudo-Hermitian Ricci form $\rho_H^{\mathcal{W}} \in \wedge_H^{2,+}(M)$ and the pseudo-Hermitian scalar curvature $s^{\mathcal{W}}$ are respectively defined by:

$$Ric_H^{\mathcal{W}} = c_H(R_H^{\mathcal{W}}), \quad \rho_H^{\mathcal{W}} = -\widehat{R_H^{\mathcal{W}}}\omega_\theta, \quad s^{\mathcal{W}} = tr_H Ric_H^{\mathcal{W}},$$

with $c_H(Q_H)$ calculated with respect to a g_θ-orthonormal frame of H.
Now, let $I_H^{\mathbb{C}} \in S^2(\wedge_H^{2,+}(M)) \cap Ker\, b$ given by:

$$I_H^{\mathbb{C}} = \frac{1}{8}\Big(g_{\theta_H} \owedge g_{\theta_H} + \omega_\theta \owedge \omega_\theta + 2\omega_\theta \odot \omega_\theta\Big).$$

We have the decompositions (cf. [13]):

$$R_H^{\mathcal{W}} = \frac{s^{\mathcal{W}}}{d(d+1)} I_H^{\mathbb{C}} + \frac{1}{d+2}\Big(\frac{1}{2}(Ric_{H_0}^{\mathcal{W}} \owedge g_{\theta_H} - \rho_{H_0}^{\mathcal{W}} \owedge \omega_\theta) - \rho_{H_0}^{\mathcal{W}} \odot \omega_\theta\Big) + C_H^{\mathcal{M}}$$

$$R_{H_0}^{\mathcal{W}} = \frac{s^{\mathcal{W}}}{d(d+1)} I_{H_0}^{\mathbb{C}} + \frac{1}{2(d+2)}\Big(Ric_{H_0}^{\mathcal{W}} \owedge g_{\theta_H} - \rho_{H_0}^{\mathcal{W}} \owedge \omega_\theta\Big)_0 + C_H^{\mathcal{M}},$$

where $Ric_{H_0}^{\mathcal{W}}$ (respectively $\rho_{H_0}^{\mathcal{W}}$) is the traceless part of $Ric_H^{\mathcal{W}}$ (respectively the primitive part of $\rho_H^{\mathcal{W}}$) and $C_H^{\mathcal{M}} \in S^2(\wedge_{H_0}^{2,+}(M)) \cap Ker\, b \cap Ker\ c_H$.

Remark 2.1 *The tensor $C_H^{\mathcal{M}}$, introduced by Chern and Moser in [13], is called the Chern-Moser tensor. Note that $C_H^{\mathcal{M}}$ is a pseudo-conformal invariant.*

Now, we define the pseudo-Hermitian sectional curvature (resp. pseudo-Hermitian complex sectional curvature) of a 2-plane $P = \mathbb{R}\{X,Y\} \subset H$ (resp. $P = \mathbb{C}\{Z,W\} \subset H^{\mathbb{C}}$) by:

$$K^{\mathcal{W}}(P) = \frac{\langle \widehat{R_H^{\mathcal{W}}}(X \wedge Y), X \wedge Y\rangle}{\langle X \wedge Y, X \wedge Y\rangle}, \quad K^{\mathcal{W}^{\mathbb{C}}}(P) = \frac{(\widehat{R_H^{\mathcal{W}^{\mathbb{C}}}}(Z \wedge W), \overline{Z \wedge W})}{(Z \wedge W, \overline{Z \wedge W})},$$

where $(\ ,\ )$ and $\widehat{R_H^{\mathcal{W}^{\mathbb{C}}}}$ are the natural extensions to $\wedge^2 H^{\mathbb{C}}$ of $\langle\ ,\ \rangle$ and $\widehat{R_H^{\mathcal{W}}}$.
The holomorphic pseudo-Hermitian sectional curvature of a holomorphic 2-plane $P = \mathbb{R}\{X, JX\} \subset H$ is defined by:

$$HK^{\mathcal{W}}(P) = \frac{\langle \widehat{R_H^{\mathcal{W}}}(X \wedge JX), X \wedge JX\rangle}{\langle X \wedge JX, X \wedge JX\rangle}.$$

We say that a strictly pseudoconvex CR manifold $(M, \theta, \xi, J, g_\theta)$ has constant holomorphic pseudo-Hermitian sectional curvature if $HK^{\mathcal{W}}(P)$ is constant for any holomorphic 2-plane $P \subset H$ and for any point of M. In this case, we have $R_H^{\mathcal{W}} = \dfrac{s^{\mathcal{W}}}{d(d+1)} I_H^{\mathbb{C}}$ with $s^{\mathcal{W}}$ constant (cf. [2]). Also we call $I_H^{\mathbb{C}}$ the holomorphic pseudo-Hermitian curvature tensor.

Let (E, g^E, ∇^E) be a riemannian vector bundle over a contact metric manifold M and let $\Omega^*(M; E)$ (resp $\Omega_H^*(M; E)$) be the bundle of E-valued forms (resp. horizontal E-valued forms) on M. We assume that M is endowed with the generalized Tanaka-Webster connection ∇. Remember (cf. [25]) that for any $\sigma \in \Omega^p(M; E)$,

$$\begin{aligned}
(\nabla_X \sigma)(X_1, \ldots, X_p) &= \nabla_X^E \sigma(X_1, \ldots, X_p) - \sum_{i=1}^{p} \sigma(X_1, \ldots, \nabla_X X_i, \ldots, X_p) \\
(d^{\nabla^E}\sigma)(X_1, \ldots, X_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} \nabla_{X_i}^E \sigma(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}) \\
&+ \sum_{i<j} (-1)^{i+j} \sigma([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}) \\
&= \sum_{i=1}^{p+1} (-1)^{i+1} (\nabla_{X_i} \sigma)(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}) \\
&+ \sum_{i<j} (-1)^{i+j} \sigma(T(X_i, X_j), X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}).
\end{aligned}$$

Now, for any $\sigma_H \in \Omega_H^p(M; E)$, we define

$$\begin{aligned}
(d_H^{\nabla^E}\sigma_H)(X_1, \ldots, X_{p+1}) &:= (d^{\nabla^E}\sigma_H)_H(X_1, \ldots, X_{p+1}) \\
&= \sum_{i=1}^{p+1} (-1)^{i+1} (\nabla_{X_i}\sigma_H)(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}) \\
(\mathcal{L}_\xi^{\nabla^E}\sigma_H)(X_1, \ldots, X_p) &:= (i(\xi)(d^{\nabla^E}\sigma_H))(X_1, \ldots, X_p) \\
&= (\nabla_\xi \sigma_H)(X_1, \ldots, X_p) + \sum_{i=1}^{p} (-1)^i \sigma_H(\tau(X_i), X_1, \ldots, \hat{X}_i, \ldots, X_p)
\end{aligned}$$

$$
\begin{aligned}
(\delta_H^{\nabla^E}\sigma_H)(X_1,\ldots,X_{p-1}) &= -tr_H(\nabla_{.}\,\sigma_H)(\,.\,,X_1,\ldots,X_{p-1})\\
(R_H(X,Y)\sigma_H)(X_1,\ldots,X_p) &= R_H^E(X,Y)\sigma_H(X_1,\ldots,X_p)-\sum_{i=1}^{p}\sigma_H(X_1,\ldots,R_H(X,Y)X_i,\ldots,X_p),
\end{aligned}
$$

where R^E is the curvature of ∇^E.
Note that for any $\sigma_H \in \Omega_H^p(M;E)$, we have:

$$
{d_H^{\nabla^E}}^2\sigma_H = -L_H(\mathcal{L}_\xi^{\nabla^E}\sigma_H) - R_H^E\wedge\sigma_H,
$$

where $R_H^E\wedge\sigma_H$ is the wedge product of the horizontal $End(E)$-valued 2-form R_H^E with the horizontal E-valued p-form σ_H. Now the horizontal $End(H)$-valued 2-form R_H satisfies the second Bianchi identity:

$$
\begin{aligned}
(\nabla_X R_H)(Y,Z)+(\nabla_Z R_H)(X,Y)+(\nabla_Y R_H)(Z,X) &= -\omega_\theta(X,Y)(i(\xi)R)(Z)+\omega_\theta(Z,X)(i(\xi)R)(Y)\\
&\quad -\omega_\theta(Y,Z)(i(\xi)R)(X)\\
(\nabla_\xi R_H)(X,Y)-(\nabla_X i(\xi)R)(Y)+(\nabla_Y i(\xi)R)(X) &= R_H(\tau(X),Y)+R_H(X,\tau(Y)). \qquad (5)
\end{aligned}
$$

Remark 2.2 *The Bianchi identity (5) is equivalent to $d_H^\nabla R_H = -L_H(i(\xi)R)$ and $\mathcal{L}_\xi^\nabla R_H = d_H^\nabla(i(\xi)R)$ where ∇ is the connection on $\wedge_H^2(M)$ induced by the Tanaka-Webster connection on H.*

3 Contact sub-symmetric spaces

Definition 3.1 *A contact (locally) sub-symmetric space is a contact metric manifold $(M,\theta,\xi,J,g_\theta)$ such that for every point $x_0\in M$ there exists an isometry (resp a local isometry) ψ, called the sub-symmetry at x_0, satisfying $\psi(x_0)=x_0$ and $d\psi(x_0)/H_{x_0} = -id$.*

Theorem 3.1 *[6] Let $(M,\theta,\xi,J,g_\theta)$ be a contact metric manifold endowed with its generalized Tanaka-Webster connection ∇ and let R (respectively T) be the curvature (respectively torsion) of ∇. Then:*
(i) M is a contact locally sub-symmetric space if and only if $\nabla_H R = \nabla_H T = 0$.
(ii) If M is a contact locally sub-symmetric space, ∇-complete and simply-connected then M is a contact sub-symmetric space.
(iii) If M is a contact sub-symmetric space then $M = G/K$ where G is the closed subgroup of $I(M,g_\theta)$ generated by all the sub-symmetries $\psi(x_0)$, $x_0\in M$, and K is the isotropy subgroup at a base point (i.e. M is an homogeneous manifold).

Note that the conditions $\nabla_H R = \nabla_H T = 0$ are equivalent to $\nabla_H R_H = 0$, $i(\xi)R = 0$, $\nabla\omega_\theta = 0$ and $\nabla_H\tau = 0$.

Definition 3.2 *[18] A contact sub-symmetric space M is said to be irreducible if the Lie algebra $\mathfrak{Hol}(M)$ of the holonomy group $Hol(M)$ acts irreducibly on H.*

The simply-connected contact sub-symmetric spaces have been classified by Bieliavsky, Falbel and Gorodski in [6].

Theorem 3.2 *Every simply-connected contact sub-symmetric space of dimension* ≥ 5 *has the following type:*

holonomy trivial	*torsionless*	$\mathcal{H}_{2p+1}$	
		Compact type	*Non-compact type*
holonomy irreducible	*torsionless*	*compact Hermitian (CH):* $SU(p+q)/SU(p)\times SU(q)$ $SO(2p)/SU(p)$ $Sp(p)/SU(p)$ $SO(p+2)/SO(p) \quad (p\geq 3)$ $E_{6(-78)}/Spin(10)$ $E_{7(-133)}/E_6$	*non-compact Hermitian (NCH):* $SU(p,q)/SU(p)\times SU(q)$ $SO^*(2p)/SU(p)$ $Sp(p,\mathbb{R})/SU(p)$ $SO_0(p,2)/SO(p) \quad (p\geq 3)$ $E_{6(-14)}/Spin(10)$ $E_{7(-25)}/E_6$
	with torsion $(p\geq 3)$	$SO(p+1)\ltimes \mathbb{R}^{p+1}/SO(p)$	$SO_0(p,1)\ltimes \mathbb{R}^{p+1}/SO(p)$ $SO_0(p+1,1)/SO(p)$
holonomy not irreducible	*torsionless*	$\mathcal{H}_{2p+1}\times_G CH \quad (G\subset U(p))$	$\mathcal{H}_{2p+1}\times_G NCH$
	with torsion	$SO(4)/SO(2)$ $SO(3)\ltimes \mathbb{R}^3/SO(2)$	$SO_0(2,2)/SO(2)$ $SO_0(2,1)\ltimes \mathbb{R}^3/SO(2)$ $SO_0(3,1)/SO(2)$

Remark 3.1 *The contact sub-symmetric spaces of compact Hermitian type (respectively non-compact Hermitian type) arise from* S^1*-fibrations over irreducible Hermitian symmetric spaces of compact type (respectively non-compact type). The previous list is not complete, because* S^1*-fibrations over not irreducible Hermitian symmetric spaces also produce examples of contact sub-symmetric spaces. We do not consider these examples in the following.*

A contact locally sub-symmetric space $(M,\theta,\xi,J,g_\theta)$ is always a strictly pseudoconvex CR manifold (since $\nabla\omega_\theta=0$). Now we recall the notion of homogeneous strictly pseudoconvex CR manifold and symmetric strictly pseudoconvex CR manifold (cf. [20],[24]).

Let (M,H,J,θ,g_θ) and $(N,H',J',\theta',g_{\theta'})$ be strictly pseudoconvex CR manifolds then a map $\phi: M\to N$ such that $d\phi(H)\subset H'$ and $J'\circ d\phi_H = d\phi_H\circ J$ is called a CR map from M to N (the definition is also valid in the general context of CR manifolds [1]). A CR automorphism $\phi: M\to M$ is a diffeomorphism and a CR map from M to M. The group of CR automorphisms $Aut_{CR}(M)$ is a Lie group. A CR automorphism $\phi: M\to M$ is called a pseudo-Hermitian transformation if $\phi^*\theta=\theta$. The group of pseudo-Hermitian transformations $PsH(M,\theta)$ is a Lie subgroup of $Aut_{CR}(M)$ and also a Lie subgroup of $I(M,g_\theta)$.

Definition 3.3 *[24] A strictly pseudoconvex CR manifold $(M, H, J, \theta, g_\theta)$ is called homogeneous if there exists a closed subgroup G of $PsH(M, \theta)$ which acts transitively on M.*

Definition 3.4 *A (locally) symmetric strictly pseudoconvex CR manifold is a strictly pseudoconvex CR manifold $(M, H, J, \theta, g_\theta)$ such that for every point $x_0 \in M$ there exists a pseudo-Hermitian transformation (resp a local pseudo-Hermitian transformation) ψ, called the pseudo-Hermitian symmetry at x_0, satisfying $\psi(x_0) = x_0$ and $d\psi(x_0)/H_{x_0} = -id$.*

If $(M, H, J, \theta, g_\theta)$ is a symmetric strictly pseudoconvex CR manifold then $M = G/K$ where G is the closed subgroup of $PsH(M, \theta)$ generated by all the pseudo-Hermitian symmetries $\psi(x_0)$, $x_0 \in M$, and K is the isotropy subgroup at a base point. Also M is an homogeneous strictly pseudoconvex CR manifold. Note that the contact sub-symmetric spaces torsionless are symmetric Sasakian manifolds.

Let $(M, \theta, \xi, J, g_\theta)$ be a simply-connected contact sub-symmetric space and Γ be a cocompact discrete subgroup of $PsH(M, \theta)$ acting freely on M then M/Γ is a compact contact locally sub-symmetric space.

Now we investigate the properties of the pseudo-Hermitian curvature tensor on a contact locally sub-symmetric space.

Proposition 3.1 *Let $(M, \theta, \xi, J, g_\theta)$ be a contact locally sub-symmetric space endowed with its Tanaka-Webster connection ∇. Then we have $\nabla R_H^{\mathcal{W}} = 0$ and consequently $s^{\mathcal{W}}$ is constant.*

Proof. Since $\nabla_H R_H = 0$ and $\nabla J = 0$, we have $\nabla_H R_H^+ = 0$. Now, we must prove that $\nabla_\xi R_H^+ = 0$. Equation (5) together with the assumption $i(\xi)R = 0$ gives, for any $X, Y \in H$,

$$(\nabla_\xi R_H)(X, Y) = R_H(\tau(X), Y) + R_H(X, \tau(Y)).$$

Since $J \circ \tau = -\tau \circ J$, we deduce that

$$(\nabla_\xi R_H^+)(X, Y) = R_H^-(\tau(X), Y) + R_H^-(X, \tau(Y)). \tag{6}$$

If $\tau = 0$, we have automatically $\nabla_\xi R_H^+ = 0$. Now, if $\tau \neq 0$, the assumption $\nabla_H \tau = 0$ implies that $|\tau|^2$ is a strictly positive constant and that $\tau^2 = \dfrac{|\tau|^2}{2d} id_H$ (cf. lemma 1 of [9]). This assumption together with (3) implies that $R_H^-(\tau(X), Y) = -R_H^-(X, \tau(Y))$ and then (6) becomes $\nabla_\xi R_H^+ = 0$. Hence $\nabla R_H^+ = 0$ and $\nabla R_H^{\mathcal{W}} = 0$. $\square$

In [9], Boeckx and Cho prove that a contact metric manifold M endowed with its generalized Tanaka-Webster connection ∇ satisfying the conditions $\nabla_H J \circ \tau = 0$ and $\tau \neq 0$ is a strictly pseudoconvex CR manifold (i.e. J is integrable) and a (k,μ)-space (cf. [8]). Moreover, Cho gives, in [14], a formula for the riemannian curvature tensor of M if M has constant holomorphic pseudo-Hermitian sectional curvature. Now we obtain a formula for the pseudo-Hermitian curvature tensor of a strictly pseudoconvex CR manifold satisfying $\nabla_H \tau = 0$.

Theorem 3.3 *Let $(M, \theta, \xi, J, g_\theta)$ be a strictly pseudoconvex CR manifold endowed with its Tanaka-Webster connection ∇. Assume that $\nabla_H \tau = 0$ and $\tau \neq 0$, then the pseudo-Hermitian curvature tensor and the Chern-Moser tensor of M are given by:*

$$R_H^{\mathcal{W}} = \frac{s^{\mathcal{W}}}{d^2}\Big(I_H^{\mathbb{C}} + \frac{2d}{|\tau|^2}\mathcal{T}_H\Big), \quad C_H^{\mathcal{M}} = \frac{s^{\mathcal{W}}}{d^2}\Big(\frac{1}{d+1}I_{H_0}^{\mathbb{C}} + \frac{2d}{|\tau|^2}\mathcal{T}_{H_0}\Big), \tag{7}$$

with $\mathcal{T}_H = \frac{1}{8}\Big(A_\theta \owedge A_\theta + B_\theta \owedge B_\theta\Big)$. Moreover, if $d \geq 2$, then M is a contact locally sub-symmetric space.

The proof of the theorem needs the following Lemma.

Lemma 3.1 *Let $(M, \theta, \xi, J, g_\theta)$ be a strictly pseudoconvex CR manifold such that $\nabla_H \tau = 0$ and $\tau \neq 0$. Then $\rho_H^{\mathcal{W}} = -\frac{s^{\mathcal{W}}}{2d}\omega_\theta$ (i.e. M is pseudo-Einstein), and $\nabla_\xi \tau = -\frac{s^{\mathcal{W}}}{d^2} J \circ \tau$.*

Proof. First recall that the assumptions $\nabla_H \tau = 0$ and $\tau \neq 0$ imply that $|\tau|^2$ is a strictly positive constant and that $\tau^2 = \frac{|\tau|^2}{2d} id_H$. Now, we have for any $X, Y \in H$,

$$R_H(X,Y)\tau = B_H(X,Y)\tau - \omega_\theta(X,Y)\nabla_\xi \tau,$$

with $B_H(X,Y) = \nabla_Y \nabla_X - \nabla_{\nabla_Y X} - (\nabla_X \nabla_Y - \nabla_{\nabla_X Y})$. Since $\nabla_H \tau = 0$, we have $R_H(X,Y)\tau = -\omega_\theta(X,Y)\nabla_\xi \tau$ and also $R_H^+(X,Y)\tau = -\omega_\theta(X,Y)\nabla_\xi \tau$. We obtain

$$g_\theta(R_H^+(X,Y)\tau(Z), J\tau(W)) - g_\theta(\tau(R_H^+(X,Y)Z), J\tau(W)) = -\omega_\theta(X,Y) g_\theta((\nabla_\xi \tau)(Z), J\tau(W)).$$

Hence,

$$R_H^{\mathcal{W}}(X,Y,\tau(Z),J\tau(W)) + \frac{|\tau|^2}{2d} R_H^{\mathcal{W}}(X,Y,Z,JW) = -\omega_\theta(X,Y) g_\theta((\nabla_\xi \tau)(Z), J\tau(W)).$$

Let $\{\epsilon_i\}$ be a local g_θ-orthonormal frame of H, then

$$\sum_{1 \leq i \leq 2d} R_H^{\mathcal{W}}(X,Y,\tau(\epsilon_i),J\tau(\epsilon_i)) + \frac{|\tau|^2}{2d} R_H^{\mathcal{W}}(X,Y,\epsilon_i,J\epsilon_i) = -\omega_\theta(X,Y) \sum_{1 \leq i \leq 2d} g_\theta((\nabla_\xi \tau)(\epsilon_i), J\tau(\epsilon_i)). \tag{8}$$

Since $\tau^2 = \dfrac{|\tau|^2}{2d} id_H$, then (8) becomes

$$\frac{|\tau|^2}{d} \sum_{1\leq i\leq 2d} R_H^{\mathcal{W}}(\epsilon_i, J\epsilon_i, X, Y) = -\omega_\theta(X,Y) \sum_{1\leq i\leq 2d} g_\theta((\nabla_\xi \tau)(\epsilon_i), J\tau(\epsilon_i))$$

which is

$$\rho_H^{\mathcal{W}} = -\wedge_H R_H^{\mathcal{W}} = \omega_\theta \frac{d}{|\tau|^2}(\nabla_\xi A_\theta, B_\theta).$$

Since $\wedge_H \rho_H^{\mathcal{W}} = -\dfrac{s^{\mathcal{W}}}{2}$, we deduce that $\rho_H^{\mathcal{W}} = -\dfrac{s^{\mathcal{W}}}{2d}\omega_\theta$ and then M is pseudo-Einstein. Now we have

$$(\wedge_H R_H^+)(\tau(X)) - \tau((\wedge_H R_H^+)(X)) = -d(\nabla_\xi \tau)(X).$$

The assumption $\rho_H^{\mathcal{W}} = -\dfrac{s^{\mathcal{W}}}{2d}\omega_\theta$ gives

$$(\wedge_H R_H^+)(\tau(X)) - \tau((\wedge_H R_H^+)(X)) = \frac{s^{\mathcal{W}}}{d} J\circ\tau(X)$$

and then $\nabla_\xi \tau = -\dfrac{s^{\mathcal{W}}}{d^2} J\circ\tau$. $\square$

Proof of Theorem 3.3 Using (2), we obtain that for any $X, Y, Z \in H$

$$g_\theta(R(X,\xi)Y, Z) = g_\theta((\nabla_Z \tau)(X), Y) - g_\theta((\nabla_Y \tau)(X), Z).$$

By the assumption $\nabla_H \tau = 0$, we have $i(\xi)R = 0$. Now, equation (5) together with $i(\xi)R = 0$, $J\circ\tau = -\tau\circ J$ and $\tau^2 = \dfrac{|\tau|^2}{2d} id_H$ yields

$$R_H^+(\tau(X), \tau(Y)) + \frac{|\tau|^2}{2d} R_H^+(X,Y) = (\nabla_\xi R_H^-)(X, \tau(Y)). \tag{9}$$

By (3), we have

$$\begin{aligned}(\nabla_\xi R_H^-)(X,Y) &= -\frac{1}{2}\Big(((\nabla_\xi\tau)(X))^* \wedge (JY)^* - ((\nabla_\xi\tau)(Y))^* \wedge (JX)^* \\ &- ((J(\nabla_\xi\tau)(X))^* \wedge Y^* + ((J(\nabla_\xi\tau)(Y))^* \wedge X^*\Big).\end{aligned}$$

We have $\nabla_\xi \tau = -\dfrac{s^{\mathcal{W}}}{d^2} J\circ\tau$ (Lemma 3.1), it follows that

$$(\nabla_\xi R_H^-)(X, \tau(Y)) = \frac{s^{\mathcal{W}}}{4d^3}|\tau|^2\Big(X^*\wedge Y^* + (JX)^*\wedge(JY)^*\Big) + \frac{s^{\mathcal{W}}}{2d^2}\Big((\tau(X))^*\wedge(\tau(Y))^* + (J\tau(X))^*\wedge(J\tau(Y))^*\Big).$$

Then (9) becomes

$$\begin{aligned} R^{\mathcal{W}}_H(\tau(X),\tau(Y),Z,W)+\frac{|\tau|^2}{2d}R^{\mathcal{W}}_H(X,Y,Z,W) &= \frac{s^{\mathcal{W}}}{8d^3}|\tau|^2\Big(g_{\theta_H}\owedge g_{\theta_H}+\omega_\theta\owedge\omega_\theta\Big)(X,Y,Z,W) \\ &+ \frac{s^{\mathcal{W}}}{4d^2}\Big(A_\theta\owedge A_\theta+B_\theta\owedge B_\theta\Big)(X,Y,Z,W). \quad (10) \end{aligned}$$

Now, we have

$$\begin{aligned} g_\theta(R^+_H(X,Y)\tau(Z),\tau(W))-g_\theta(\tau(R^+_H(X,Y)Z),\tau(W)) &= -\omega_\theta(X,Y)g_\theta((\nabla_\xi\tau)(Z),\tau(W)) \\ &= -\frac{s^{\mathcal{W}}}{2d^3}|\tau|^2\omega_\theta(X,Y)\omega_\theta(Z,W). \end{aligned}$$

Hence

$$R^{\mathcal{W}}_H(\tau(X),\tau(Y),Z,W)-\frac{|\tau|^2}{2d}R^{\mathcal{W}}_H(X,Y,Z,W)=-\frac{s^{\mathcal{W}}}{4d^3}|\tau|^2(\omega_\theta\odot\omega_\theta)(X,Y,Z,W). \quad (11)$$

We deduce from (10) and (11) the following expression for the pseudo-Hermitian curvature

$$\begin{aligned} R^{\mathcal{W}}_H(X,Y,Z,W) &= \frac{s^{\mathcal{W}}}{8d^2}\Big(g_{\theta_H}\owedge g_{\theta_H}+\omega_\theta\owedge\omega_\theta+2\omega_\theta\odot\omega_\theta\Big)(X,Y,Z,W) \\ &+ \frac{s^{\mathcal{W}}}{4d|\tau|^2}\Big(A_\theta\owedge A_\theta+B_\theta\owedge B_\theta\Big)(X,Y,Z,W). \end{aligned}$$

The expression for $C^{\mathcal{M}}_H$ directly follows from the decomposition of $R^{\mathcal{W}}_H$. Now we assume $d>1$. Since $\nabla_H\tau=0$, we have the formula $\delta_H Ric^{\mathcal{W}}_H=-\frac{1}{2}d_H s^{\mathcal{W}}$. Also M pseudo-Einstein and $d>1$ yields to $s^{\mathcal{W}}$ constant. Since $\nabla_H g_\theta=\nabla_H\omega_\theta=\nabla_H A_\theta=\nabla_H B_\theta=0$ and $s^{\mathcal{W}}$ is constant, then we have by the previous formula for $R^{\mathcal{W}}_H$ and (4) that $\nabla_H R_H=0$. Consequently M is a contact locally sub-symmetric space. □

Corollary 3.1 *The pseudo-Hermitian curvature tensor $R^{\mathcal{W}}_H$ of a contact locally sub-symmetric space M has the following form. If M is holonomy irreducible and torsionless, in this case M is the total space of a S^1-fibration π over an irreducible Hermitian locally symmetric space B and $R^{\mathcal{W}}_H$ is given by $R^{\mathcal{W}}_H=\pi^*R^B$ where R^B is the curvature of B. If M has torsion, in this case $R^{\mathcal{W}}_H$ is given by formula (7). Note that, in each case, M is pseudo-Einstein with $s^{\mathcal{W}}$ constant.*

Remark 3.2 *Note that the contact sub-symmetric spaces of non-compact Hermitian type have nonpositive pseudo-Hermitian sectional curvature whereas the contact sub-symmetric spaces of compact Hermitian type have nonnegative pseudo-Hermitian sectional curvature.*

4 Mok-Siu-Yeung type formulas for horizontal maps between strictly pseudoconvex CR manifolds

Let $(M,\theta,\xi,J,g_\theta,\nabla)$ be a contact metric manifold endowed with the (generalized) Tanaka-Webster connection and let (E,g^E,∇^E) be a Riemannian vector bundle over M. For any $Q_H\in\wedge^2_H(M)\otimes End(E)$ and $\sigma_H\in\wedge^1_H(M)\otimes E$, we define $Q_H(\sigma_H)\in\wedge^1_H(M)\otimes E$ by:

$$Q_H(\sigma_H)(X)=\sum_i Q_H(\epsilon_i,X)\sigma_H(\epsilon_i),$$

where $\{\epsilon_i\}$ is a local g_θ-orthonormal frame of H. For any $Q_H\in S^2(\wedge^2_H(M))$ and $s_H\in S^2_H(M)\otimes E$ (respectively $\sigma_H\in\wedge^2_H(M)\otimes E$), we define $\overset{\circ}{Q}_H\ s_H\in S^2_H(M)\otimes E$ (respectively $\widehat{Q_H}\sigma_H\in\wedge^2_H(M)\otimes E$) by:

$$\begin{aligned}(\overset{\circ}{Q}_H\ s_H)(X,Y) &= \sum_{i,j} Q_H(\epsilon_i,X,Y,\epsilon_j)s_H(\epsilon_i,\epsilon_j),\\ (\widehat{Q_H}\sigma_H)(X,Y) &= \frac{1}{2}\sum_{i,j} Q_H(\epsilon_i,\epsilon_j,X,Y)\sigma_H(\epsilon_i,\epsilon_j).\end{aligned}$$

Proposition 4.1 *Let $(M,\theta,\xi,J,g_\theta,\nabla)$ be a compact contact metric manifold and let $Q_H\in\Gamma(S^2(\wedge^2_H(M)))$ satisfying as horizontal $\wedge^2_H(M)$-valued 2-form the assumptions $\delta^\nabla_H Q_H=0$ and $\wedge_H Q_H=0$. Then, for any $\sigma_H\in\Omega^1_H(M;E)$, we have:*

$$\begin{aligned}\int_M\langle\overset{\circ}{Q}_H\ \nabla^{\mathcal{S}}\sigma_H,\nabla^{\mathcal{S}}\sigma_H\rangle+\langle(b(\widehat{Q_H)-}Q_H)d^{\nabla^E}_H\sigma_H,d^{\nabla^E}_H\sigma_H\rangle v_{g_\theta} &= 2\int_M\langle(\widehat{Q_H}R^E_H)(\sigma_H),\sigma_H\rangle\\ &\quad-\langle(c_H(\widehat{R_H}\circ\widehat{Q_H}))^{\mathcal{S}},\sigma^*_H g^E\rangle v_{g_\theta},\end{aligned}\tag{12}$$

where, for any horizontal 2-tensor μ_H, $\mu^{\mathcal{S}}_H(X,Y)=\mu_H(X,Y)+\mu_H(Y,X)$.

Proof. Let $Q_H\in\Gamma(S^2(\wedge^2_H(M)))$ satisfying $\delta^\nabla_H Q_H=0$ and $\wedge_H Q_H=0$, then formula (8) of [26] gives for any $\sigma_H\in\Omega^1_H(M;E)$

$$(\delta_H Q_H(\nabla))\sigma_H=\mathcal{R}^Q_H\sigma_H,$$

where $Q_H(\nabla)$ and $\mathcal{R}^Q_H$ are given in a local orthonormal frame $\{\epsilon_i\}$ of H by:

$$Q_H(\nabla)(X)=\sum_i\widehat{Q_H}(\epsilon_i\wedge X).\nabla_{\epsilon_i}=\frac{1}{2}\sum_{i,k,l}Q_H(\epsilon_i,X,\epsilon_k,\epsilon_l)\epsilon_k.\epsilon_l.\nabla_{\epsilon_i},$$

and

$$\mathcal{R}^Q_H = -\frac{1}{2}\sum_{i,j}\widehat{Q_H}(\epsilon_i\wedge\epsilon_j).R_H(\epsilon_i,\epsilon_j) = -\frac{1}{4}\sum_{i,j,k,l}Q_H(\epsilon_i,\epsilon_j,\epsilon_k,\epsilon_l)\epsilon_k.\epsilon_l.R_H(\epsilon_i,\epsilon_j).$$

By integrating, we obtain

$$\int_M\langle Q_H(\nabla)\sigma_H,\nabla\sigma_H\rangle v_{g_\theta} = \int_M\langle\mathcal{R}^Q_H\sigma_H,\sigma_H\rangle v_{g_\theta}.$$

We have

$$\langle Q_H(\nabla)\sigma_H,\nabla\sigma_H\rangle = -\frac{1}{2}\sum_{i,j,k,l}Q_H(\epsilon_i,\epsilon_j,\epsilon_k,\epsilon_l)\langle\epsilon_l.\nabla_{\epsilon_i}\sigma_H,\epsilon_k.\nabla_{\epsilon_j}\sigma_H\rangle.$$

Now, for any $X,Y\in TM$ and any $\sigma,\gamma\in\Omega^1(M;E)$, we have (cf. [25]):

$$\langle X.\sigma,Y.\gamma\rangle = g_\theta(X,Y)\langle\sigma,\gamma\rangle + \langle i(X)\sigma,i(Y)\gamma\rangle - \langle i(Y)\sigma,i(X)\gamma\rangle. \tag{13}$$

We deduce from (13) that

$$\langle Q_H(\nabla)\sigma_H,\nabla\sigma_H\rangle = -\sum_{i,j,k,l}Q_H(\epsilon_i,\epsilon_j,\epsilon_k,\epsilon_l)\langle(\nabla_{\epsilon_i}\sigma_H)(\epsilon_l),(\nabla_{\epsilon_j}\sigma_H)(\epsilon_k)\rangle \tag{14}$$

Now we have for any $\sigma_H\in\Omega^1_H(M;E)$ and any $X,Y\in H$

$$(\nabla_X\sigma_H)(Y) = \frac{1}{2}\Big((\nabla^{\mathcal{S}}\sigma_H)(X,Y) + (d^{\nabla^E}_H\sigma_H)(X,Y)\Big),$$

with $(\nabla^{\mathcal{S}}\sigma_H)(X,Y) = (\nabla_X\sigma_H)(Y) + (\nabla_Y\sigma_H)(X)$ and $(d^{\nabla^E}_H\sigma_H)(X,Y) = (\nabla_X\sigma_H)(Y) - (\nabla_X\sigma_H)(Y)$. Then (14) becomes

$$\begin{aligned}
\langle Q_H(\nabla)\sigma_H,\nabla\sigma_H\rangle &= -\frac{1}{4}\Big(\sum_{i,j,k,l}Q_H(\epsilon_i,\epsilon_j,\epsilon_k,\epsilon_l)\langle(\nabla^{\mathcal{S}}\sigma_H)(\epsilon_i,\epsilon_l),(\nabla^{\mathcal{S}}\sigma_H)(\epsilon_j,\epsilon_k)\rangle\\
&\quad+\frac{1}{2}\sum_{i,j,k,l}(Q_H(\epsilon_i,\epsilon_j,\epsilon_k,\epsilon_l)+Q_H(\epsilon_k,\epsilon_i,\epsilon_j,\epsilon_l))\langle(d^{\nabla^E}_H\sigma_H)(\epsilon_i,\epsilon_l),(d^{\nabla^E}_H\sigma_H)(\epsilon_j,\epsilon_k)\rangle\\
&\quad+2\sum_{i,j,k,l}Q_H(\epsilon_i,\epsilon_j,\epsilon_k,\epsilon_l)\langle(\nabla^{\mathcal{S}}\sigma_H)(\epsilon_i,\epsilon_l),(d^{\nabla^E}_H\sigma_H)(\epsilon_j,\epsilon_k)\rangle\Big)\\
&= -\frac{1}{4}\Big(\sum_{i,j,k,l}Q_H(\epsilon_i,\epsilon_j,\epsilon_k,\epsilon_l)\langle(\nabla^{\mathcal{S}}\sigma_H)(\epsilon_i,\epsilon_l),(\nabla^{\mathcal{S}}\sigma_H)(\epsilon_j,\epsilon_k)\rangle\\
&\quad+\frac{1}{2}\sum_{i,j,k,l}(b(Q_H)(\epsilon_i,\epsilon_l,\epsilon_j,\epsilon_k)-Q_H(\epsilon_i,\epsilon_l,\epsilon_j,\epsilon_k))\langle(d^{\nabla^E}_H\sigma_H)(\epsilon_i,\epsilon_l),(d^{\nabla^E}_H\sigma_H)(\epsilon_j,\epsilon_k)\rangle
\end{aligned}$$

$$
\begin{aligned}
&+2\sum_{i,j,k,l} Q_H(\epsilon_i,\epsilon_j,\epsilon_k,\epsilon_l)\langle(\nabla^{\mathcal{S}}\sigma_H)(\epsilon_i,\epsilon_l),(d_H^{\nabla^E}\sigma_H)(\epsilon_j,\epsilon_k)\rangle\Big)\\
=\ & -\frac{1}{4}\Big(\sum_{j,k}\langle(\overset{\circ}{Q_H}\ \nabla^{\mathcal{S}}\sigma_H)(\epsilon_j,\epsilon_k),(\nabla^{\mathcal{S}}\sigma_H)(\epsilon_j,\epsilon_k)\rangle\\
&+\sum_{j,k}\langle(\widehat{(b(Q_H)-Q_H)}d_H^{\nabla^E}\sigma_H)(\epsilon_j,\epsilon_k),(d_H^{\nabla^E}\sigma_H)(\epsilon_j,\epsilon_k)\rangle\\
&+2\sum_{j,k}\langle(\overset{\circ}{Q_H}\ \nabla^{\mathcal{S}}\sigma_H)(\epsilon_j,\epsilon_k),(d_H^{\nabla^E}\sigma_H)(\epsilon_j,\epsilon_k)\rangle\Big)\\
=\ & -\frac{1}{2}\Big(\langle\overset{\circ}{Q_H}\ \nabla^{\mathcal{S}}\sigma_H,\nabla^{\mathcal{S}}\sigma_H\rangle+\langle\widehat{(b(Q_H)-Q_H)}d_H^{\nabla^E}\sigma_H,d_H^{\nabla^E}\sigma_H\rangle\Big).
\end{aligned}
$$

For the second term, we have

$$
\begin{aligned}
\langle\mathcal{R}_H^Q\sigma_H,\sigma_H\rangle &= \frac{1}{4}\sum_{i,j,k,l} Q_H(\epsilon_i,\epsilon_j,\epsilon_k,\epsilon_l)\langle\epsilon_l.R_H(\epsilon_i,\epsilon_j)\sigma_H,\epsilon_k.\sigma_H\rangle\\
&= -\frac{1}{2}\sum_{i,j,k,l} Q_H(\epsilon_i,\epsilon_j,\epsilon_k,\epsilon_l)\langle(R_H(\epsilon_i,\epsilon_j)\sigma_H)(\epsilon_k),\sigma_H(\epsilon_l)\rangle.
\end{aligned}
$$

Since $(R_H(X,Y)\sigma_H)(Z)=R_H^E(X,Y)\sigma_H(Z)-\sigma_H(R_H(X,Y)Z)$, we have

$$
\begin{aligned}
\langle\mathcal{R}_H^Q\sigma_H,\sigma_H\rangle &= -\frac{1}{2}\sum_{i,j,k,l} Q_H(\epsilon_i,\epsilon_j,\epsilon_k,\epsilon_l)\langle R_H^E(\epsilon_i,\epsilon_j)\sigma_H(\epsilon_k),\sigma_H(\epsilon_l)\rangle\\
&\quad+\frac{1}{2}\sum_{i,j,k,l,m} Q_H(\epsilon_i,\epsilon_j,\epsilon_k,\epsilon_l)R_H(\epsilon_i,\epsilon_j,\epsilon_k,\epsilon_m)\langle\sigma_H(\epsilon_l),\sigma_H(\epsilon_m)\rangle\\
&= -\sum_{k,l}\langle(\widehat{Q_H}R_H^E)(\epsilon_k,\epsilon_l)\sigma_H(\epsilon_k),\sigma_H(\epsilon_l)\rangle\\
&\quad+\frac{1}{2}\sum_{l,m}\Big(c_H(\widehat{R_H}\circ\widehat{Q_H})(\epsilon_l,\epsilon_m)+c_H(\widehat{R_H}\circ\widehat{Q_H})(\epsilon_m,\epsilon_l)\Big)(\sigma_H^*g^E)(\epsilon_l,\epsilon_m)\\
&= -\langle(\widehat{Q_H}R_H^E)(\sigma_H),\sigma_H\rangle+\langle(c_H(\widehat{R_H}\circ\widehat{Q_H}))^{\mathcal{S}},\sigma_H^*g^E\rangle.
\end{aligned}
$$

Hence the formula. □

Assume that $(M,\theta,\xi,J,g_\theta,\nabla)$ and $(N,\theta',\xi',J',g_{\theta'},\nabla')$ are contact metric manifolds endowed with their Tanaka-Webster connections and let $\phi:M\to N$ be a differential map. Let ϕ^*TN be the pull-back bundle of TN endowed with the metric and the connection induced by those of TN. The covariant derivative of the ϕ^*TN-valued 1-form $d\phi_H$ is given by:

$$(\nabla_X d\phi_H)(Y)=\nabla_X^{'\phi^*TN}d\phi_H(Y)-d\phi_H(\nabla_X Y),$$

where $\nabla^{'\phi^*TN}$ denotes the connection induced by ∇' on ϕ^*TN.

Definition 4.1 *Let $H = Ker\,\theta$ and $H^{'} = Ker\,\theta^{'}$. A map $\phi : M \to N$ such that $d\phi(H) \subset H^{'}$ is called a horizontal map from M to N. We denote by $\mathcal{H}(M,N)$ the subspace of horizontal maps from M to N.*

Note that a horizontal map ϕ satisfies $\phi^*\theta^{'} = f\theta$ with $f \in C^\infty(M,\mathbb{R})$.

Lemma 4.1 *For any horizontal map $\phi : M \to N$, we have:*

$$d_H^{\nabla^{'}} d\phi_H = -\omega_\theta \otimes (d\phi(\xi))_{H^{'}}, \tag{15}$$

and

$$\mathcal{L}_\xi^{\nabla^{'}} d\phi_H = \nabla_H^{'\phi^*TN}(d\phi(\xi))_{H^{'}} - f\,\tau^{'} \circ d\phi_H. \tag{16}$$

Proof. For any map $\phi : M \to N$, we have $d^{\nabla^{'}} d\phi = -\phi^*T^{'}$ where $T^{'}$ is the torsion of $\nabla^{'}$. Hence

$$\begin{aligned} d^{\nabla^{'}} d\phi_H &= d^{\nabla^{'}}(d\phi - \theta \otimes d\phi(\xi)) = -\phi^*T^{'} - \omega_\theta \otimes d\phi(\xi) + \theta \wedge \nabla^{'\phi^*TN} d\phi(\xi) \\ &= -(\phi^*T^{'})_H - \omega_\theta \otimes d\phi(\xi) + \theta \wedge (\nabla^{'\phi^*TN} d\phi(\xi) - i(\xi)\phi^*T^{'}). \end{aligned}$$

We deduce that

$$d_H^{\nabla^{'}} d\phi_H = (d^{\nabla^{'}} d\phi_H)_H = -(\phi^*T^{'})_H - \omega_\theta \otimes d\phi(\xi)$$

and

$$\mathcal{L}_\xi^{\nabla^{'}} d\phi_H = i(\xi)(d^{\nabla^{'}} d\phi_H) = \nabla_H^{'\phi^*TN} d\phi(\xi) - i(\xi)\phi^*T^{'}.$$

Now, let $\phi : M \to N$ be a horizontal map, then we have $\phi^*\theta^{'} = f\theta$ and $\phi^*\omega_{\theta^{'}} = f\omega_\theta - \theta \wedge df_H$. Since $T^{'} = -\omega_{\theta^{'}} \otimes \xi^{'} + \theta^{'} \wedge \tau^{'}$, we deduce that $\phi^*T^{'} = -f\omega_\theta \otimes \xi^{'} + \theta \wedge (df_H \otimes \xi^{'} + f\,\tau^{'} \circ d\phi_H)$ and consequently

$$d_H^{\nabla^{'}} d\phi_H = \omega_\theta \otimes (f\xi^{'} - d\phi(\xi)) = -\omega_\theta \otimes (d\phi(\xi))_{H^{'}}.$$

Now,

$$\begin{aligned} \mathcal{L}_\xi^{\nabla^{'}} d\phi_H &= \nabla_H^{'\phi^*TN} d\phi(\xi) - df_H \otimes \xi^{'} - f\,\tau^{'} \circ d\phi_H \\ &= \nabla_H^{'\phi^*TN}(d\phi(\xi) - f\xi^{'}) - f\,\tau^{'} \circ d\phi_H = \nabla_H^{'\phi^*TN}(d\phi(\xi))_{H^{'}} - f\,\tau^{'} \circ d\phi_H. \ \square \end{aligned}$$

In the following, for any horizontal symmetric 2-tensor μ_H, we denote by μ_{H_0} its traceless part.

Proposition 4.2 *(Mok-Siu-Yeung type formulas for horizontal maps between strictly pseudoconvex CR manifolds) Let $(M,\theta,\xi,J,g_\theta,\nabla)$ and $(N,\theta',\xi',J',g_{\theta'},\nabla')$ be strictly pseudoconvex CR manifolds with the assumption M compact. For any $Q^+_{H_0}\in\Gamma(S^2(\wedge^{2,+}_{H_0}(M)))$ (resp. $Q^-_H\in\Gamma(S^2(\wedge^{2,-}_H(M)))$) satisfying $\delta^\nabla_H Q^+_{H_0}=0$ and $(c_H(Q^+_{H_0}))_0=0$ (resp. $\delta^\nabla_H Q^-_H=0$ and $(c_H(Q^-_H))_0=0$) and any horizontal map ϕ from M to N, we have:*

$$\begin{aligned}&\int_M \langle \overset{\circ}{Q}{}^+_{H_0}\,(\nabla^{\mathcal S}d\phi_H)_0,(\nabla^{\mathcal S}d\phi_H)_0\rangle-\frac{tr_H\widehat{Q^+_{H_0}}}{d^2}\Big(|\delta^{\nabla'}_H d\phi_H|^2+d^2|(d\phi(\xi))_{H'}|^2\Big)v_{g_\theta}\\&=4\int_M 2\langle Q^+_{H_0},(\phi^*R^{\mathcal W}_{H'})_H\rangle-\frac{1}{2}\langle(c_H(\widehat{R^{\mathcal W}_H}\circ\widehat{Q^+_{H_0}}))^{\mathcal S},(\phi^*g_{\theta'})_H\rangle-\langle\overset{\circ}{Q}{}^+_{H_0}\,(\phi^*B_{\theta'})_H,(\phi^*g_{\theta'})_H\rangle v_{g_\theta},\end{aligned}\tag{17}$$

$$\begin{aligned}&\int_M \langle \overset{\circ}{Q}{}^-_{H}\,(\nabla^{\mathcal S}d\phi_H)_0,(\nabla^{\mathcal S}d\phi_H)_0\rangle-\frac{tr_H\widehat{Q^-_{H}}}{d^2}\Big(|\delta^{\nabla'}_H d\phi_H|^2-d^2|(d\phi(\xi))_{H'}|^2\Big)v_{g_\theta}\\&=4\int_M 2\langle Q^-_{H},(\phi^*R^{\mathcal W}_{H'})_H\rangle-\langle\overset{\circ}{Q}{}^-_{H}\,((\phi^*B_{\theta'})_H-B_\theta)+\frac{tr_H\widehat{Q^-_H}}{d}B_\theta,(\phi^*g_{\theta'})_H\rangle v_{g_\theta}.\end{aligned}\tag{18}$$

Proof. Let $\phi:M\to N$ be a horizontal map. For any $Q_H\in\Gamma(S^2(\wedge^2_H(M)))$, we obtain, using the relation $\nabla^{\mathcal S}d\phi_H=(\nabla^{\mathcal S}d\phi_H)_0-\frac{1}{d}g_{\theta_H}\otimes\delta^{\nabla'}_H d\phi_H$, that

$$\begin{aligned}\langle\overset{\circ}{Q}{}_H\,\nabla^{\mathcal S}d\phi_H,\nabla^{\mathcal S}d\phi_H\rangle&=\langle\overset{\circ}{Q}{}_H\,(\nabla^{\mathcal S}d\phi_H)_0,(\nabla^{\mathcal S}d\phi_H)_0\rangle+\frac{2}{d}\langle c_H(Q_H)\otimes\delta^{\nabla'}_H d\phi_H,(\nabla^{\mathcal S}d\phi_H)_0\rangle\\&\quad-\frac{tr_H\widehat{Q_H}}{d^2}|\delta^{\nabla'}_H d\phi_H|^2.\end{aligned}$$

The assumption $(c_H(Q_H))_0=0$ implies that $\langle c_H(Q_H)\otimes\delta^{\nabla'}_H d\phi_H,(\nabla^{\mathcal S}d\phi_H)_0\rangle=0$. Hence we have

$$\langle\overset{\circ}{Q}{}_H\,\nabla^{\mathcal S}d\phi_H,\nabla^{\mathcal S}d\phi_H\rangle=\langle\overset{\circ}{Q}{}_H\,(\nabla^{\mathcal S}d\phi_H)_0,(\nabla^{\mathcal S}d\phi_H)_0\rangle-\frac{tr_H\widehat{Q_H}}{d^2}|\delta^{\nabla'}_H d\phi_H|^2.\tag{19}$$

Since $\phi:M\to N$ is horizontal then (15) yields $d^{\nabla'}_H d\phi_H=-\omega_\theta\otimes(d\phi(\xi))_{H'}$. Now we have

$$\begin{aligned}\langle(b(\widehat{Q_H)-Q_H})d^{\nabla'}_H d\phi_H,d^{\nabla'}_H d\phi_H\rangle&=\langle(b(\widehat{Q_H)-Q_H})\omega_\theta,\omega_\theta\rangle|(d\phi(\xi))_{H'}|^2\\&=\langle b(Q_H)-Q_H,\omega_\theta\odot\omega_\theta\rangle|(d\phi(\xi))_{H'}|^2\\&=(tr_H\widehat{Q^-_H}-tr_H\widehat{Q^+_H})|(d\phi(\xi))_{H'}|^2.\end{aligned}$$

For $Q^+_{H_0}\in\Gamma(S^2(\wedge^{2,+}_{H_0}(M)))$ and $Q^-_H\in\Gamma(S^2(\wedge^{2,-}_H(M)))$, we have

$$\langle(b(\widehat{Q^+_{H_0})-Q^+_{H_0}})d^{\nabla'}_H d\phi_H,d^{\nabla'}_H d\phi_H\rangle=-tr_H\widehat{Q^+_{H_0}}\tag{20}$$

and

$$\langle (b(\widehat{Q_H^- - Q_H^-})d_H^{\nabla'}d\phi_H, d_H^{\nabla'}d\phi_H\rangle = tr_H\widehat{Q_H^-}. \tag{21}$$

Now for $Q_H \in \Gamma(S^2(\wedge^2_H(M)))$, we have $\langle(\widehat{Q_H}R_H^{\phi^*TN})(d\phi_H), d\phi_H\rangle = 2\langle Q_H, (\phi^*R_{H'})^{\mathcal{S}}_H\rangle$. Since $R_{H'} = R^{\mathcal{W}}_{H'} - \frac{1}{2}(\omega_{\theta'} \owedge A_{\theta'} - g_{\theta'} \owedge B_{\theta'})$, we have

$$(\phi^*R_{H'})_H = (\phi^*R^{\mathcal{W}}_{H'})_H + \frac{1}{2}(\phi^*g_{\theta'})_H \owedge (\phi^*B_{\theta'})_H - \frac{1}{2}(\phi^*\omega_{\theta'})_H \owedge (\phi^*A_{\theta'})_H.$$

Since $(\phi^*\omega_{\theta'})_H \owedge (\phi^*A_{\theta'})_H \in \Gamma(\wedge^2(\wedge^2_H(M)))$, we have

$$\begin{aligned}\langle(\widehat{Q_H}R_H^{\phi^*TN})(d\phi_H), d\phi_H\rangle &= 4\langle Q_H, (\phi^*R^{\mathcal{W}}_{H'})_H\rangle + 2\langle Q_H, (\phi^*g_{\theta'})_H \owedge (\phi^*B_{\theta'})_H\rangle \\ &= 4\langle Q_H, (\phi^*R^{\mathcal{W}}_{H'})_H\rangle - 2\langle \overset{\circ}{Q}_H\ (\phi^*B_{\theta'})_H, (\phi^*g_{\theta'})_H\rangle \end{aligned} \tag{22}$$

The relations $g_{\theta_H}\widehat{\owedge} B_\theta \circ \omega_\theta\widehat{\owedge}\omega_\theta = 2\omega_\theta\widehat{\owedge} A_\theta$ and $g_{\theta_H}\widehat{\owedge} B_\theta \circ g_{\theta_H}\widehat{\owedge} g_{\theta_H} = 2g_{\theta_H}\widehat{\owedge} B_\theta$ yield to

$$\widehat{R_H} = \widehat{R^{\mathcal{W}}_H} + \frac{1}{2}g_{\theta_H}\widehat{\owedge} B_\theta \circ (g_{\theta_H}\widehat{\owedge} g_{\theta_H})^-.$$

Since for any $T^{\pm}_H, Q^{\pm}_H \in S^2(\wedge^{2,\pm}_H(M))$, $\widehat{T^{\pm}_H} \circ \widehat{Q^{\mp}_H} = 0$ and $(g_{\theta_H}\widehat{\owedge} g_{\theta_H})^- \circ \widehat{Q^-_H} = 2\widehat{Q^-_H}$ then

$$\langle (c_H(\widehat{R_H} \circ \widehat{Q^+_{H_0}}))^{\mathcal{S}}, (\phi^*g_{\theta'})_H\rangle = \langle (c_H(\widehat{R^{\mathcal{W}}_H} \circ \widehat{Q^+_{H_0}}))^{\mathcal{S}}, (\phi^*g_{\theta'})_H\rangle, \tag{23}$$

and

$$\langle (c_H(\widehat{R_H} \circ \widehat{Q^-_H}))^{\mathcal{S}}, (\phi^*g_{\theta'})_H\rangle = \langle (c_H(g_{\theta_H}\widehat{\owedge} B_\theta \circ \widehat{Q^-_H}))^{\mathcal{S}}, (\phi^*g_{\theta'})_H\rangle.$$

Now, we have

$$\langle (c_H(g_{\theta_H}\widehat{\owedge} B_\theta \circ \widehat{Q^-_H}))^{\mathcal{S}}, (\phi^*g_{\theta'})_H\rangle = \langle (c_H(\overset{\circ}{Q^-_H}) \circ \overset{\circ}{B_\theta}\ )^{\mathcal{S}} - 2\ \overset{\circ}{Q^-_H}\ B_\theta, (\phi^*g_{\theta'})_H\rangle,$$

where $\overset{\circ}{\mu_H}$ is the symmetric endomorphism associated by g_{θ_H} to the symmetric 2-tensor μ_H. Using the assumption $(c_H(Q^-_H))_0 = 0$, we deduce that

$$\langle (c_H(\widehat{R_H} \circ \widehat{Q^-_H}))^{\mathcal{S}}, (\phi^*g_{\theta'})_H\rangle = 2\langle \frac{tr_H\widehat{Q^-_H}}{d}B_\theta - \overset{\circ}{Q^-_H}\ B_\theta, (\phi^*g_{\theta'})_H\rangle. \tag{24}$$

We obtain the formulas by replacing (19),(20),(21),(22),(23) and (24) in (12). □

As applications of formulas (17) and (18), we recover, in a different way, the Siu formula given in [25] and we derive a Tanaka-Weitzenbock formula. First we have the following lemma whose proof is left to the reader.

Lemma 4.2 *We have for any $s_H \in S^2_H(M)\otimes E$ the relations:*

$$\begin{aligned} g_{\theta_H} \overset{\circ}{\barwedge} g_{\theta_H}\ s_H &= 2(s_H - g_{\theta_H}\otimes tr_H s_H), \quad \omega_\theta \overset{\circ}{\barwedge} \omega_\theta\ s_H = \omega_\theta \overset{\circ}{\odot} \omega_\theta\ s_H = -2(s^+_H - s^-_H),\\ c_H(g_{\theta_H}\barwedge g_{\theta_H}) &= 2(2d-1)g_{\theta_H}, \quad c_H(\omega_\theta \barwedge \omega_\theta) = c_H(\omega_\theta\odot\omega_\theta) = 2g_{\theta_H},\\ tr_H\, \widehat{g_{\theta_H}\barwedge g_{\theta_H}} &= 2d(2d-1), \quad tr_H\, \widehat{\omega_\theta\barwedge\omega_\theta} = tr_H\, \widehat{\omega_\theta\odot\omega_\theta} = 2d. \end{aligned}$$

Proposition 4.3 *For any horizontal map ϕ from a compact Sasakian manifold M to a Sasakian manifold N, we have:*

$$\int_M |(\nabla^{\mathcal{S}} d\phi_H)^+_0|^2 - \Big(1-\frac{1}{d}\Big)|\delta^{\nabla'}_H d\phi_H|^2 + d(d-1)|(d\phi(\xi))_{H'}|^2 v_{g_\theta} = 4\int_M tr^{2,0}_H((\widehat{\phi^* R^{\mathcal{W}}_{H'}})^{\mathbb{C}}_H) v_{g_\theta}, \quad (25)$$

$$\begin{aligned} &\int_M \frac{1}{d}|(\nabla^{\mathcal{S}} d\phi_H)^+_0|^2 + \Big(1-\frac{1}{d}\Big)|(\nabla^{\mathcal{S}} d\phi_H)^-|^2 - \Big(1-\frac{1}{d}\Big)|\delta^{\nabla'}_H d\phi_H|^2 - (d^2-1)|(d\phi(\xi))_{H'}|^2 v_{g_\theta}\\ &\quad = 4\int_M \frac{1}{d} tr^{2,0}_H((\widehat{\phi^* R^{\mathcal{W}}_{H'}})^{\mathbb{C}}_H) + \Big(1-\frac{1}{d}\Big) tr^{1,1}_H((\widehat{\phi^* R^{\mathcal{W}}_{H'}})^{\mathbb{C}}_H) - \frac{1}{2}\Big(1-\frac{1}{d}\Big)\langle d\phi_H \circ \overset{\circ}{Ric}{}^{\mathcal{W}}_H, d\phi_H\rangle v_{g_\theta}, \quad (26) \end{aligned}$$

where, in an adapted frame $\{\epsilon_1,\dots\epsilon_d, J\epsilon_1,\dots J\epsilon_d\}$ of H,

$$tr^{2,0}_H((\widehat{\phi^* R^{\mathcal{W}}_{H'}})^{\mathbb{C}}_H) = \sum_{i,j\le d}((\widehat{\phi^* R^{\mathcal{W}}_{H'}})^{\mathbb{C}}_H(Z_i\wedge Z_j), \overline{Z_i\wedge Z_j})$$

and

$$tr^{1,1}_H((\widehat{\phi^* R^{\mathcal{W}}_{H'}})^{\mathbb{C}}_H) = \sum_{i,j\le d}((\widehat{\phi^* R^{\mathcal{W}}_{H'}})^{\mathbb{C}}_H(Z_i\wedge \overline{Z_j}), \overline{Z_i\wedge \overline{Z_j}}),$$

with $Z_i = \frac{1}{\sqrt{2}}(\epsilon_i - \sqrt{-1}J\epsilon_i)$.

Proof. Let $Q^-_H \in \Gamma(S^2(\wedge^{2,-}_H(M)))$ and $Q^+_{H_0} \in \Gamma(S^2(\wedge^{2,+}_{H_0}(M)))$ defined by:

$$Q^-_H = (g_{\theta_H}\barwedge g_{\theta_H})^- = \frac{1}{2}(g_{\theta_H}\barwedge g_{\theta_H} - \omega_\theta\barwedge\omega_\theta)$$

and

$$Q^+_{H_0} = (g_{\theta_H}\barwedge g_{\theta_H})^+_0 = \frac{1}{2}(g_{\theta_H}\barwedge g_{\theta_H} + \omega_\theta\barwedge\omega_\theta - \frac{2}{d}\omega_\theta\odot\omega_\theta).$$

Then we have $\nabla_H Q^+_{H_0} = \nabla_H Q^-_H = 0$ (since $\nabla g_{\theta_H} = \nabla\omega_\theta = 0$). Moreover, using Lemma 4.2, we have $c_H(Q^-_H) = 2(d-1)g_{\theta_H}$ and $c_H(Q^+_{H_0}) = 2d\Big(1-\frac{1}{d^2}\Big)g_{\theta_H}$. Also $(c_H(Q^-_H))_0 = (c_H(Q^+_{H_0}))_0 = 0$. Moreover, we have $(c_H(\widehat{R^{\mathcal{W}}_H}\circ\widehat{Q^+_{H_0}}))^{\mathcal{S}} = 4\Big(1-\frac{1}{d}\Big)Ric^{\mathcal{W}}_H$. Let ϕ be a horizontal map from M to N, by Lemma 4.2, we have

$$\langle \overset{\circ}{Q}{}^-_H\,(\nabla^{\mathcal{S}} d\phi_H)_0, (\nabla^{\mathcal{S}} d\phi_H)_0\rangle = 2|(\nabla^{\mathcal{S}} d\phi_H)^+_0|^2$$

$$
\begin{aligned}
\langle \overset{\circ}{Q}{}^{+}_{H_0}\,(\nabla^{\mathcal{S}} d\phi_H)_0, (\nabla^{\mathcal{S}} d\phi_H)_0\rangle &= 2\Big(\frac{1}{d}|(\nabla^{\mathcal{S}} d\phi_H)_0^{+}|^2 + \Big(1-\frac{1}{d}\Big)|(\nabla^{\mathcal{S}} d\phi_H)^{-}|^2\Big)\\
tr_H\widehat{Q^{-}_H} &= 2d(d-1), \quad tr_H\widehat{Q^{+}_{H_0}} = 2\Big(d^2-1\Big)\\
\langle (c_H(\widehat{R^{\mathcal{W}}_H}\circ\widehat{Q^{+}_{H_0}}))^{\mathcal{S}}, (\phi^* g_{\theta'})_H\rangle &= 2\Big(1-\frac{1}{d}\Big)\langle d\phi_H\circ \overset{\circ}{Ric}{}^{\mathcal{W}}_H, d\phi_H\rangle\\
\langle Q^{-}_H, (\phi^* R^{\mathcal{W}}_{H'})_H\rangle &= \langle (g_{\theta_H}\owedge g_{\theta_H})^{-}, (\phi^* R^{\mathcal{W}}_{H'})_H\rangle = tr_H(\widehat{\phi^* R^{\mathcal{W}}_{H'}})^{-}_H\\
\langle Q^{+}_{H_0}, (\phi^* R^{\mathcal{W}}_{H'})_H\rangle &= \langle (g_{\theta_H}\owedge g_{\theta_H})^{+} - \frac{1}{d}\omega_\theta\odot\omega_\theta, (\phi^* R^{\mathcal{W}}_{H'})_H\rangle\\
&= \Big(1-\frac{1}{d}\Big)tr_H(\widehat{\phi^* R^{\mathcal{W}}_{H'}})^{+}_H + \frac{1}{d}tr_H(\widehat{\phi^* R^{\mathcal{W}}_{H'}})^{-}_H.
\end{aligned}
$$

In an adapted frame $\{\epsilon_1,\dots,\epsilon_d, J\epsilon_1,\dots,J\epsilon_d\}$ of H, we have

$$
tr_H(\widehat{\phi^* R^{\mathcal{W}}_{H'}})^{\pm}_H = \sum_{i,j\leq d}((\phi^* R^{\mathcal{W}}_{H'})^{\pm}_H(\epsilon_i,\epsilon_j,\epsilon_i,\epsilon_j) + (\phi^* R^{\mathcal{W}}_{H'})^{\pm}_H(\epsilon_i,J\epsilon_j,\epsilon_i,J\epsilon_j)).
$$

Now we have, for any $T_H\in S^2(\wedge^2_H(M))\cap Ker\, b$, the relations

$$
(\widehat{T_H}^{\mathbb{C}}(Z\wedge W), \overline{Z\wedge W}) = T^{-}_H(X,Y,X,Y) + T^{-}_H(X,JY,X,JY)
$$

and

$$
(\widehat{T_H}^{\mathbb{C}}(Z\wedge \overline{W}), \overline{Z\wedge \overline{W}}) = T^{+}_H(X,Y,X,Y) + T^{+}_H(X,JY,X,JY),
$$

with $Z=\frac{1}{\sqrt{2}}(X-\sqrt{-1}JX)$, $\quad W=\frac{1}{\sqrt{2}}(Y-\sqrt{-1}JY)$. Since $(\phi^* R^{\mathcal{W}}_{H'})_H\in S^2(\wedge^2_H(M))\cap Ker\, b$, we deduce that

$$
\langle Q^{-}_H, (\phi^* R^{\mathcal{W}}_{H'})_H\rangle = \sum_{i,j\leq d}((\widehat{\phi^* R^{\mathcal{W}}_{H'}})^{\mathbb{C}}_H(Z_i\wedge Z_j), \overline{Z_i\wedge Z_j})
$$

and

$$
\begin{aligned}
\langle Q^{+}_{H_0}, (\phi^* R^{\mathcal{W}}_{H'})_H\rangle &= \sum_{i,j\leq d}\Big(\frac{1}{d}((\widehat{\phi^* R^{\mathcal{W}}_{H'}})^{\mathbb{C}}_H(Z_i\wedge Z_j), \overline{Z_i\wedge Z_j})\\
&+ \Big(1-\frac{1}{d}\Big)((\widehat{\phi^* R^{\mathcal{W}}_{H'}})^{\mathbb{C}}_H(Z_i\wedge \overline{Z_j}), \overline{Z_i\wedge \overline{Z_j}})\Big),
\end{aligned}
$$

with $Z_i=\frac{1}{\sqrt{2}}(\epsilon_i-\sqrt{-1}J\epsilon_i)$. By replacing in (17) and (18) together with the assumptions M,N Sasakian yields the formulas. □

5 Horizontal pseudoharmonic maps, CR-pluriharmonic maps and Rumin complex

. Pseudoharmonic maps

In [3] and [5] Barletta, Dragomir and Urakawa have introduced the notion of pseudoharmonic maps from a compact contact metric manifold into a Riemannian manifold. Now we extend this notion to horizontal maps between contact metric manifolds.

Assume that $(M,\theta,\xi,J,g_\theta,\nabla)$ and $(N,\theta',\xi',J',g_{\theta'},\nabla')$ are contact metric manifolds endowed with their Tanaka-Webster connections and that M is compact. For any differential map $\phi : M \to N$, we define $d\phi_{H,H'}(X) = (d\phi_H(X))_{H'}$ with $X \in H$ and the horizontal energy $E_{H,H'}(\phi)$ by:

$$E_{H,H'}(\phi) = \frac{1}{2}\int_M |d\phi_{H,H'}|^2 v_{g_\theta}.$$

Proposition 5.1 *For any variation ϕ_t of ϕ, we have:*

$$\frac{d}{dt}E_{H,H'}(\phi_t)_{|t=0} = \int_M g_{\theta'}(\delta_H^{\nabla'} d\phi_{H,H'} + i(\phi^*\theta')\tau' \circ d\phi_{H,H'} - \xi' tr_H(\phi^* A_{\theta'})_H, v)v_{g_\theta},$$

with $v = \dfrac{\partial\phi}{\partial t}_{|t=0}$.

Proof. Let $\{\phi_t\}_{|t|<\epsilon}$ be a variation of ϕ. We consider the map $\Phi :]-\epsilon,\epsilon[\times M \to N$ given by $\Phi(t,x) = \phi_t(x)$ and the pull-back bundle $\Phi^*TN \to]-\epsilon,\epsilon[\times M$ of TN by Φ. Let $\{\epsilon_i\}$ be a local g_θ-orthonormal frame of H, then we have

$$\frac{1}{2}\frac{d}{dt}|d\phi_{t\,H,H'}|^2 = \frac{1}{2}\frac{\partial}{\partial t}\sum_i g_{\theta'}((d\Phi(\epsilon_i))_{H'},(d\Phi(\epsilon_i))_{H'}) = \sum_i g_{\theta'}(\nabla'^{\Phi^*TN}_{\frac{\partial}{\partial t}}(d\Phi(\epsilon_i))_{H'},(d\Phi(\epsilon_i))_{H'}).$$

We have

$$(d^{\nabla'}d\Phi)(\frac{\partial}{\partial t},\epsilon_i) = \nabla'^{\Phi^*TN}_{\frac{\partial}{\partial t}}d\Phi(\epsilon_i) - \nabla'^{\Phi^*TN}_{\epsilon_i}d\Phi(\frac{\partial}{\partial t}) - d\Phi([\frac{\partial}{\partial t},\epsilon_i]) = -T'(d\Phi(\frac{\partial}{\partial t}),d\Phi(\epsilon_i)),$$

where T' is the torsion of ∇'. Since $[\frac{\partial}{\partial t},\epsilon_i] = 0$ and ∇'^{Φ^*TN} preserves H', we obtain

$$\nabla'^{\Phi^*TN}_{\frac{\partial}{\partial t}}(d\Phi(\epsilon_i))_{H'} - \nabla'^{\Phi^*TN}_{\epsilon_i}(d\Phi(\frac{\partial}{\partial t}))_{H'} = -(T'(d\Phi(\frac{\partial}{\partial t}),d\Phi(\epsilon_i)))_{H'}.$$

Using $T' = -\omega_{\theta'}\otimes\xi' + \theta'\wedge\tau'$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|d\phi_{t\,H,H'}|^2 &= \sum_i g_{\theta'}\Big(\nabla'^{\Phi^*TN}_{\epsilon_i}(d\Phi(\frac{\partial}{\partial t}))_{H'} - \theta'(d\Phi(\frac{\partial}{\partial t}))\tau'(d\Phi(\epsilon_i))\\ &+ \theta'(d\Phi(\epsilon_i))\tau'(d\Phi(\frac{\partial}{\partial t})),(d\Phi(\epsilon_i))_{H'}\Big)\\ &= \sum_i\Big(\epsilon_i g_{\theta'}((d\Phi(\frac{\partial}{\partial t}))_{H'},(d\Phi(\epsilon_i))_{H'}) - g_{\theta'}((d\Phi(\frac{\partial}{\partial t}))_{H'},(d\Phi(\nabla_{\epsilon_i}\epsilon_i))_{H'})\Big)\end{aligned}$$

$$
\begin{aligned}
- & \sum_i g_{\theta'}((d\Phi(\frac{\partial}{\partial t}))_{H'}, \nabla'^{\Phi^*TN}_{\epsilon_i}(d\Phi(\epsilon_i))_{H'} - (d\Phi(\nabla_{\epsilon_i}\epsilon_i))_{H'}) \\
- & (\Phi^*\theta')(\frac{\partial}{\partial t})\sum_i A_{\theta'}((d\Phi(\epsilon_i))_{H'}, (d\Phi(\epsilon_i))_{H'}) \\
+ & \sum_i (\Phi^*\theta')(\epsilon_i)A_{\theta'}((d\Phi(\frac{\partial}{\partial t}))_{H'}, (d\Phi(\epsilon_i))_{H'}) \\
= & -\delta_H\alpha_{(d\Phi(\frac{\partial}{\partial t}))_{H'}} - g_{\theta'}\Big((d\Phi(\frac{\partial}{\partial t}))_{H'}, \sum_i ((\nabla'^{\Phi^*TN}_{\epsilon_i}d\Phi)(\epsilon_i))_{H'}\Big) \\
- & (\Phi^*\theta')(\frac{\partial}{\partial t})\sum_i A_{\theta'}((d\Phi(\epsilon_i))_{H'}, (d\Phi(\epsilon_i))_{H'}) \\
+ & \sum_i (\Phi^*\theta')(\epsilon_i)A_{\theta'}((d\Phi(\frac{\partial}{\partial t}))_{H'}, (d\Phi(\epsilon_i))_{H'}),
\end{aligned}
$$

with $\alpha_{(d\Phi(\frac{\partial}{\partial t}))_{H'}}(X) = g_{\theta'}((d\Phi(\frac{\partial}{\partial t}))_{H'}, (d\Phi(X))_{H'})$. We deduce that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}|d\phi_{t\,H,H'}|^2_{|t=0} &= -\delta_H\alpha_{v_{H'}} - g_{\theta'}\Big(\sum_i ((\nabla'^{\phi^*TN}_{\epsilon_i}d\phi)(\epsilon_i))_{H'}, v_{H'}\Big) \\
&\quad -\theta'(v)\sum_i A_{\theta'}(d\phi_{H,H'}(\epsilon_i), d\phi_{H,H'}(\epsilon_i)) + \sum_i (\phi^*\theta')(\epsilon_i)A_{\theta'}(d\phi_{H,H'}(\epsilon_i), v_{H'}) \\
&= -\delta_H\alpha_{v_{H'}} + g_{\theta'}(\delta'^{\nabla}_H d\phi_{H,H'} + i(\phi^*\theta')\tau' \circ d\phi_{H,H'} - \xi' tr_H(\phi^*A_{\theta'})_H, v).
\end{aligned}
$$

The result follows by integrating. □

Definition 5.1 *A map $\phi : M \to N$ is called a pseudoharmonic map if it is a critical point of $E_{H,H'}$.*

A map $\phi : M \to N$ is pseudoharmonic if and only if

$$
\delta'^{\nabla}_H d\phi_{H,H'} + i(\phi^*\theta')\tau' \circ d\phi_{H,H'} = 0 \quad \text{and} \quad tr_H(\phi^*A_{\theta'})_H = 0.
$$

A horizontal map $\phi : M \to N$ is pseudoharmonic if and only if $\delta'^{\nabla}_H d\phi_H = 0$ and $tr_H(\phi^*A_{\theta'})_H = 0$.

. CR-pluriharmonic maps

Let $(M, \theta, \xi, J, g_\theta)$ be a strictly pseudoconvex CR manifold of dimension $2d+1$. A real function h on M is called a CR-pluriharmonic function if h is the real part of a CR function on M.

We have the following equivalent characterizations for the CR-pluriharmonic functions.

Theorem 5.1 *(Lee[21])*
The following assertions are equivalent:
(i) h is CR-pluriharmonic.
(ii) There exists a real function λ such that $d(J^*dh_H + \lambda\theta) = 0$.
(iii) $(d_H J^*dh_H)_0 = 0$ and $(\mathcal{L}_\xi + \frac{1}{d} d_H \delta_{H,J}) J^*dh_H = \mathcal{L}_\xi J^*dh_H + \frac{1}{d} d_H \delta_H dh_H = 0$,

where $(d_H J^*dh_H)_0$ is the primitive part of $d_H J^*dh_H$ and $\delta_{H,J} = [\wedge_H, d_H]$.

Note that, if $d > 1$, then the assumption $(d_H J^*dh_H)_0 = 0$ implies that $\mathcal{L}_\xi J^*dh_H + \frac{1}{d} d_H \delta_H dh_H = 0$ and, if $d = 1$, then the assumption $(d_H J^*dh_H)_0 = 0$ is always satisfied for any h.

Definition 5.2 *Let $(M, \theta, \xi, J, g_\theta, \nabla)$ and $(N, \theta', \xi', J', g_{\theta'}, \nabla')$ be strictly pseudoconvex CR manifolds endowed with their Tanaka-Webster connections together with $\dim M > 3$. A horizontal map $\phi : M \to N$ such that $(d_H^{\nabla'} J^* d\phi_H)_0 = 0$ is called a CR-pluriharmonic map from M to N.*

Proposition 5.2 *(i) A map $\phi : M \to N$ is CR-pluriharmonic if and only if*

$$(\nabla^{\mathcal{S}} d\phi_H)_0^+ := (\nabla^{\mathcal{S}} d\phi_H)^+ + \frac{1}{d} g_{\theta_H} \otimes \delta_H^{\nabla'} d\phi_H = 0.$$

(ii) Any CR-pluriharmonic map $\phi : M \to N$ satisfies:

$$\mathcal{L}_\xi^{\nabla'} J^* d\phi_H + \frac{1}{d} d_H^{\nabla'} \delta_H^{\nabla'} d\phi_H = \frac{2}{d-1} tr_H \left(R_H^{\phi^* TN}\right)^- (\quad, .) d\phi_H(.) - f\, \tau' \circ J^* d\phi_H.$$

(iii) Any CR map $\phi : M \to N$ is CR-pluriharmonic and we have

$$\delta_H^{\nabla'} J^* d\phi_H = J' \delta_H^{\nabla'} d\phi_H = d(d\phi(\xi))_{H'}. \tag{27}$$

Proof. Recall that for $\gamma_H \in \Omega^2_H(M)$, its primitive part $\gamma_{H_0} \in \Omega^2_{H_0}(M)$ is given by $\gamma_{H_0} = \gamma_H - \frac{1}{d} L_H \wedge_H \gamma_H$. We have

$$(d_H^{\nabla'} J^* d\phi_H)_0 = d_H^{\nabla'} J^* d\phi_H - \frac{1}{d} L_H \wedge_H d_H^{\nabla'} J^* d\phi_H = d_H^{\nabla'} J^* d\phi_H - \frac{1}{d} L_H \delta_{H,J}^{\nabla'} J^* d\phi_H.$$

Now since $\delta_{H,J}^{\nabla'} J^* d\phi_H = \delta_{H,J}^{\nabla'} (d\phi_H \circ J) = \delta_H^{\nabla'} d\phi_H$, we deduce that

$$(d_H^{\nabla'} J^* d\phi_H)_0 = d_H^{\nabla'} J^* d\phi_H - \frac{1}{d} \omega_\theta \otimes \delta_H^{\nabla'} d\phi_H. \tag{28}$$

Now, using (15), we have

$$\begin{aligned} (d_H^{\nabla'} J^* d\phi_H)(JX, Y) &= (\nabla_{JX} d\phi_H)(JY) + (\nabla_Y d\phi_H)(X) = \frac{1}{2}\Big((\nabla^{\mathcal{S}} d\phi_H)(X, Y) + (\nabla^{\mathcal{S}} d\phi_H)(JX, JY)\Big) \\ &= (\nabla^{\mathcal{S}} d\phi_H)^+(X, Y). \end{aligned}$$

Hence by (28) we obtain

$$(d_H^{\nabla^{'}} J^* d\phi_H)_0(JX,Y) = (\nabla^{\mathcal{S}} d\phi_H)^+(X,Y) + \frac{1}{d} g_{\theta_H}(X,Y)\delta_H^{\nabla^{'}} d\phi_H.$$

The assumption (i) directly follows. Now we have

$$\delta_{H,J}^{\nabla^{'}}((d_H^{\nabla^{'}} J^* d\phi_H)_0) = \wedge_H d_H^{\nabla^{'}}((d_H^{\nabla^{'}} J^* d\phi_H)_0) = \wedge_H d_H^{\nabla^{'}{}^2} J^* d\phi_H - \frac{1}{d} \wedge_H d_H^{\nabla^{'}} L_H \delta_H^{\nabla^{'}} d\phi_H.$$

Since $d_H^{\nabla^{'}{}^2} J^* d\phi_H = -L_H(\mathcal{L}_\xi^{\nabla^E} J^* d\phi_H) - R_H^{\phi^* TN} \wedge J^* d\phi_H$ and $[d_H^{\nabla^{'}}, L_H] = 0$, we obtain

$$\begin{aligned}
\delta_{H,J}^{\nabla^{'}}((d_H^{\nabla^{'}} J^* d\phi_H)_0) &= -\wedge_H (R_H^{\phi^* TN} \wedge J^* d\phi_H) - \wedge_H L_H(\mathcal{L}_\xi^{\nabla^E} J^* d\phi_H) - \frac{1}{d} \wedge_H L_H d_H^{\nabla^{'}} \delta_H^{\nabla^{'}} d\phi_H \\
&= -\wedge_H (R_H^{\phi^* TN} \wedge J^* d\phi_H) - (d-1)(\mathcal{L}_\xi^{\nabla^E} J^* d\phi_H + \frac{1}{d} d_H^{\nabla^{'}} \delta_H^{\nabla^{'}} d\phi_H). \quad (29)
\end{aligned}$$

Let $\{\epsilon_i\}$ be a local g_θ-orthonormal frame of H, then we have

$$\wedge_H(R_H^{\phi^* TN} \wedge J^* d\phi_H) = \frac{1}{2}\sum_i \Big(R_{H'}(d\phi_H(\epsilon_i), d\phi_H(J\epsilon_i)) d\phi_H(JX) + 2R_{H'}(d\phi_H(\epsilon_i), d\phi_H(X)) d\phi_H(\epsilon_i) \Big)$$

Using (1), we obtain that

$$\sum_i R_{H'}(d\phi_H(\epsilon_i,)d\phi_H(J\epsilon_i))d\phi_H(JX) = 2\sum_i R_{H'}(d\phi_H(JX), d\phi_H(J\epsilon_i))d\phi_H(\epsilon_i) + 2f(d-1)\tau^{'}((d\phi_H \circ J)(X)).$$

Hence we have

$$\begin{aligned}
\wedge_H (R_H^{\phi^* TN} \wedge J^* d\phi_H) &= -\sum_i \Big(R_{H'}(d\phi_H(X), d\phi_H(\epsilon_i)) d\phi_H(\epsilon_i) - R_{H'}(d\phi_H(JX), d\phi_H(J\epsilon_i)) d\phi_H(\epsilon_i) \Big) \\
&+ f(d-1)\tau^{'}((d\phi_H \circ J)(X)) \\
&= -2\sum_i (R_H^{\phi^* TN})^-(X, \epsilon_i) d\phi_H(\epsilon_i) + f(d-1)(\tau^{'} \circ J^* d\phi_H)(X). \quad (30)
\end{aligned}$$

The assumption ϕ CR-pluriharmonic together with (29) and (30) gives the formula. Hence (ii). Let $\phi : M \to N$ be a CR map then $J^{'} \circ d\phi_H = d\phi_H \circ J$. Consequently, we have

$$J^{'} \circ (d_H^{\nabla^{'}} d\phi_H)_0 = (d_H^{\nabla^{'}}(d\phi_H \circ J))_0$$

and

$$J^{'} \delta_H^{\nabla^{'}} d\phi_H = \delta_H^{\nabla^{'}}(d\phi_H \circ J).$$

By (15) we have $(d_H^{\nabla^{'}} d\phi_H)_0 = 0$ and we obtain that $(d_H^{\nabla^{'}}(d\phi_H \circ J)_0 = 0$. Now we have

$$\delta_H^{\nabla^{'}}(d\phi_H \circ J) = -\delta_{H,J}^{\nabla^{'}}(d\phi_H) = -\wedge_H (d_H^{\nabla^{'}} d\phi_H) = d(d\phi(\xi))_{H'}. \quad \square$$

The following theorem holds for pseudoharmonic maps between Sasakian manifolds.

Theorem 5.2 *Let M and N be Sasakian manifolds. Assume that M is compact and N has nonpositive pseudo-Hermitian complex sectional curvature. Then:*
(i) Any horizontal pseudoharmonic map ϕ from M to N is CR-pluriharmonic.
(ii) If the pseudo-Hermitian Ricci tensor of M is nonnegative, then any horizontal pseudoharmonic map ϕ from M to N satisfies $\nabla d\phi = 0$ and $|d\phi| = const.$
(iii) If the pseudo-Hermitian Ricci tensor of M is positive, then any horizontal pseudoharmonic map ϕ from M to N satisfies $d\phi_H = 0$ and consequently $rg_x(\phi) \leq 1$, where $rg_x(\phi)$ is the rank of ϕ at a point x of M.

Proof. Let ϕ be a horizontal pseudoharmonic map from M to N. If N has nonpositive pseudo-Hermitian complex sectional curvature then (25) yields to $(d\phi(\xi))_{H'} = 0$ and $(\nabla^{\mathcal{S}} d\phi_H)_0^+ = 0$. In particular, ϕ is CR-pluriharmonic. Moreover, if $Ric_H^{\mathcal{W}}$ is nonnegative, it follows from (26) that $(\nabla^{\mathcal{S}} d\phi_H)^- = 0$. Consequently $\nabla^{\mathcal{S}} d\phi_H = 0$ and $d_H^{\nabla'} d\phi_H = 0$. We deduce that $\nabla_H d\phi_H = 0$. The assumptions M and N torsionless together with $(d\phi(\xi))_{H'} = 0$, yield by (16) that $\nabla_\xi d\phi_H = 0$. Taking into account that $d\phi(\xi) = f\xi'$, we obtain that $i(\xi)\phi^*\omega_{\theta'} = -df_H = 0$ and so f is constant. We immediately deduce that $\nabla d\phi = \nabla d\phi_H + \theta \otimes \nabla d\phi(\xi) = 0$ and so $|d\phi| = const.$ If $Ric_H^{\mathcal{W}}$ is positive, it directly follows from $\langle d\phi_H \circ \overset{\circ}{Ric}_H^{\mathcal{W}}, d\phi_H \rangle = 0$ that $d\phi_H = 0$. Since $d\phi(\xi) = f\xi'$, we deduce that the rank of ϕ is less than equal 1 at each point of M. □

. Horizontal maps and twisted Rumin pseudo-complex

Let M be a strictly pseudoconvex CR manifold of dimension $2d+1$. We recall that the Rumin complex [27] is the complex:

$$0 \to \mathbb{R} \to C^\infty(M) \overset{d_{\mathcal{R}}}{\to} \mathcal{R}^1(M) \overset{d_{\mathcal{R}}}{\to} \ldots \overset{d_{\mathcal{R}}}{\to} \mathcal{R}^d(M) \overset{D_{\mathcal{R}}}{\to} \mathcal{R}^{d+1}(M) \overset{d_{\mathcal{R}}}{\to} \ldots \overset{d_{\mathcal{R}}}{\to} \mathcal{R}^{2d+1}(M) \to 0,$$

where

$$\begin{aligned} \mathcal{R}^p(M) &= \Omega^p_{H_0}(M) \quad \text{for } p \leq d \\ &= \mathcal{F}^p_\xi(M) \quad \text{for } p \geq d+1, \end{aligned}$$

and

$$\begin{aligned} d_{\mathcal{R}}\gamma_H = (d_H\gamma_H)_0 = (d_H - \tfrac{1}{d-p+1} L_H \delta_{H,J})\gamma_H \quad &\text{for } \gamma_H \in \mathcal{R}^p(M) \quad (p \leq d-1) \\ D_{\mathcal{R}}\gamma_H = \theta \wedge (\mathcal{L}_\xi + d_H\delta_{H,J})\gamma_H \quad &\text{for } \gamma_H \in \mathcal{R}^d(M) \\ d_{\mathcal{R}}\gamma_\xi = \theta \wedge i(\xi)(d\gamma_\xi) \quad &\text{for } \gamma_\xi \in \mathcal{R}^p(M) \quad (p \geq d+1). \end{aligned}$$

The formal adjoints of $d_{\mathcal{R}}$ and $D_{\mathcal{R}}$ for the usual scalar product are denoted by $\delta_{\mathcal{R}}$ and $D^*_{\mathcal{R}}$. The laplacians associated to this complex are defined by:

$$\begin{aligned} \triangle_{\mathcal{R}} = (d-p)d_{\mathcal{R}}\delta_{\mathcal{R}} + (d-p+1)\delta_{\mathcal{R}}d_{\mathcal{R}} \quad &\text{on } \mathcal{R}^p(M) \quad (p \leq d-1) \\ \triangle_{\mathcal{R}} = D^*_{\mathcal{R}}D_{\mathcal{R}} + (d_{\mathcal{R}}\delta_{\mathcal{R}})^2 \quad &\text{on } \mathcal{R}^d(M) \\ \triangle_{\mathcal{R}} = D_{\mathcal{R}}D^*_{\mathcal{R}} + (\delta_{\mathcal{R}}d_{\mathcal{R}})^2 \quad &\text{on } \mathcal{R}^{d+1}(M) \\ \triangle_{\mathcal{R}} = (d-p+1)d_{\mathcal{R}}\delta_{\mathcal{R}} + (d-p)\delta_{\mathcal{R}}d_{\mathcal{R}} \quad &\text{on } \mathcal{R}^p(M) \quad (p \geq d+2). \end{aligned}$$

The fondamental fact is that, if M is compact, then (cf. [27]):

$$H^*_{dR}(M,\mathbb{R}) = H^*_{\mathcal{R}}(M,\mathbb{R}) = Ker\,\triangle_{\mathcal{R}},$$

where $H^*_{dR}(M,\mathbb{R})$ and $H^*_{\mathcal{R}}(M,\mathbb{R})$ are respectively the cohomologies of the De Rham complex and the Rumin complex.

Let (E,∇^E) be a vector bundle over M then the previous definitions of $\mathcal{R}^p(M)$, $d_{\mathcal{R}}$ and $D_{\mathcal{R}}$ can be extended to E-twisted bundles. Also we define the sequence:

$$0 \to \mathbb{R} \to C^\infty(M) \overset{d^{\nabla^E}_{\mathcal{R}}}{\to} \mathcal{R}^1(M;E) \overset{d^{\nabla^E}_{\mathcal{R}}}{\to} \ldots \overset{d^{\nabla^E}_{\mathcal{R}}}{\to} \mathcal{R}^d(M;E) \overset{D^{\nabla^E}_{\mathcal{R}}}{\to} \mathcal{R}^{d+1}(M;E) \overset{d^{\nabla^E}_{\mathcal{R}}}{\to} \ldots \overset{d^{\nabla^E}_{\mathcal{R}}}{\to} \mathcal{R}^{2d+1}(M;E) \to 0.$$

Note that $d^{\nabla^E\,2}_{\mathcal{R}}\sigma_H = -(R^E_H \wedge \sigma_H)_0$ for $\sigma_H \in \mathcal{R}^p(M;E)$ $(p \leq d-2)$. Also the previous sequence is not a complex excepted if E is flat. Also we call this sequence the twisted Rumin pseudo-complex.

Let N be a strictly pseudoconvex CR manifold, $\phi : M \to N$ a horizontal map and $E = \phi^*TN$ the pull-back bundle endowed with the connection ∇' induced by the Tanaka-Wester connection of TN. Then $d\phi_H \in \Omega^1_H(M;\phi^*TN)$ satisfies $d^{\nabla'}_{\mathcal{R}}\,d\phi_H = (d^{\nabla'}_H\,d\phi_H)_0 = 0$. Moreover, if ϕ is a pseudoharmonic map then $\delta^{\nabla'}_{\mathcal{R}}\,d\phi_H = \delta^{\nabla'}_H\,d\phi_H = 0$. Consequently, if $d > 1$, we have $\triangle^{\nabla'}_{\mathcal{R}}\,d\phi_H = ((d-1)\,d^{\nabla'}_{\mathcal{R}}\,\delta^{\nabla'}_{\mathcal{R}} + d\,\delta^{\nabla'}_{\mathcal{R}}\,d^{\nabla'}_{\mathcal{R}})d\phi_H = 0$. If $d = 1$ and N is torsionless, then we have $D^{\nabla'}_{\mathcal{R}}\,d\phi_H = \theta \wedge (\mathcal{L}^{\nabla'}_\xi\,d\phi_H + d^{\nabla'}_H\,\delta^{\nabla'}_{H,J}d\phi_H) = 0$. The assumption ϕ pseudoharmonic yields to $\triangle^{\nabla'}_{\mathcal{R}}\,d\phi_H = D^{\nabla'\,*}_{\mathcal{R}}D^{\nabla'}_{\mathcal{R}}\,d\phi_H + (d^{\nabla'}_{\mathcal{R}}\,\delta^{\nabla'}_{\mathcal{R}})^2\,d\phi_H = 0$.

Remark 5.1 *The condition $\phi : M \to N$ CR-pluriharmonic is equivalent to $d^{\nabla'}_{\mathcal{R}}\,J^*d\phi_H = 0$. Moreover, if $\dim M = 3$, it seems natural in view of the Theorem 5.1 to define the CR-pluriharmonicity of a horizontal map $\phi : M \to N$ by the condition $D^{\nabla'}_{\mathcal{R}}\,J^*d\phi_H = 0$.*

6 Rigidity results for horizontal pseudoharmonic maps defined on contact locally sub-symmetric spaces

Now we derive Mok-Siu-Yeung type formulas for horizontal maps from compact contact locally sub-symmetric spaces into strictly pseudoconvex CR manifolds. In this section we assume that M is a contact locally sub-symmetric space of dimension $2d+1 \geq 5$. First we consider the case M torsionless.

Lemma 6.1 *Let $(M,\theta,\xi,J,g_\theta)$ be a contact locally sub-symmetric space torsionless with $s^{\mathcal{W}}$ non-zero. The tensor $Q^+_{H_0} \in \Gamma(S^2(\wedge^{2,+}_{H_0}(M)))$ given by $Q^+_{H_0} = c_0 I^{\mathbb{C}}_{H_0} + C^{\mathcal{M}}_H$ with $c_0 = -\dfrac{8d}{d-1}\dfrac{|C^{\mathcal{M}}_H|^2}{s^{\mathcal{W}}}$ is parallel and satisfies $\langle Q^+_{H_0}, R^{\mathcal{W}}_{H_0}\rangle = 0$ and $(c_H(Q^+_{H_0}))_0 = 0$.*

Proof. First recall that $I^{\mathbb{C}}_{H_0} = \frac{1}{4}(g_{\theta_H} \owedge g_{\theta_H})^+_0$. Now we determine c_0 in a such way that $Q^+_{H_0} = c_0 I^{\mathbb{C}}_{H_0} + C^{\mathcal{M}}_H$ satisfies $\langle Q^+_{H_0}, R^{\mathcal{W}}_{H_0}\rangle = 0$. We have

$$\langle Q^+_{H_0}, R^{\mathcal{W}}_{H_0}\rangle = \frac{c_0}{4} tr_H \widehat{R^{\mathcal{W}}_{H_0}} + \langle C^{\mathcal{M}}_H, R^{\mathcal{W}}_{H_0}\rangle.$$

Now, we have

$$R^{\mathcal{W}}_{H_0} = \frac{s^{\mathcal{W}}}{d(d+1)} I^{\mathbb{C}}_{H_0} + \frac{1}{2(d+2)}\Big(Ric^{\mathcal{W}}_{H_0} \owedge g_{\theta_H} - \rho^{\mathcal{W}}_{H_0} \owedge \omega_\theta\Big)_0 + C^{\mathcal{M}}_H.$$

Since $tr_H\Big(\widehat{Ric^{\mathcal{W}}_{H_0} \owedge g_{\theta_H}} - \rho^{\mathcal{W}}_{H_0} \owedge \omega_\theta\Big)_0 = tr_H \widehat{C^{\mathcal{M}}_H} = 0$ and $tr_H\, \widehat{I^{\mathbb{C}}_{H_0}} = \dfrac{d^2-1}{2}$, we deduce that $tr_H \widehat{R^{\mathcal{W}}_{H_0}} = \dfrac{d-1}{2d} s^{\mathcal{W}}$. Using $\langle C^{\mathcal{M}}_H, R^{\mathcal{W}}_{H_0}\rangle = |C^{\mathcal{M}}_H|^2$, we obtain that

$$\langle Q^+_{H_0}, R^{\mathcal{W}}_{H_0}\rangle = c_0 \frac{d-1}{8d} s^{\mathcal{W}} + |C^{\mathcal{M}}_H|^2.$$

By taking $c_0 = -\dfrac{8d}{d-1}\dfrac{|C^{\mathcal{M}}_H|^2}{s^{\mathcal{W}}}$ we obtain that $\langle Q^+_{H_0}, R^{\mathcal{W}}_{H_0}\rangle = 0$. Now we have $c_H(Q^+_{H_0}) = c_0 \dfrac{d}{2}\Big(1 - \dfrac{1}{d^2}\Big) g_{\theta_H}$ and then $\big(c_H(Q^+_{H_0})\big)_0 = 0$. Since M is a contact locally sub-symmetric space, then $\nabla R^{\mathcal{W}}_H = 0$ and $s^{\mathcal{W}}$ constant yield to $\nabla C^{\mathcal{M}}_H = 0$. Hence we have $|C^{\mathcal{M}}_H|^2$ and c_0 constant. The parallelism of $Q^+_{H_0} = 0$ directly follows. □

Proposition 6.1 *For any horizontal map ϕ from a compact contact locally sub-symmetric space M, holonomy irreducible and torsionless, to a Sasakian manifold N, we have:*

$$\begin{aligned}
&\int_M \frac{c_0}{2d}|(\nabla^{\mathcal{S}} d\phi_H)^+_0|^2 + \langle \overset{\circ}{C^{\mathcal{M}}_H}(\nabla^{\mathcal{S}} d\phi_H)^+_0, (\nabla^{\mathcal{S}} d\phi_H)^+_0\rangle + \frac{c_0}{2}\Big(1 - \frac{1}{d}\Big)|(\nabla^{\mathcal{S}} d\phi_H)^-|^2 \\
&+\langle \overset{\circ}{C^{\mathcal{M}}_H}(\nabla^{\mathcal{S}} d\phi_H)^-, (\nabla^{\mathcal{S}} d\phi_H)^-\rangle - \frac{c_0}{2}\Big(1 - \frac{1}{d^2}\Big)|\delta^{\nabla'}_H d\phi_H|^2 - \frac{c_0}{2}(d^2-1)|(d\phi(\xi))_{H'}|^2 v_{g_\theta} \\
&= 2\int_M c_0\Big(\frac{1}{d} tr^{2,0}_H((\widehat{\phi^* R^{\mathcal{W}}_{H'}})^{\mathbb{C}}_H) + \Big(1 - \frac{1}{d}\Big) tr^{1,1}_H((\widehat{\phi^* R^{\mathcal{W}}_{H'}})^{\mathbb{C}}_H)\Big) + 4\langle C^{\mathcal{M}}_H, (\phi^* R^{\mathcal{W}}_{H'})_H\rangle v_{g_\theta} \quad (31)
\end{aligned}$$

where $c_0 = -\dfrac{8d}{d-1}\dfrac{|C^{\mathcal{M}}_H|^2}{s^{\mathcal{W}}}$.

Proof. Let $Q^+_{H_0} = c_0 I^{\mathbb{C}}_{H_0} + C^{\mathcal{M}}_H$ with c_0 defined in Lemma 6.1. The horizontal J-invariant symmetric 2-tensor $(c_H(\widehat{R^{\mathcal{W}}_H} \circ \widehat{Q^+_{H_0}}))^{\mathcal{S}}$ is parallel. We deduce from the irreducibility of M that $(c_H(\widehat{R^{\mathcal{W}}_H} \circ \widehat{Q^+_{H_0}}))^{\mathcal{S}} = \lambda g_{\theta_H}$ with $\lambda \in C^\infty(M, \mathbb{R})$. Now,

$$\lambda = \frac{1}{2d} tr_H(c_H(\widehat{R^{\mathcal{W}}_H} \circ \widehat{Q^+_{H_0}}))^{\mathcal{S}} = \frac{2}{d} tr_H(\widehat{R^{\mathcal{W}}_H} \circ \widehat{Q^+_{H_0}}) = \frac{4}{d}\langle R^{\mathcal{W}}_H, Q^+_{H_0}\rangle = \frac{4}{d}\langle R^{\mathcal{W}}_{H_0}, Q^+_{H_0}\rangle = 0.$$

Hence $(c_H(\widehat{R^{\mathcal{W}}_H \circ Q^+_{H_0}}))^{\mathcal{S}} = 0$. If ϕ is a horizontal map from M to N, we have by Lemma 4.2

$$\begin{aligned}
\langle \overset{\circ}{Q}{}^+_{H_0}\,(\nabla^{\mathcal{S}} d\phi_H)_0, (\nabla^{\mathcal{S}} d\phi_H)_0\rangle &= \frac{c_0}{2}\Big(\frac{1}{d}|(\nabla^{\mathcal{S}} d\phi_H)^+_0|^2 + \Big(1-\frac{1}{d}\Big)|(\nabla^{\mathcal{S}} d\phi_H)^-|^2\Big)\\
&+ \langle \overset{\circ}{C}{}^{\mathcal{M}}_H\,(\nabla^{\mathcal{S}} d\phi_H)_0, (\nabla^{\mathcal{S}} d\phi_H)_0\rangle\\
tr_H \widehat{Q^+_{H_0}} &= \frac{c_0}{2}\Big(d^2-1\Big)\\
\langle Q^+_{H_0}, (\phi^* R^{\mathcal{W}}_{H'})_H\rangle &= \frac{c_0}{4}\Big(\frac{1}{d} tr^{2,0}_H(\widehat{(\phi^* R^{\mathcal{W}}_{H'})}^{\mathbb{C}}_H) + \Big(1-\frac{1}{d}\Big) tr^{1,1}_H(\widehat{(\phi^* R^{\mathcal{W}}_{H'})}^{\mathbb{C}}_H)\Big) + \langle C^{\mathcal{M}}_H, (\phi^* R^{\mathcal{W}}_{H'})_H\rangle.
\end{aligned}$$

By replacing in (17), we obtain the formula. □

Now we consider the case of contact locally sub-symmetric spaces with torsion. Let M be a contact locally sub-symmetric spaces with torsion, we recall that we have $\tau^2 = \frac{|\tau|^2}{2d} id_H$. We always may assume that $\frac{|\tau|^2}{2d} = 1$ also τ becomes a paracomplex structure on H. Now $(\tau, J\circ\tau, J)$ defines a so called bi-paracomplex stucture (cf. [16]) on M. Any horizontal 2-tensor t_H on M decomposes into $t_H = t_{H_+} + t_{H_-}$, where $t_{H_\pm} := \frac{1}{2}(t_H \pm \tau^* t_H)$ are respectively the τ-invariant part and the τ-anti-invariant part of t_H. Let $\wedge^2_{H_\pm}(M)$ be the bundle of τ-(anti)invariant horizontal antisymmetric 2-tensors. For $Q_H \in S^2(\wedge^2_H(M))$, we define $Q_{H_\pm} \in S^2(\wedge^2_{H_\pm}(M))$ by

$$\begin{aligned}
Q_{H_\pm}(X,Y,Z,W) &= \frac{1}{4}(Q_H(X,Y,Z,W) \pm Q_H(\tau(X),\tau(Y),Z,W) \pm Q_H(X,Y,\tau(Z),\tau(W))\\
&+ Q_H(\tau(X),\tau(Y),\tau(Z),\tau(W))).
\end{aligned}$$

The tensors $(Q_H)^\pm_\pm \in S^2((\wedge^2_H(M))^\pm_\pm)$ are defined by $(Q_H)^\pm_\pm = (Q^\pm_H)_\pm = (Q_{H_\pm})^\pm$.

Lemma 6.2 *Let M be a contact locally sub-symmetric space with torsion and $s_H \in S^2_H(M)\otimes E$. We have the relations:*

$$\begin{aligned}
A_\theta \overset{\circ}{\owedge} A_\theta\, s_H &= 2(\tau^* s_H - A_\theta \otimes tr_H c_H(A_\theta \otimes s_H)),\\
B_\theta \overset{\circ}{\owedge} B_\theta\, s_H &= 2((J\circ\tau)^* s_H - B_\theta \otimes tr_H c_H(B_\theta \otimes s_H)),\\
c_H(A_\theta \owedge A_\theta) &= c_H(B_\theta \owedge B_\theta) = -\frac{|\tau|^2}{d} g_{\theta_H}, \quad tr_H\, \widehat{A_\theta \owedge A_\theta} = tr_H\, \widehat{B_\theta \owedge B_\theta} = -|\tau|^2.
\end{aligned}$$

Proposition 6.2 *For any horizontal map ϕ from a compact contact locally sub-symmetric space with torsion M to a Sasakian manifold N, we have:*

$$\begin{aligned}
&\int_M |(\nabla^{\mathcal{S}} d\phi_{H_0})^+_+|^2 + |(\nabla^{\mathcal{S}} d\phi_H)^+_-|^2 + |(\nabla^{\mathcal{S}} d\phi_H)^-_+ + \frac{1}{d} A_\theta \otimes \delta^{\nabla'}_H (d\phi_H \circ \tau)|^2 \\
&-|(\nabla^{\mathcal{S}} d\phi_H)^-_- + \frac{1}{d} B_\theta \otimes \delta^{\nabla'}_H (d\phi_H \circ J \circ \tau)|^2 \\
&+\Big(1 - \frac{1}{d}\Big)\Big(|\delta^{\nabla'}_H (d\phi_H \circ J \circ \tau)|^2 - |\delta^{\nabla'}_H (d\phi_H \circ \tau)|^2 + d^2 |(d\phi(\xi))_{H'}|^2 - |\delta^{\nabla'}_H d\phi_H|^2\Big) v_{g_\theta} \\
&= 8 \int_M tr_H((\widehat{\phi^* R^{\mathcal{W}}_{H'}})_H)^-_+ v_{g_\theta},
\end{aligned}$$

$$\begin{aligned}
&\int_M \Big(1 + \frac{2}{d}\Big)|(\nabla^{\mathcal{S}} d\phi_H)^+_-|^2 + \Big(1 - \frac{2}{d}\Big)|(\nabla^{\mathcal{S}} d\phi_H)^-_+ + \frac{1}{d} A_\theta \otimes \delta^{\nabla'}_H (d\phi_H \circ \tau)|^2 \\
&+\Big(1 - \frac{2}{d}\Big)|(\nabla^{\mathcal{S}} d\phi_H)^-_- + \frac{1}{d} B_\theta \otimes \delta^{\nabla'}_H (d\phi_H \circ J \circ \tau)|^2 - \Big(1 - \frac{2}{d}\Big)|(\nabla^{\mathcal{S}} d\phi_{H_0})^+_+|^2 \\
&\Big(1 - \frac{1}{d}\Big)\Big(1 + \frac{2}{d}\Big)\Big(|\delta^{\nabla'}_H (d\phi_H \circ J \circ \tau)|^2 + |\delta^{\nabla'}_H (d\phi_H \circ \tau)|^2 - d^2 |(d\phi(\xi))_{H'}|^2 - |\delta^{\nabla'}_H d\phi_H|^2\Big) v_{g_\theta} \\
&= 8 \int_M \Big(1 - \frac{1}{d}\Big) tr_H(\widehat{\phi^* R^{\mathcal{W}}_{H'}})^+_H + \frac{1}{d} tr_H(\widehat{\phi^* R^{\mathcal{W}}_{H'}})^-_H - tr_H((\widehat{\phi^* R^{\mathcal{W}}_{H'}})_H)^+_+ v_{g_\theta}, \qquad (32)
\end{aligned}$$

Proof. Let the tensors $Q^-_H \in \Gamma(S^2(\wedge^{2,-}_H(M))$ and $Q^+_{H_0} \in \Gamma(S^2(\wedge^{2,+}_{H_0}(M)))$ defined by:

$$Q^-_H = (g_{\theta_H} \owedge g_{\theta_H})^-_+ = \frac{1}{4}(g_{\theta_H} \owedge g_{\theta_H} - \omega_\theta \owedge \omega_\theta + A_\theta \owedge A_\theta - B_\theta \owedge B_\theta)$$

and

$$Q^+_{H_0} = \frac{1}{4}((g_{\theta_H} \owedge g_{\theta_H})_0)^+_- = (I^{\mathbb{C}}_{H_0})_- = \frac{1}{2}(I^{\mathbb{C}}_{H_0} - \mathcal{T}_{H_0}),$$

with $\mathcal{T}_{H_0} = \frac{1}{8}\Big(A_\theta \owedge A_\theta + B_\theta \owedge B_\theta + \frac{2}{d}\omega_\theta \odot \omega_\theta\Big)$. Since M is a contact locally sub-symmetric space, we have $\nabla_H A_\theta = \nabla_H B_\theta = 0$ and then $\nabla_H Q^-_H = \nabla_H Q^+_{H_0} = 0$. From Lemmas 4.2 and 6.2, we have

$$c_H(Q^-_H) = (d-1) g_{\theta_H}, \quad c_H(Q^+_{H_0}) = \frac{(d-1)(d+2)}{4d} g_{\theta_H},$$

and

$$tr_H \widehat{Q^-_H} = d(d-1), \quad tr_H \widehat{Q^+_{H_0}} = \frac{(d-1)(d+2)}{4}.$$

It directly follows that $(c_H(Q^-_H))_0 = (c_H(Q^+_{H_0}))_0 = 0$, and that, $\frac{tr_H \widehat{Q^-_H}}{d} B_\theta - \overset{\circ}{Q^-_H} B_\theta = 0$. Now since $R^{\mathcal{W}}_{H_0} = \frac{2s^{\mathcal{W}}}{d^2}(I^{\mathbb{C}}_{H_0})_+$ then

$$\widehat{R^{\mathcal{W}}_H} \circ \widehat{Q^+_{H_0}} = \widehat{R^{\mathcal{W}}_{H_0}} \circ \widehat{Q^+_{H_0}} = \frac{2s^{\mathcal{W}}}{d^2} \widehat{(I^{\mathbb{C}}_{H_0})_+} \circ \widehat{(I^{\mathbb{C}}_{H_0})_-} = 0.$$

Also $(c_H(\widehat{R^{\mathcal{W}}_{H_0}} \circ \widehat{Q^+_{H_0}}))^{\mathcal{S}} = 0$. Now, let ϕ be a horizontal map from M to N, by Lemmas 4.2 and 6.2, we have

$$\begin{aligned}
\langle \overset{\circ}{Q}{}^-_H (\nabla^{\mathcal{S}} d\phi_H)_0, (\nabla^{\mathcal{S}} d\phi_H)_0 \rangle &= |(\nabla^{\mathcal{S}} d\phi_{H_0})^+_+|^2 + |(\nabla^{\mathcal{S}} d\phi_H)^+_-|^2 + |(\nabla^{\mathcal{S}} d\phi_H)^-_+|^2 - |(\nabla^{\mathcal{S}} d\phi_H)^-_-|^2 \\
&\quad -\frac{1}{4}|tr_H c_H(A_\theta \otimes \nabla^{\mathcal{S}} d\phi_H)|^2 + \frac{1}{4}|tr_H c_H(B_\theta \otimes \nabla^{\mathcal{S}} d\phi_H)|^2 \\
\langle \overset{\circ}{Q}{}^+_{H_0} (\nabla^{\mathcal{S}} d\phi_H)_0, (\nabla^{\mathcal{S}} d\phi_H)_0 \rangle &= \frac{1}{4}\Big(\Big(1+\frac{2}{d}\Big)|(\nabla^{\mathcal{S}} d\phi_H)^+_-|^2 + \Big(1-\frac{2}{d}\Big)|(\nabla^{\mathcal{S}} d\phi_H)^-_+|^2 \\
&\quad +\Big(1-\frac{2}{d}\Big)|(\nabla^{\mathcal{S}} d\phi_H)^-_-|^2 - \Big(1-\frac{2}{d}\Big)|(\nabla^{\mathcal{S}} d\phi_{H_0})^+_+|^2 \\
&\quad +\frac{1}{4}|tr_H c_H(A_\theta \otimes \nabla^{\mathcal{S}} d\phi_H)|^2 + \frac{1}{4}|tr_H c_H(B_\theta \otimes \nabla^{\mathcal{S}} d\phi_H)|^2\Big).
\end{aligned}$$

Now we have

$$tr_H c_H(A_\theta \otimes \nabla^{\mathcal{S}} d\phi_H) = -2\delta^{\nabla'}_H (d\phi_H \circ \tau) \quad \text{and} \quad tr_H c_H(B_\theta \otimes \nabla^{\mathcal{S}} d\phi_H) = -2\delta^{\nabla'}_H (d\phi_H \circ J \circ \tau).$$

Moreover

$$|(\nabla^{\mathcal{S}} d\phi_H)^-_+|^2 = |(\nabla^{\mathcal{S}} d\phi_H)^-_+ + \frac{1}{d} A_\theta \otimes \delta^{\nabla'}_H (d\phi_H \circ \tau)|^2 + \frac{1}{d}|\delta^{\nabla'}_H (d\phi_H \circ \tau)|^2$$

and

$$|(\nabla^{\mathcal{S}} d\phi_H)^-_-|^2 = |(\nabla^{\mathcal{S}} d\phi_H)^-_- + \frac{1}{d} B_\theta \otimes \delta^{\nabla'}_H (d\phi_H \circ J \circ \tau)|^2 + \frac{1}{d}|\delta^{\nabla'}_H (d\phi_H \circ J \circ \tau)|^2.$$

Then

$$\begin{aligned}
\langle \overset{\circ}{Q}{}^-_H (\nabla^{\mathcal{S}} d\phi_H)_0, (\nabla^{\mathcal{S}} d\phi_H)_0 \rangle &= |(\nabla^{\mathcal{S}} d\phi_{H_0})^+_+|^2 + |(\nabla^{\mathcal{S}} d\phi_H)^+_-|^2 + |(\nabla^{\mathcal{S}} d\phi_H)^-_+ + \frac{1}{d} A_\theta \otimes \delta^{\nabla'}_H (d\phi_H \circ \tau)|^2 \\
&\quad -|(\nabla^{\mathcal{S}} d\phi_H)^-_- + \frac{1}{d} B_\theta \otimes \delta^{\nabla'}_H (d\phi_H \circ J \circ \tau)|^2 \\
&\quad \Big(1-\frac{1}{d}\Big)\Big(|\delta^{\nabla'}_H (d\phi_H \circ J \circ \tau)|^2 - |\delta^{\nabla'}_H (d\phi_H \circ \tau)|^2\Big) \\
\langle \overset{\circ}{Q}{}^+_{H_0} (\nabla^{\mathcal{S}} d\phi_H)_0, (\nabla^{\mathcal{S}} d\phi_H)_0 \rangle &= \frac{1}{4}\Big(\Big(1+\frac{2}{d}\Big)|(\nabla^{\mathcal{S}} d\phi_H)^+_-|^2 - \Big(1-\frac{2}{d}\Big)|(\nabla^{\mathcal{S}} d\phi_{H_0})^+_+|^2 \\
&\quad +\Big(1-\frac{2}{d}\Big)|(\nabla^{\mathcal{S}} d\phi_H)^-_+ + \frac{1}{d} A_\theta \otimes \delta^{\nabla'}_H (d\phi_H \circ \tau)|^2 \\
&\quad +\Big(1-\frac{2}{d}\Big)|(\nabla^{\mathcal{S}} d\phi_H)^-_- + \frac{1}{d} B_\theta \otimes \delta^{\nabla'}_H (d\phi_H \circ J \circ \tau)|^2 \\
&\quad +\Big(1-\frac{1}{d}\Big)\Big(1+\frac{2}{d}\Big)\Big(|\delta^{\nabla'}_H (d\phi_H \circ J \circ \tau)|^2 + |\delta^{\nabla'}_H (d\phi_H \circ \tau)|^2\Big)\Big).
\end{aligned}$$

We have also

$$\begin{aligned}
\langle Q^-_H, (\phi^* R^{\mathcal{W}}_{H'})_H\rangle &= \langle (g_{\theta_H} \owedge g_{\theta_H})^-_+, (\phi^* R^{\mathcal{W}}_{H'})_H\rangle = tr_H(\widehat{((\phi^* R^{\mathcal{W}}_{H'})_H)^-_+}\\
\langle Q^+_{H_0}, (\phi^* R^{\mathcal{W}}_{H'})_H\rangle &= \frac{1}{4}\langle (g_{\theta_H} \owedge g_{\theta_H})^+_- - \frac{1}{d}\omega_\theta \odot \omega_\theta, (\phi^* R^{\mathcal{W}}_{H'})_H\rangle\\
&= \frac{1}{4}\Big(\Big(1-\frac{1}{d}\Big) tr_H(\widehat{((\phi^* R^{\mathcal{W}}_{H'})_H)^+_-} - \frac{1}{d} tr_H(\widehat{((\phi^* R^{\mathcal{W}}_{H'})_H)^+_+} + \frac{1}{d} tr_H(\widehat{(\phi^* R^{\mathcal{W}}_{H'})^-_H}\Big)\\
&= \frac{1}{4}\Big(\Big(1-\frac{1}{d}\Big) tr_H(\widehat{\phi^* R^{\mathcal{W}}_{H'})^+_H} + \frac{1}{d} tr_H(\widehat{\phi^* R^{\mathcal{W}}_{H'})^-_H} - tr_H(\widehat{((\phi^* R^{\mathcal{W}}_{H'})_H)^+_+}\Big).
\end{aligned}$$

By replacing in (17) and (18), we obtain the formulas. □

Now we deduce some rigidity results for the contact sub-symmetric space of non-compact type.

In the following we denote respectively by $\mathfrak{g}, \mathfrak{k}, \mathfrak{l}$ the Lie algebras of the Lie groups G, K, L. Let $\tilde{M} = G/K$ be a simply-connected contact sub-symmetric space of non-compact Hermitian type. Then $\tilde{M}$ is the total space of a S^1-fibration π over an irreducible Hermitian symmetric space of non-compact type $\tilde{B} = G/L$. To the Hermitian symmetric space G/L, it is naturally associated an irreducible Hermitian orthogonal involutive Lie algebra $(\mathfrak{g}, s, \beta_{/\mathfrak{p}})$, where s is an involutive automorphism of $\mathfrak{g}$ such that $\mathfrak{l}$=+1-eigenspace of s, $\mathfrak{p}$=−1-eigenspace of s and $\beta_{/\mathfrak{p}}$ is the $ad_{\mathfrak{l}}$-invariant inner product on $\mathfrak{p}$ given by the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{p}$ (we refer to Falbel-Gorodski[17] for the precise definition). Also it follows a so-called irreducible subtorsionless Hermitian sub-orthogonal involutive Lie algebra $(\mathfrak{g}, s, \mathfrak{k}, \beta_{/\mathfrak{p}})$ associated to the sub-symmetric space G/K. Concerning $(\mathfrak{g}, s, \mathfrak{k}, \beta_{/\mathfrak{p}})$, the following facts hold [17]:
$\mathfrak{k} = [\mathfrak{l}, \mathfrak{l}]$, $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{l}$ and $\mathfrak{l} =< \xi^* > \oplus \mathfrak{k}$ with ξ^* in the center of $\mathfrak{l}$.
The ideal $\mathfrak{k}$ of $\mathfrak{l}$ is either a simple ideal or $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ with $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are simple ideals of $\mathfrak{l}$.
The Killing form β is negative definite on $\mathfrak{l}$ and we have the orthogonal decomposition of $\mathfrak{g}$ relatively to β,

$$\mathfrak{g} =< \xi^* > \oplus \mathfrak{k} \oplus \mathfrak{p}.$$

The endomorphism $J^* = ad_{\xi^*/\mathfrak{p}}$ of $\mathfrak{p}$ defines a $ad_{\mathfrak{l}}$-invariant complex structure on $\mathfrak{p}$.
The $ad_{\mathfrak{l}}$-invariant skew-symmetric bilinear form $\beta(J^*., .)$ on $\mathfrak{p}$ is non-degenerate and coincides with $\Pi^*(d\theta)$ where $\Pi : G \to G/K$ is the natural projection.
Now the curvature $\tilde{R}^{\mathcal{W}}_H$ of $\tilde{M}$ is given by $\tilde{R}^{\mathcal{W}}_H = \pi^* R^{\tilde{B}}$ where $R^{\tilde{B}}$ is the curvature of $\tilde{B}$. Also $\tilde{M}$ has nonpositive pseudo-Hermitian sectional curvature and negative pseudo-Hermitian scalar curvature. The Lie algebra expression of $\tilde{R}^{\mathcal{W}}_H$ is given (cf. [18]), for any $X^*_i \in \mathfrak{p}$, by:

$$\tilde{R}^{\mathcal{W}}_H(d\Pi(X^*_1), d\Pi(X^*_2), d\Pi(X^*_3), d\Pi(X^*_4)) = \beta([X^*_1, X^*_2], [X^*_3, X^*_4]).$$

Let $c_0^{'} = -4\frac{|\tilde{R}_H^{\mathcal{W}}|^2}{\tilde{s}^{\mathcal{W}}} = -4\frac{|R^{\tilde{B}}|^2}{s^{\tilde{B}}} (> 0)$ and $\kappa(\tilde{M})$ be the lowest eigenvalue of the quadratic form $s_{H_0} \to \langle \overset{\circ}{\tilde{R}}{}_H^{\mathcal{W}}\ s_{H_0}, s_{H_0}\rangle = \langle \pi^* \overset{\circ}{R}{}^{\tilde{B}}\ s_{H_0}, s_{H_0}\rangle$ associated to $\tilde{R}_H^{\mathcal{W}}$ for horizontal traceless symmetric 2-tensors. The following tabular, coming from those obtained in [10],[12] and [22] for the irreducible Hermitian symmetric spaces of non-compact type, gives the values of $c_0^{'}$ and $\kappa(\tilde{M})$ for the simply-connected contact sub-symmetric spaces of non-compact Hermitian type.

Type	d	$c_0^{'}$	$\kappa(\tilde{M})$
$SU(p,q)/SU(p)\times SU(q)$	pq	$\frac{pq+1}{(p+q)^2}$	$-\frac{1}{p+q}$
$SO^*(2p)/SU(p)$	$\frac{p(p-1)}{2}$	$\frac{1}{4}+\frac{3-p}{4(p-1)^2}$	$-\frac{1}{2(p-1)}$
$Sp(p,\mathbb{R})/SU(p)$	$\frac{p(p+1)}{2}$	$\frac{1}{4}+\frac{3+p}{4(p+1)^2}$	$-\frac{1}{p+1}$
$SO_0(p,2)/SO(p)$	p	$\frac{3}{2p}-\frac{1}{p^2}$	$-\frac{1}{p}$
$E_{6(-14)}/Spin(10)$	16	$\frac{3}{16}$	$-\frac{1}{12}$
$E_{7(-25)}/E_6$	27	$\frac{29}{162}$	$-\frac{1}{18}$

Theorem 6.1 *Let $\tilde{M}$ be a simply-connected contact sub-symmetric space of non-compact Hermitian type other than $SU(d,1)/SU(d)$ and let Γ be a cocompact discrete subgroup of $PsH(\tilde{M})$. Any horizontal pseudoharmonic map ϕ from $M = \tilde{M}/\Gamma$ to a Sasakian manifold N with nonpositive pseudo-Hermitian complex sectional curvature satisfies $\nabla d\phi = 0$.*

Proof. Let ϕ be a horizontal pseudoharmonic map from M to N. Since M and N are torsionless, then (25) together with the assumption on the curvature of N yields to $(d\phi(\xi))_{H'} = 0$, $(\nabla^{\mathcal{S}} d\phi_H)_0^+ = 0$ and $tr_H^{2,0}(\widehat{(\phi^* R_{H'}^{\mathcal{W}})}_H^{\mathbb{C}}) = 0$. Now, using the irreducibility of M, we have by equation (31):

$$\begin{aligned}&\int_M \frac{c_0}{2}\Big(1-\frac{1}{d}\Big)|(\nabla^{\mathcal{S}} d\phi_H)^-|^2 + \langle \overset{\circ}{C}{}_H^{\mathcal{M}}\ (\nabla^{\mathcal{S}} d\phi_H)^-, (\nabla^{\mathcal{S}} d\phi_H)^-\rangle v_{g_\theta}\\ &= 2\int_M c_0\Big(1-\frac{1}{d}\Big) tr_H^{1,1}(\widehat{(\phi^* R_{H'}^{\mathcal{W}})}_H^{\mathbb{C}}) + 4\langle C_H^{\mathcal{M}}, (\phi^* R_{H'}^{\mathcal{W}})_H\rangle v_{g_\theta}. \qquad (33)\end{aligned}$$

Now, since M is pseudo-Einstein, then $C_H^{\mathcal{M}} = R_H^{\mathcal{W}} - \frac{s^{\mathcal{W}}}{d(d+1)} I_H^{\mathbb{C}}$ and $|C_H^{\mathcal{M}}|^2 = |R_H^{\mathcal{W}}|^2 - \frac{s^{\mathcal{W}^2}}{4d(d+1)}$. We deduce by Lemma 4.2 that

$$\frac{c_0}{2}\Big(1-\frac{1}{d}\Big)|(\nabla^{\mathcal{S}} d\phi_H)^-|^2 + \langle \overset{\circ}{C}{}_H^{\mathcal{M}}\ (\nabla^{\mathcal{S}} d\phi_H)^-, (\nabla^{\mathcal{S}} d\phi_H)^-\rangle = c_0^{'}|(\nabla^{\mathcal{S}} d\phi_H)^-|^2 + \langle \overset{\circ}{R}{}_H^{\mathcal{W}}\ (\nabla^{\mathcal{S}} d\phi_H)^-, (\nabla^{\mathcal{S}} d\phi_H)^-\rangle \qquad (34)$$

and

$$c_0\Big(1-\frac{1}{d}\Big)tr_H^{1,1}(\widehat{(\phi^*R^{\mathcal{W}}_{H'})^{\mathbb{C}}_H})+4\langle C^{\mathcal{M}}_H,(\phi^*R^{\mathcal{W}}_{H'})_H\rangle=2c_0^{'}tr_H^{1,1}(\widehat{(\phi^*R^{\mathcal{W}}_{H'})^{\mathbb{C}}_H})+4\langle R^{\mathcal{W}}_H,(\phi^*R^{\mathcal{W}}_{H'})_H\rangle. \tag{35}$$

By the comparison between $c_0^{'}$ and $\kappa(\tilde{M})$, we deduce that (34) is positive excepted if $(\nabla^{\mathcal{S}}d\phi_H)^{-}=0$. Moreover, using the Lie algebra expression of $\tilde{R}^{\mathcal{W}}_H$, we can prove as Jost-Yau do in [19] p 257-273, that (35) is always nonpositive. Also it follows from (33) that $(\nabla^{\mathcal{S}}d\phi_H)^{-}=0$. Consequently $\nabla^{\mathcal{S}}d\phi_H=0$ and $d^{\nabla^{'}}_H d\phi_H=0$. The end of the proof follows from the proof of Theorem 5.2. □

Let I be an open interval of $\mathbb{R}$ containing 0, recall that a regular curve $c:I\to M$ on a strictly pseudoconvex CR manifold M is called a parabolic geodesic (cf. [15]) if $\dot{c}(0)\in H_{c(0)}$ and if there exists $\alpha\in\mathbb{R}$ such that $\nabla_{\dot{c}(t)}\dot{c}(t)=\alpha\xi_{c(t)}$ for $t\in I$. As a consequence of the previous theorem, we have:

Corollary 6.1 *Let $M=\tilde{M}/\Gamma$ as above. Any horizontal pseudoharmonic map ϕ from M to a Sasakian manifold N with nonpositive pseudo-Hermitian complex sectional curvature maps parabolic geodesics of M to parabolic geodesics of N.*

Proof. For any curve $c:I\subset\mathbb{R}\to M$ and any map ϕ from M to N, we have

$$\nabla^{'}_{\dot{(\phi\circ c)}(t)}(\dot{\phi\circ c})(t)=(\nabla_{\dot{c}(t)}d\phi)(\dot{c}(t))+d\phi(\nabla_{\dot{c}(t)}\dot{c}(t)).$$

If ϕ is a horizontal pseudoharmonic map from M to N, we have $\nabla d\phi=0$. Consequently, for a parabolic geodesic $c:I\subset\mathbb{R}\to M$, we obtain that $\nabla^{'}_{\dot{(\phi\circ c)}(t)}(\dot{\phi\circ c})(t)=\alpha d\phi(\xi_{c(t)})=\alpha f\xi^{'}_{(\phi\circ c)(t)}$. Hence $\phi\circ c$ is a parabolic geodesic of N. □

Corollary 6.2 *Let $M=\tilde{M}/\Gamma$ as above. Any horizontal pseudoharmonic map ϕ from M to a Tanaka-Webster flat Sasakian manifold N satisfies $d\phi_H=0$.*

Proof. Let ϕ be a horizontal pseudoharmonic map from M to N. Since N is Tanaka-Webster flat then Theorem 6.1 together with equation (26), yields to $\langle d\phi_H\circ \overset{\circ}{Ric}{}^{\mathcal{W}}_H,d\phi_H\rangle=0$. Now, since M is pseudo-Einstein with $s^{\mathcal{W}}<0$, then $d\phi_H=0$. □

If M is a compact strictly pseudoconvex CR manifold, then $b_1(M)=dim\,Ker\,\triangle_{\mathcal{R}}$. Also, we have using formulas similar to (25),(26) and (31):

Corollary 6.3 *Let $\tilde{M}$ be a simply-connected contact sub-symmetric space of non-compact Hermitian type other than $SU(d,1)/SU(d)$ then $b_1(\tilde{M}/\Gamma)=0$.*

Remark 6.1 *We can observe that the previous corollary directly follows from the two following facts. First M is the total space of a S^1-fibration over a compact irreducible Hermitian locally symmetric space of non-compact type B and then $b_1(M)=b_1(B)$. Second by the Matsushima Theorem [22] we have $b_1(B)=0$.*

7 Rigidity results for CR maps defined on contact locally sub-symmetric spaces

In this section we suppose that N is a strictly pseudoconvex CR manifold and ϕ is a CR map from M to N.

Proposition 7.1 *(i) For any CR map ϕ from a compact contact locally sub-symmetric space torsionless M to N, we have:*

$$\int_M \frac{c_0}{2}\Big(1-\frac{1}{d}\Big)|(\nabla^{\mathcal{S}}d\phi_H)^-|^2+\langle \overset{\circ}{C^{\mathcal{M}}_H}(\nabla^{\mathcal{S}}d\phi_H)^-,(\nabla^{\mathcal{S}}d\phi_H)^-\rangle - c_0(d^2-1)|(d\phi(\xi))_{H'}|^2 v_{g_\theta}$$
$$= 2\int_M c_0\Big(1-\frac{1}{d}\Big)(HBK'^{\mathcal{W}}_{\phi})_H + 4\langle C^{\mathcal{M}}_H,(\phi^* R^{\mathcal{W}}_{H'})_H\rangle v_{g_\theta}. \tag{36}$$

(ii) For any CR map ϕ from a compact contact locally sub-symmetric space with torsion M to N, we have:

$$\int_M \Big(1-\frac{2}{d}\Big)\Big(|(\nabla^{\mathcal{S}}d\phi_H)^-_+ + \frac{1}{d}A_\theta\otimes\delta^{\nabla'}_H(d\phi_H\circ\tau)|^2+|(\nabla^{\mathcal{S}}d\phi_H)^-_- + \frac{1}{d}B_\theta\otimes J'\delta^{\nabla'}_H(d\phi_H\circ\tau)|^2\Big)$$
$$+2\Big(1-\frac{1}{d}\Big)\Big(1+\frac{2}{d}\Big)\Big(|\delta^{\nabla'}_H(d\phi_H\circ\tau)|^2-d^2|(d\phi(\xi))_{H'}|^2\Big)v_{g_\theta} = 8\int_M\Big(1-\frac{1}{d}\Big)(HBK'^{\mathcal{W}}_{\phi})_H-(K'^{\mathcal{W}}_{\phi})_H v_{g_\theta}, \tag{37}$$

with

$$(K'^{\mathcal{W}}_{\phi})_H=\sum_{i,j\leq d} R^{\mathcal{W}}_{H'}(d\phi_H(\epsilon_i),d\phi_H(\epsilon_j),d\phi_H(\epsilon_i),d\phi_H(\epsilon_j)),$$

and

$$(HBK'^{\mathcal{W}}_{\phi})_H=\sum_{i,j\leq d} R^{\mathcal{W}}_{H'}(d\phi_H(\epsilon_i),J'd\phi_H(\epsilon_i),d\phi_H(\epsilon_j),J'd\phi_H(\epsilon_j)).$$

Proof. Let ϕ be a CR map from M to N then $J'\circ d\phi_H = d\phi_H\circ J$. Also we have $\delta^{\nabla'}_H d\phi_H = -dJ'(d\phi(\xi))_{H'}$ by (27) and $\delta^{\nabla'}_H(d\phi_H\circ J\circ\tau)=J'\delta^{\nabla'}_H(d\phi_H\circ\tau)$. Moreover,

$$(\phi^* g_{\theta'})_H(X,Y)=(\phi^*\omega_{\theta'})_H(X,JY)=f\omega_\theta(X,JY)=fg_{\theta_H}(X,Y)$$

and

$$(\phi^*B_{\theta'})_H(JX,JY)=B_{\theta'}(J'd\phi_H(X),J'd\phi_H(Y))=-B_{\theta'}(d\phi_H(X),d\phi_H(Y))=-(\phi^*B_{\theta'})_H(X,Y).$$

Hence we have $(\phi^*g_{\theta'})_H=fg_{\theta_H}$ and $(\phi^*B_{\theta'})^+_H=0$. It follows that for each $Q^+_{H_0}$ defined in Propositions 6.1 and 6.2, we have

$$\langle(c_H(\widehat{R^{\mathcal{W}}_H}\circ\widehat{Q^+_{H_0}}))^{\mathcal{S}},(\phi^*g_{\theta'})_H\rangle=\frac{1}{2}ftr_H(c_H(\widehat{R^{\mathcal{W}}_H}\circ\widehat{Q^+_{H_0}}))^{\mathcal{S}}=4\langle R^{\mathcal{W}}_{H_0},Q^+_{H_0}\rangle=0,$$

and

$$\langle \overset{\circ}{Q}{}^{+}_{H_0}\,(\phi^*B_{\theta'})_H, (\phi^*g_{\theta'})_H\rangle = \frac{1}{2} f tr_H(\overset{\circ}{Q}{}^{+}_{H_0}\,(\phi^*B_{\theta'})^-_H) = 0.$$

Now, we have $(\phi^*R^{\mathcal{W}}_{H'})^-_H = 0$ and $(\phi^*R^{\mathcal{W}}_{H'})^+_H = (\phi^*R^{\mathcal{W}}_{H'})_H$. Consequently $tr_H(\widehat{\phi^*R^{\mathcal{W}}_{H'}})^-_H = 0$ and

$$tr_H(\widehat{\phi^*R^{\mathcal{W}}_{H'}})^+_H = \sum_{i,j\leq d} R^{\mathcal{W}}_{H'}(d\phi_H(\epsilon_i), J^{'}d\phi_H(\epsilon_i), d\phi_H(\epsilon_j), J^{'}d\phi_H(\epsilon_j)).$$

We can choose an adapted frame $\{\epsilon_1, \dots \epsilon_d, J\epsilon_1, \dots J\epsilon_d\}$ of H such that $\tau(\epsilon_i) = \epsilon_i$ and $\tau(J\epsilon_i) = -J\epsilon_i$, also we obtain

$$\begin{aligned} tr_H((\widehat{\phi^*R^{\mathcal{W}}_{H'}})_H)^+_+ &= \sum_{i,j\leq d} (((\phi^*R^{\mathcal{W}}_{H'})_H)_+(\epsilon_i,\epsilon_j,\epsilon_i,\epsilon_j) + ((\phi^*R^{\mathcal{W}}_{H'})_H)_+(\epsilon_i, J\epsilon_j, \epsilon_i, J\epsilon_j)) \\ &= \sum_{i,j\leq d} R^{\mathcal{W}}_{H'}(d\phi_H(\epsilon_i), d\phi_H(\epsilon_j), d\phi_H(\epsilon_i), d\phi_H(\epsilon_j)). \end{aligned}$$

Now the formulas directly follow from equations (17) and (18) combinated to (31) and (32). □

A strictly pseudoconvex CR manifold with constant holomorphic pseudo-Hermitian sectional curvature is called a pseudo-Hermitian space form. The sphere $S^{2d+1} = SU(d+1)/SU(d)$ viewed as the total space of the Hopf fibration over $\mathbb{C}P^d$ and its non-compact dual $SU(d,1)/SU(d)$, are examples of Sasakian pseudo-Hermitian space forms with respectively $s^{\mathcal{W}} > 0$ and $s^{\mathcal{W}} < 0$. The Heisenberg group $\mathcal{H}_{2d+1}$ with its standard pseudo-Hermitian structure is an example of flat Sasakian pseudo-Hermitian space form whereas the unit tangent bundle of the hyperbolic space H^{d+1}, T_1H^{d+1} with its standard pseudo-Hermitian structure is an example of flat non-Sasakian pseudo-Hermitian space form (cf. [14]). Note that all these examples are examples of contact sub-symmetric spaces.

Theorem 7.1 *(i) Let $\tilde{M}$ be a simply-connected contact sub-symmetric space of non-compact Hermitian type other than $SU(d,1)/SU(d)$ and let Γ be a cocompact discrete subgroup of $PsH(\tilde{M})$. Then any horizontal pseudoharmonic CR map ϕ from $M = \tilde{M}/\Gamma$ to a strictly pseudoconvex CR manifold N with nonpositive pseudo-Hermitian complex sectional curvature satisfies $\nabla_H d\phi = 0$.*
(ii) Let $\tilde{M}$ be a simply-connected contact sub-symmetric space of non-compact type other than $\mathcal{H}_{2p+1} \times_G NCH$, then any horizontal pseudoharmonic CR map ϕ from $M = \tilde{M}/\Gamma$ to a pseudo-Hermitian space form N with $s^{\mathcal{W}'} < 0$ is constant.

Proof. A horizontal pseudoharmonic CR map ϕ from M to N satisfies, by Proposition 5.2, $\delta^{\nabla'}_H d\phi_H = (d\phi(\xi))_{H'} = 0$ and $(\nabla^{\mathcal{S}} d\phi_H)^+ = 0$. Since M is torsionless, then it directly follows from (36) that $(\nabla^{\mathcal{S}} d\phi_H)^- = 0$. We deduce from the previous assumptions that

$\nabla_H d\phi = 0$. Hence (i) is proved. Now we assume that N is a pseudo-Hermitian space form, then we have $R^{\mathcal{W}}_{H'} = \dfrac{s^{\mathcal{W}'}}{d'(d'+1)} I^{\mathbb{C}}_{H'}$. Since ϕ is a CR map, we obtain

$$(\phi^* R^{\mathcal{W}}_{H'})^+_H = (\phi^* R^{\mathcal{W}}_{H'})_H = \frac{s^{\mathcal{W}'}}{d'(d'+1)} (\phi^* I^{\mathbb{C}}_{H'})_H = f^2 \frac{s^{\mathcal{W}'}}{d'(d'+1)} I^{\mathbb{C}}_H.$$

Hence we have $\langle C^{\mathcal{M}}_H, (\phi^* R^{\mathcal{W}}_{H'})_H \rangle = 0$ and

$$(HBK'^{\mathcal{W}}_{\phi})_H = tr_H \widehat{(\phi^* R^{\mathcal{W}}_{H'})^+_H} = f^2 \frac{s^{\mathcal{W}'}}{d'(d'+1)} tr_H \widehat{I^{\mathbb{C}}_H} = \frac{f^2}{2} \frac{d(d+1)}{d'(d'+1)} s^{\mathcal{W}'},$$

$$(K'^{\mathcal{W}}_{\phi})_H = tr_H \widehat{((\phi^* R^{\mathcal{W}}_{H'})_H)^+_+} = f^2 \frac{s^{\mathcal{W}'}}{d'(d'+1)} tr_H \widehat{(I^{\mathbb{C}}_H)_+} = \frac{f^2}{4} \frac{d(d-1)}{d'(d'+1)} s^{\mathcal{W}'},$$

and

$$\Big(1 - \frac{1}{d}\Big)(HBK'^{\mathcal{W}}_{\phi})_H - (K'^{\mathcal{W}}_{\phi})_H = \frac{f^2}{4} \frac{(d-1)(d+2)}{d'(d'+1)} s^{\mathcal{W}'}.$$

Since $s^{\mathcal{W}'} < 0$, the right hand sides of (36) and (37) are nonpositive, whereas the left hand sides are nonnegative. We deduce from (36) that $(HBK'^{\mathcal{W}}_{\phi})_H = 0$ and from (37) that $\Big(1 - \dfrac{1}{d}\Big)(HBK'^{\mathcal{W}}_{\phi})_H - (K'^{\mathcal{W}}_{\phi})_H = 0$. In each case, we obtain that $f = 0$. Since $(\phi^* g_{\theta'})_H = f g_{\theta_H}$ and $d\phi(\xi) = f\xi'$, then ϕ is constant. □

A regular curve $c : I \to M$ on a strictly pseudoconvex CR manifold M is called a Carnot-Caratheodory geodesic (cf. [4],[27],[28]) if $\dot{c}(t) \in H_{c(t)}$ and if there exists a function $\alpha : I \to \mathbb{R}$ with $\dot{\alpha}(t) = A_\theta(\dot{c}(t), \dot{c}(t))$ such that $\nabla_{(\dot{c}(t))_H} (\dot{c}(t))_H = -\alpha(t) J(\dot{c}(t))_H$ for $t \in I$. Also, we have

Corollary 7.1 *Let $\tilde{M}$ be a simply-connected contact sub-symmetric space of non-compact Hermitian type other than $SU(d,1)/SU(d)$ and $M = \tilde{M}/\Gamma$. Then any horizontal pseudo-harmonic CR map ϕ from M to a Sasakian manifold N with nonpositive pseudo-Hermitian complex sectional curvature maps Carnot-Caratheodory geodesics of M to Carnot-Caratheodory geodesics of N.*

Proof. Let $c : I \subset \mathbb{R} \to M$ be a Carnot-Caratheodory geodesic, then $\nabla_{(\dot{c}(t))_H} (\dot{c}(t))_H = -\alpha J(\dot{c}(t))_H$ with α constant (since M is torsionless). Let ϕ be a horizontal pseudoharmonic CR map from M to N, we obtain using $\nabla_H d\phi = 0$ that $\nabla_{(\dot{(\phi \circ c)}(t))_{H'}} ((\phi \circ c)(t))_{H'} = -\alpha d\phi_H (J(\dot{c}(t))_H) = -\alpha J' d\phi_H ((\dot{c}(t))_H) = -\alpha J' ((\dot{\phi \circ c})(t))_{H'}$. Also $\phi \circ c$ is a Carnot-Caratheodory geodesic of N. □

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